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A Shadow rate model with time-varying lower bound of interest rates

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Abstract

Typically a constant – or zero – lower bound for interest rates is applied in shadow rate term structure models. However, euro area yield curve data suggest that a time-varying lower bound might be appropriate for the euro area. I show that this indeed is the case, i.e. a shadow rate model with time-varying lower bound outperforms the constant lower bound model in euro area data. I argue that the time-variation in the lower bound is related to the deposit facility rate and, thus, to monetary policy. This time-variation in the lower bound gives a new channel via which monetary policy may affect the yield curve in a shadow rate model. I show that the intensity of this channel depends on how tightly the lower bound restricts the yield curve, and I argue that this channel has recently become important for the euro area.

JEL Codes: E43, E44, E52

Keywords: term structure models, shadow rates, policy liftoff, monetary policy, zero lower bound

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1 Introduction

The term structure of interest rates is typically described by a Gaussian affine dynamic term structure model. However, these models ignore the lower bound for nominal interest rates. As a result, these models do not describe the dynamics of near-zero interest rates.[1] One – currently widely used – approach to setting an appropriate lower bound for the interest rate in a dynamic term structure model is based on the shadow-rate concept, first introduced by Black (1995).

The lower bound for the nominal interest rate in a shadow rate model is a parameter which can be calibrated or estimated. On theoretical grounds it can be argued that the lower bound of nominal interest could be set to zero or to some negative number.[2] Studies have used zero, negative and positive numbers for the lower bound, typically values from 0.25 to -0.25 percent.[3] So, there is no consensus on the level of the lower bound, but it is considered to be some constant. This might be an appropriate assumption at least for models applied to US data since short-term interest rates and monetary policy in US support the idea of a constant lower bound. That is, the Federal Reserve has kept its target for the federal funds rate – which is the key short-term interest – positive and constant for a long period of time.[4]

The constant lower bound might be an appropriate choice for the US but it might not be for other economies. In the presence of high excess liquidity, euro area short-term interest rates tend to anchor to the deposit facility rate of Eurosystem and to anticipate its movements (see European Central Bank, 2014). Moreover, the deposit facility rate and its expected path provides a lower bound for the interest rates in the euro area since no bank would lend overnight at a rate

[1]Especially, Gaussian affine models fail in two respects. First, these models potentially place positive probabilities on future interest rates which are "too" negative. Secondly, these models do not capture the stylized fact that policy rates tend to stay at a low level or at the lower bound for a prolonged periods of time. See, Kim and Singleton (2012) for more details.

[2]The existence of cash provides a zero risk-free nominal return when negative interest rates enable an arbitrage between currency and short-term loans. However, costs associated with storing, transferring and spending large amounts of currency make negative interest rates possible and many European yields have been negative. For instance, in Switzerland, short-term rates have been around -0.8 percent.

[3]For models applied to US data, see, Krippner (2015b); Wu and Xia (2016); Christensen and Rudebusch (2016); Bauer and Rudebusch (2016) and Kim and Priebsch (2013). For euro area, see Lemke and Vladu (2014); Pericoli and Taboga (2015), and for the UK, see Andreasen and Meldrum (2015).

[4]Federal Reserve set the target range for federal funds at 0 - 0.25 percent on December 16, 2008 and the liftoff happened on December 16, 2015. The fed funds rate has been positive during this period although there has been some variation inside the range set by Federal Reserve (see, for example, Gagnon and Sack 2014).

lower than the rate it can get on deposits with the central bank.[5] However, the deposit facility rate is not constant since it is a monetary policy instrument that might be changed from time to time. Further, it hard to say what could be the lower bound for the rate since the Governing Council of ECB has not set a lower bound for it. These factors suggest that the lower bound for interest in the euro area might be time-varying when the use of a constant lower bound is not appropriate. Moreover, time-variation in the lower bound – due to monetary policy – gives a new channel for monetary policy to affect the yield curve. Obviously, this channel is not present in a shadow rate model with a constant lower bound when the effects of this monetary policy instruments on the yield curve are not correctly captured.

In this paper I incorporate a time-varying lower bound for nominal interest rates in a shadow rate model by Krippner (2015b) with three factors. This is a shadow rate arbitrage free dynamic Nelson and Siegel model ála Christensen, Diebold, and Rudebusch (2011) in continuous time. I impose the time-varying lower bound in a straightforward manner and I assume it to be exogenous in the model. There is two reasons for this. First, this is (one of) the first paper(s) to consider a time-varying lower bound, and the first step is to see if the time-varying lower bound is suitable for this class of models. Second, the motivation for introducing the time-varying lower bound in the model is purely empirical since theory does not provide guidance on how the time-varying lower bound should be incorporated in the model. Hence, a good starting point to study models with time-varying lower bounds is to have the lower bound exogenous in the model. Since there is no clear-cut procedure to set the time-varying lower bound, I consider different time-varying lower bounds.

I show that the model with time-varying lower bound outperforms the model with a constant lower bound in many dimensions. First, a shadow rate model with a time-varying lower bound fits the data better than the model with a constant lower bound. Second, a model with a time-varying lower bound produces plausible estimates for the monetary policy stance at the lower bound of interest rates. I consider two measures: time to liftoff and the short-term shadow rate. In the case of the liftoff measure, the model with time-varying lower bound produces a robust

[5] For the banks' depositors, the lowest interest rate that they could face is the interest rate just below the deposit facility rate.

and plausible measure of the time it takes for the short-term interest rate to return to positive territory.[6] Moreover, the model with time-varying lower bound can also produce a short-term shadow rate which is highly negatively correlated with the liftoff measure when this describes the stance of monetary policy in a similar fashion. I also estimate the same model with reasonable constant lower bounds, but these models do not generate highly negatively correlated short-term shadow rates with the liftoff measure. In summary, the shadow rate model with time-varying lower bound outperforms the constant lower bound model and it should be favored for the euro area.

Finally, I analyze the effects of monetary policy on the yield curve when monetary policy is implemented by changing the lower bound. I show that Krippner's shadow rate model can be seen as a censored regression model or Tobit model which facilitates the interpretation of the model. Moreover, this representation enables one to derive a simple equation which decomposes variations in the forward rate into fluctuations in the shadow forward rate or lower bound. The significance of these two sources of variation depends on how tightly the lower bound restricts the dynamics of interest rates: the more tightly the lower bound restricts the interest rates, the more effective are changes in the lower bound for changing interest rates. Moreover, I show that the lower bound has recently restricted also long-term interest rates in the euro area, which implies that changes in the deposit facility rate or its expected path affect long-term interest rates. Paradoxically, the more tightly the lower bound restricts the yield curve, the more effectively can the yield curve be affected by monetary policy.

Currently there is a well-established literature on shadow rate models. Krippner (2015b); Wu and Xia (2016); Christensen and Rudebusch (2016); Bauer and Rudebusch (2016) and Kim and Priebsch (2013) applied shadow rate models to US data; for euro area, see Lemke and Vladu (2014); Pericoli and Taboga (2015), and for the UK, see Andreasen and Meldrum (2015).[7] The main contribution of this paper to this literature is in introducing the concept of a time-varying lower bound of interest rates and to show that it outperforms the constant lower bound model with

[6]This is a typical result in the literature also for shadow rate models with constant lower bound. See, for example, Krippner (2015b); Christensen and Rudebusch (2016) and Bauer and Rudebusch (2016).

[7]For the first references to shadow rate models, see Black (1995); Bomfim (2003); Ueno, Baba, and Sakurai (2003) and Krippner (2011).

euro area data. Closest to this paper is Lemke and Vladu (2014) who did a comparative static analysis of how a shift in the lower bound would affect the yield curve and were the first to suggest that a time-varying lower bound might be appropriate for the euro area. However, they did not provide an estimated shadow rate model with time-varying lower bound.

The second contribution of this paper to this literature is to show that the shadow rate model by Krippner (2015b) can be seen as a certain type of censored regression model or Tobit model. Generally, this remark makes the interpretation of shadow rate models quite straightforward. Priebsch (2013) also shows that shadow rate models can be considered as models with censored variables, but he does not derive any easily interpreted equation connecting these models to Tobit models.

Third, this paper empirically evaluates the effects of variations in the lower bound on the yield curve. Lemke and Vladu (2014) and Bauer and Rudebusch (2016) also discuss the issue, but here the effects of time-variation in the lower bound on long-term interest rates is quantified.

The rest of the paper is organized as follows. Section 2 introduces the model and Section 3 focuses on the data and estimation of the model. Section 4 discusses the evolution of short-term interest rates in the euro area and introduces the sequences of lower bounds used in the paper. Section 5 evaluates the estimated models. Section 6 discusses the different measures of monetary policy stance produced by the models. Section 7 shows how changes in the lower bound matter for the yield curve and discusses the implications of these findings for monetary policy. Section 8 concludes the paper.

2 The model

I operate with a term structure model by Krippner (2015b) with three factors, which is a shadow rate version of Christensen, Diebold, and Rudebusch (2011). I first introduce a general affine arbitrage free shadow rate model model and then I describe its empirical counterpart. Finally, I describe in detail how the effective lower bound for interest rates is introduced in the model. I mainly follow Krippner (2015b) and introduce the model in continuous time.

2.1 A general model

2.1.1 Short-term shadow and interest rate

A shadow rate term structure model is similar to a standard dynamic term structure model except that the affine short-rate equation is replaced by a shadow rate specification. In the standard model the short-rate, r_t, is affine in the N latent factors or state variables x_t, i.e. $r_t = a_0 + b_0' x_t$. In a shadow rate model the short-rate is the maximum of the shadow rate, s_t, and an effective lower bound $\underline{r}_t$, but the shadow rate is affine Gaussian:

$$r_t = \max(s_t, \underline{r}_t) \text{ and } s_t = a_0 + b_0' x_t. \tag{1}$$

In the shadow rate model the lower bound for the interest rate is now set at $\underline{r}_t$. That is, the short-term interest rate cannot be lower than the lower bound, $\underline{r}_t$. If $s_t \geq \underline{r}_t$, the short-term rate is equal to shadow rate, but if $s_t < \underline{r}_t$ then $r_t = \underline{r}_t$. Note that the shadow rate can be negative to the needed extent, and it can be stay negative for a long time. This enables r_t to be at the lower bound for long time. The difference between the specification given by equation (1) and existing literature on shadow rate models is that the lower bound may now vary in time. The path of $\underline{r}_t(\tau)$ to the future horizon $\tau \in [\, t, \infty\,)$ is exogenous in the model. Hence, I can assume that the expected values of the lower bound are always equal to the current value $\underline{r}_t(\tau) = \underline{r}_t \; \forall \; \tau$.

2.1.2 Factor dynamics

Under physical measure, $\mathbb{P}$, x_t evolves as a correlated vector Ornstein-Uhlenbeck process:

$$dx_t = \kappa(\theta - x_t)dt + \Sigma dW_t, \quad dW_t \overset{iid}{\sim} \mathcal{N}(0, 1) \tag{2}$$

where κ is a matrix of mean-reversion parameters, θ is a vector of constants, Σ captures correlations between innovations, which are given by Wiener components dW_t. The dynamics of state variables are determined by the solution of (2):

$$x_t(\tau) = \theta + e^{\kappa\tau}(x_t - \theta) + \int_t^{t+\tau} e^{-\kappa(\tau - v)} \Sigma dW(v) dv \tag{3}$$

where $e^{\kappa\tau}$ is the matrix exponential and the factor dynamics follow the first order Gaussian VAR.

It is assumed that there exists a risk-neutral probability measure $\mathbb{Q}$ that prices all financial market assets. Hence, the expected returns for all assets under this measure are equal to the risk-adjusted short-term rate, i.e. there are no arbitrage opportunities. Under the risk-adjusted $\mathbb{Q}$ measure, x_t also follows a correlated vector Ornstein-Uhlenbeck process:

$$dx_t = \tilde{\kappa}(\tilde{\theta} - x_t)dt + \Sigma d\tilde{W}_t, \quad d\tilde{W}_t \overset{iid}{\sim} \mathcal{N}(0, 1) \tag{4}$$

where mean and persistence parameters now differ from their $\mathbb{P}$ measure counterparts. However, the dynamic process for x_t under the $\mathbb{Q}$ measure are analogous to those under the $\mathbb{P}$ measure. Moreover, these assumptions imply that the stochastic discount factor is essentially-affine, as in Duffee (2002).

Now equations (1), (2), (4) and the assumption of no-arbitrage provide a complete description of the current yield curve and its evolution through time. However, general models are not identified and must be restricted.

2.2 Empirical model

The parameters of the model must restricted to enable a unique identification for estimation. In the absence of a lower bound, Christensen, Diebold, and Rudebusch (2011) derived a set of restrictions for the general model which provided a new class of affine arbitrage free models that can be called arbitrage free Nelson-Siegel models.[8] The main idea is to restrict the general model such that the factor loadings are the same as the regressors in Nelson and Siegel (1987). To do

[8] For a review, see Diebold and Rudebusch (2013).

that, I first set $a_0 = 0$, $b_0 = [1 \quad 1 \quad 0]'$, which implies that equation (1) can be rewritten as

$$r_t = \max(s_t, \underline{r}_t) \text{ and } s_t = x_t^1 + x_t^2 \tag{5}$$

where x_t^1 is interpreted as the level factor, x_t^2 is the slope factor, and the last factor is interpreted as the curvature factor.

The only restriction imposed on the factor dynamics under $\mathbb{P}$ measure is that Σ is a is a lower triangular matrix identified by Choleski decomposition. Otherwise, there are no restrictions for the dynamics under the $\mathbb{P}$ measure. Restrictions on factor dynamics under $\mathbb{Q}$ measure concern $\tilde{\theta}$ and $\tilde{\kappa}$. Without loss of generality, it can assumed that $\tilde{\theta} = 0$ and the appropriate restrictions on $\tilde{\kappa}$ result following factor dynamics:

$$\begin{bmatrix} dx_t^1 \\ dx_t^2 \\ dx_t^3 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \phi & -\phi \\ 0 & 0 & \phi \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{bmatrix} + \Sigma d\tilde{W}_t. \tag{6}$$

Now equations (5), (2), (6) and the assumption of no-arbitrage define an identified term structure model.

2.3 Term structure and the lower bound

Equation (5) introduces non-linearity into an otherwise linear system. Hence, the well-known closed form solution for yields and bond prices cannot be found. However, Krippner (2011) provides an approximation for pricing formulas that have closed form solutions and, thus, the model is called Krippner's shadow rate model. Before introducing the approximation, it is useful to derive the shadow term structure.

2.3.1 Term structure without the lower bound: A shadow term structure

Shadow rates describe interest rates in the absence of a lower bound. This term structure is derived in a shadow rate model by the same equations as the term structure in a typical Gaussian affine term structure model. The shadow term structure is part of a shadow rate model when it is useful to introduce before the lower bound is made effective.

Without a lower bound the short-term interest rate is equal to the shadow rate when it is given by $s_t = x_t^1 + x_t^2$ and the typical term structure relationships are valid. The instantaneous shadow forward rate, $f_t^s(\tau)$, (hereafter, shadow forward rate) under the $\mathbb{Q}$ measure is given by

$$f_t^s(\tau) \quad = \quad \tilde{\mathbb{E}}\left[s_t(\tau)\,|x_t\right] - \text{ve}(\tau) \quad \text{where} \tag{7}$$

$$\tilde{\mathbb{E}}\left[s_t(\tau)\,|x_t\right] \quad = \quad b_0' e^{-\tilde{\kappa}\tau} x_t = x_t^1 + x_t^2 e^{-\phi\tau} + x_t^3 \phi\tau e^{-\phi\tau}. \tag{8}$$

Now $\tilde{\mathbb{E}}\left[s_t(\tau)\,|x_t\right]$ is the expected path of the shadow short-term rate under the $\mathbb{Q}$ measure and $\text{ve}(\tau)$ captures the effects of Jensen's inequality.[9] Moreover, shadow interest rates, $S_t(\tau)$, are given by the standard term-structure relationship:

$$S_t(\tau) = \frac{1}{\tau}\int_0^\tau f_t^s(v)dv = \frac{1}{\tau}\int_0^\tau \tilde{\mathbb{E}}\left[s_t(v)\,|x_t\right]dv - \frac{1}{\tau}\int_0^\tau \text{ve}(v)dv. \tag{9}$$

where S_t is an affine function of the state variables. Equations (7), (8) and (9) give the term structure for the shadow interest.

2.3.2 Introducing the lower bound

Black (1995) introduced the idea that cash provides an instantaneous call option on the shadow rate for investors. Consider an investor who could invest on the shadow rate, s_t, but cash provides instantaneous (net) return $\underline{r}_t$. Then, she will invest on the shadow rate if $s_t > \underline{r}_t$, otherwise she will hold cash. Hence, cash provides an instantaneous call option on the shadow rate. This interpretation of the lower bound of interest rates can be seen more clearly if equation (5) is rewritten as

$$r_t = s_t + \max\{\underline{r}_t - s_t, 0\}. \tag{10}$$

[9] Christensen, Diebold, and Rudebusch (2011) and Krippner (2015b, p. 78-79) give expression for $\text{ve}(\tau)$.

Now the intuition is that the investor will receive s_t, but the payoff provided by the call option, $\max\{\underline{r}_t - s_t, 0\}$, will compensate her such that always $r_t \geq \underline{r}_t$.[10] Equation (10) must also hold for the expected path of short-term interest when

$$r_t(\tau) = s_t(\tau) + \max\{\underline{r}_t - s_t(\tau), 0\}. \quad (11)$$

Next assume that the forward rate can be approximated by the conditional mean of the short term rate when Jensen's inequality is ignored. Hence, this approximation relies on the forward rate, which is not equal to the arbitrage-free forward rate. However, with this approximation, a lower bound consistent (instantaneous) forward rate under measure $\mathbb{Q}$, $f_t(\tau)$, can be given as

$$f_t(\tau) = \tilde{\mathbb{E}}\left[r_t(\tau)\,|x_t\right]. \quad (12)$$

Substituting equation (11) into equation (12) gives

$$\begin{aligned} f_t(\tau) &= \tilde{\mathbb{E}}\left[s_t(\tau)\,|x_t\right] + \tilde{\mathbb{E}}\left[\max\{\underline{r}_t - s_t(\tau), 0\}\,|x_t\right] \\ &= f_t^s(\tau) + z_t(\tau) \end{aligned} \quad (13)$$

where, in moving from the first line to the second, equation (7) is utilized and the assumption given by (12) is applied for shadow rates. Moreover, I define $z_t(\tau) \equiv \tilde{\mathbb{E}}\left[\max\{\underline{r}_t - s_t(\tau), 0\}\,|x_t\right]$. Now the forward rate is the sum of the shadow forward rate and the option value of cash, $z_(\tau)$. For both terms, there exists a closed form solution: $f_t^s(\tau)$ can be evaluated using equation (7) and Krippner (2015b, p. 111) shows that the option effect can be written as

$$z_t(\tau) = \left[\underline{r}_t - f_t^s(\tau)\right]\left(1 - \Phi\left[\frac{f_t^s(\tau) - \underline{r}_t}{\omega(\tau)}\right]\right) + \omega(\tau)\phi\left[\frac{f_t^s(\tau) - \underline{r}_t}{\omega(\tau)}\right] \quad (14)$$

where $\omega(\tau)^2 = \tilde{\text{var}}\left[s_t(\tau)\,|x_t\right]$,[11] $\Phi(\cdot)$ is the cumulative distribution function for the unit normal distribution and $\phi(\cdot)$ is its density function. Finally, substituting equation (14) into equation (13)

[10] Note that equation (10) provides the same results as equations (5) and (1). When $\underline{r}_t - s_t > 0$, then $r_t = \underline{r}_t$ and $r_t = s_t$ when $\underline{r}_t - s_t < 0$.

[11] Krippner (2015b, p. 135) shows how the conditional variance of shadow rate can be calculated.

yields the measurement equation for the model:

$$f_t(\tau) = \underline{r}_t + (f_t^s(\tau) - \underline{r}_t)\Phi\left[\frac{f_t^s(\tau) - \underline{r}_t}{\omega(\tau)}\right] + \omega(\tau)\phi\left[\frac{f_t^s(\tau) - \underline{r}_t}{\omega(\tau)}\right]. \tag{15}$$

It is straightforward to show that equation (15) provides the first conditional moment for a censored variable, which eases the interpretation of the equation. This representation connects this shadow rate model to a wider class of econometric models known as Tobit models.[12] To see this, notice that $\Phi\left[-(\underline{r}_t - f_t^s)/\omega\right] = 1 - \Phi\left[(\underline{r}_t - f_t^s)/\omega\right]$ and that $\phi(\cdot)$ is a symmetric distribution when equation (15) can be written as

$$f_t(\tau) = \Phi\underline{r}_t + (1 - \Phi)\left(f_t^s(\tau) + \omega(\tau)\lambda\right) \tag{16}$$

where

$$\Phi(\cdot) \quad = \quad \Phi\left[\frac{\underline{r}_t - f_t^s(\tau)}{\omega(\tau)}\right], \quad \lambda = \frac{\phi}{1-\Phi} \quad \text{and} \quad \phi(\cdot) = \phi\left[\frac{\underline{r}_t - f_t^s(\tau)}{\omega(\tau)}\right].$$

Now λ equals the hazard function for the standard normal distribution and equation (16) can be seen as an equation from a censored regression model.[13] The first term in equation (16), $\Phi\underline{r}_t$, describes the probability mass associated with the censored part of the distribution, which is assigned to the censoring point. The second term gives the expected value of the uncensored distribution. Note also that when the shadow forward rate is very negative, the forward rate equals the lower bound, and with very positive values for the shadow forward rate, the forward rate equals the shadow forward rate.[14] The main result is that $f_t(\tau) \geq f_t^s(\tau)$ since all the values

[12] Krippner (2015b) describes the model as if it would be a censored regression model and provides equation (15) but does not make a direct connection to regression models with censored variables. Priebsch (2013) shows that shadow rate models are generally models with censored variables but he do not provide an easily interpreted measurement equation.

[13] For example, Greene (2003, chap. 22) provides detailed discussion of estimation procedures with censored data. Especially, Theorem 22.3 provides the following result: if $y^\star \sim \mathcal{N}(\mu, \sigma^2)$ and $y = a$ if $y^\star \leq a$ or else $y = y^\star$, then

$$E\left[y\right] = \Phi a + (1 - \Phi)(\mu + \sigma\lambda).$$

which is the unconditional version of equation (16).

[14] That is, given time-invariant ω, equation (16) implies

$$\lim_{f_t^s \to -\infty} f_t = \underline{r}_t \quad \text{and} \quad \lim_{f_t^s \to \infty} f_t = f_t^s.$$

$f_t^s(\tau) \leq \underline{r}_t$ are reported as a higher value $\underline{r}_t$, which increases the expected value. That is, the lower bound raises the expected value of the forward rate. Note that, this mechanism applies even if $f_t(\tau) > \underline{r}_t$, Appendix A.1 gives a graphical illustration for the formation of the forward rate in the presence of a lower bound.

Finally, note that interest rates, $R_t(\tau)$, are defined similarly as in the case of the shadow rate – following the standard term-structure relationship – when equation (13) implies following:

$$R_t(\tau) = \frac{1}{\tau}\int_0^\tau f_t(v)dv = \frac{1}{\tau}\int_0^\tau f_t^s(v)dv + \frac{1}{\tau}\int_0^\tau z_t(v)dv = S_t(\tau) + Z_t(\tau) \tag{17}$$

where, $S_t(\tau)$, is the shadow interest rate defined by equation (9) and $Z_t(\tau)$ is the average option value of cash over horizon τ. The effects of the lower bound on the interest rate depends on how much the lower bound restricts on average the path of shadow forward rate over horizon τ.

Krippner (2011) was the first to derive a closed form solution for a shadow rate model with more than one factor. The key contribution was to approximate the option value of cash, which should be valued as an American call option by a series of European call options. The derivation above is different and is based on Krippner (2015b), but it comes to the same approximation. Christensen and Rudebusch (2016) show that Krippner's approximation error is less than 10 basis points for the 10-year maturity. Moreover, Wu and Xia (2016) independently derived the same approximation as Krippner, but it is for discrete time. Priebsch (2013) provides a more accurate approximation based on the second order Taylor approximation, but this method is more time-consuming to estimate.

To summarize, the model used in this paper can be described by a state-space representation where the measurement equation is given by equation (15) when the idiosyncratic error term is added and the transition equation under $\mathbb{P}$ measure is given by (2) and under $\mathbb{Q}$ by (4). Finally, interest rates are given by (17).

3 Data and estimation

3.1 Data

The data consist of monthly observations of interest rates from January 1999 to March 2016. The data are from two sources. First, and for most, interest rates are observations on overnight index swaps (OIS), and these swaps are based on EONIA (the overnight unsecured interbank rate in euro area), which is the interest rate on overnight unsecured deposits between prime banks. In OIS contracts, one counterparty receives a variable payment given by EONIA and the other counterparty receives the fixed OIS rate. Hence, OIS rates in different maturities provide a term-structure for EONIA. OIS rate data downloaded via Bloomberg can be directly interpreted as a term-structure of zero-coupon spot rates. The OIS market is liquid also for longer maturities, and overnight deposits between prime banks are considered risk-free. Therefore, OIS rates have started to be considered as proxies for risk-free interest rates.[15] Data on OIS rates are available only from January 2006 onwards.

To provide a sufficiently long sample for the estimation, the second source of the data is government bonds of Germany and France. That is, the data before January 2006 are mean yields on government zero-coupon bonds of France and Germany. These data are also available via Bloomberg. Hence, the data used in estimation includes 207 end-of-month observation and included maturities are 3 and 6 months and 1, 2, 5, 7 and 10 years. Figure 1 shows the data.

3.2 Estimation

The details of the estimation procedure are provided by Krippner (2015b), and I only briefly summarize the procedure.[16] The ML estimation of the model is done in state space form where the measurement equation is given by equation (15) with the idiosyncratic measurement error

[15] Recent yield curve estimations for euro area are typically based on OIS rates. See, for example, Lemke and Vladu (2014) and Pericoli and Taboga (2015).

[16] MATLAB programs used by Krippner (2015b) are available at the homepage of The Reserve Bank of New Zeland.

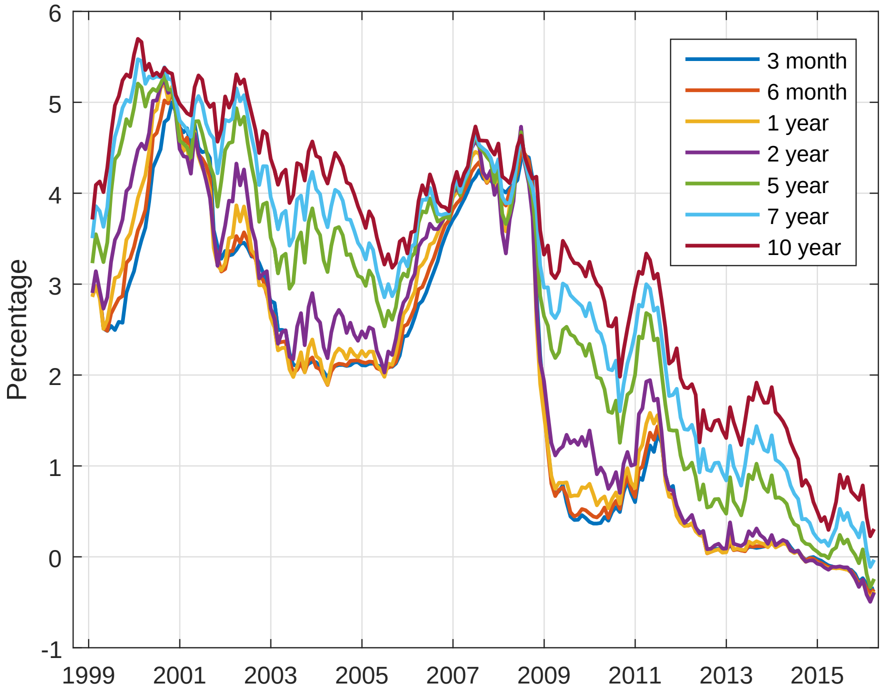


Figure 1: Monthly yield curve data from January 1999 to March 2016. Before January 2006 data is an average of France and Germany bond yields and after that the data is based on OIS rates.

term $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$ added, where ϵ_t is the vector of errors across maturities. The measurement error also serves as a residual for the model or a component of the interest rate which is not explained by the model. The transition equation under the $\mathbb{P}$ measure is given by (2), and Kalman filter is applied to the transition equation.

It should be noticed that the measurement equation is now non-linear respect the state variables since $\Phi(\cdot)$ and $\phi(\cdot)$ are non-linear respect to x_t. With a non-linear measurement equation, a non-linear Kalman filter must be applied: here an iterated extended Kalman filter is applied. In an iterated extended Kalman filter a non-linear measurement equation is approximated by first-order Taylor-approximation with the best available estimate of x_t. Then the state variable vector is updated as in Kalman filter. The first-order Taylor approximation can be now repeated with the updated state variable vector. This iteration can be done several times or until convergence in x_t.[17]

It is straightforward to include the time-varying lower bound in the estimation procedure when it is considered as an exogenous process in the model. Kalman filter is applied recursively throughout the sample when a new value for the lower bound can be applied at each step of recursion for the cross-section estimation of yields. Given the sequence for the lower bound $\{\underline{r}_t\}_{t=1}^{T}$ and the data $\{R_t\}_{t=1}^{T}$,the parameters of the model, $\mathbb{A}$, are such that the log-likelihood, $L\left(\{R_t\}_{t=1}^{T}, \mathbb{A}, \{\underline{r}_t\}_{t=1}^{T}\right)$, is maximized.

4 Time-varying lower bound of the interest rate for the euro area

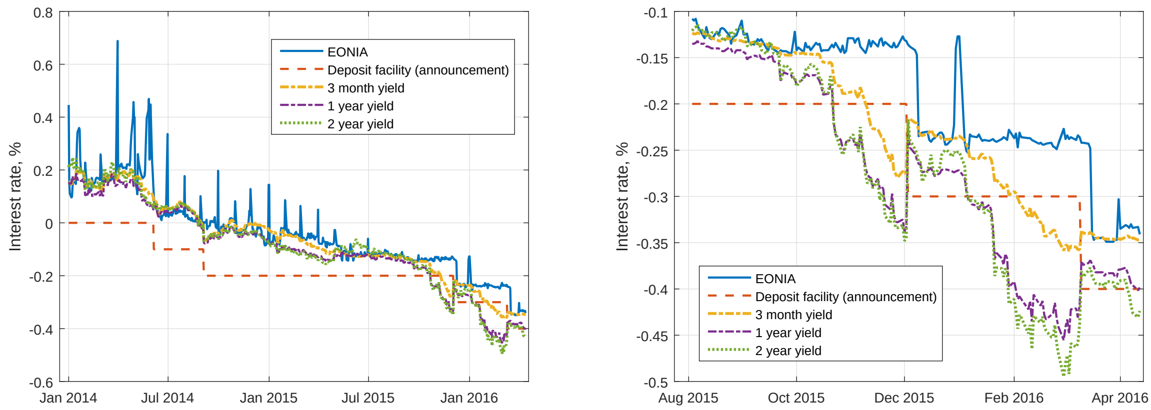
The use of a time-varying lower bound in a shadow rate model can be motivated by studying the evolution of euro area short-term interest rates, and in Section 4.1 I argue that euro area data support the idea of a time-varying lower bound. Since the lower bound sequence is assumed to be exogenous in the model, it is not a priori clear how to form the sequence. In Section 4.2 I identify many sequences for the lower bound whose performances can be then evaluated in the model.

4.1 The case for a time-varying lower bound

For euro area a constant lower bound might not be appropriate, as Lemke and Vladu (2014) argued. Figure 2 shows the evolution of selected short-term interest rates which are also the lowest interest rates in the sample. Figure 2 also includes EONIA and the Eurosystem's deposit facility rate.

From Figure 2 two stylized facts arise. First, short-term interest rates have been trending downwards also after they have reached the zero-lower bound. Second, and related to the first one, short-term interest rates tend to anchor to the deposit facility rate and anticipate its movements.

[17] Krippner (2015b, p. 118) argues that an iterated extended Kalman filter is more reliable and accurate than the typically applied extended Kalman filter. He also argues that an iterated extended Kalman filter outperforms the unscented Kalman filter.



(a) Daily data from 1.1.2014 to 12.4.2016

(b) Zoomed to 3.8.2015 - 12.4.2016

Figure 2: Short-term interest for euro area: the deposit facility rate of Eurosystem, EONIA (the overnight unsecured interbank rate in euro area) and interest rates from swap contracts associated with EONIA (OIS).

The deposit facility rate is the key policy rate due to abundant excess liquidity in the euro area (see, European Central Bank 2014 for further analysis). This can be seen, especially, for December 2015 and March 2016, when the market expected the Governing Council of ECB to cut the deposit facility rate by around 20 basis points whereas the actual decrease was 10 basis points. This caused short-term rates jump up significantly upon the announcement.

The stylized facts make it quite difficult to set a constant lower bound for interest rates in the euro area. The downward trend in short term rates is associated with the evolution of the deposit rate. Since the Governing Council of ECB have not announced a certain lower bound for the deposit rate, it is difficult to say how low the interest rates could go.[18] If a constant lower bound is used, and the lower bound is set too low, the shadow rate model comes closer to a term structure model without lower bound and has the same flaws as well. If the constant lower bound is quite high, then the residual will be persistently negative and it is hard to argue that it is a measurement error. So, it seems difficult to set a constant lower bound for euro area which can be interpreted to support the idea of a time-varying lower bound. In following Sections, I also provide statistical evidence that supports this argument.

[18]For example, in Switzerland the one week LIBOR rate was around -0.8 percent at the beginning of December 2015.

4.2 The time-varying lower bounds used in the model

The time-varying lower bound is an exogenous sequence in the model defined as $\underline{r} = \{\underline{r}_t\}_{t=1}^{T}$. This is (one of) the first(s) paper which apply a time-varying lower bound, and so I set the sequence in a straightforward manner based on the data in order to see how a shadow rate model with time-varying lower bound performs. The nature of this investigation is purely empirical, which also justifies the use of data to guide the imposition of the lower bound. Moreover, previously in the literature the constant lower bound is set by observing the data features or it is estimated. I follow these lines: I use the features of the data to set the sequence and I use a sequence which is a result of a recursive estimation of a constant lower bound shadow rate model. In every specification I assume that $\underline{r}_t \leq 0$.

To be more specific, I consider five different types of lower bounds in the model:

1. **The most negative yield in cross-section**, $\underline{r}^n$. The effective lower bound is set for every t equal to the minimum of the observed interest across maturities. That is, $\underline{r}_t^n = \min\{R_t(\tau), 0\}$ where the $R_t(\tau)$ are observed interest rates across maturities at time t and $\tau = \{0.25, 0.5, 1, 2, 3, 5, 10\}$ which gives the maturities included in the sample. This procedure ensures that at every t the observed and fitted interest rates are equal to or higher than the lower bound, but it makes the lower bound somewhat volatile since it may also rise.

2. **The most negative yield in the sample up to** t, $\underline{r}^m$. Now I assume that the effective lower bound is the lowest yield observed in the sample up to t when daily data are used. Mathematically, $\underline{r}_t^m = \min\left\{\{R_i(\tau)\}_{i=1}^{t}, 0\right\}$ where $\tau = \{0.25, 0.5, 1, 2, 3, 5, 10\}$. This lower bound also, ensures that observed or fitted interest rates are not lower than the lower bound but this lower bound is less volatile than $\underline{r}^n$ since it not increase, i.e. $\underline{r}_t^m \leq \underline{r}_{t-i}^m$, $i = 1, \ldots, t-1$.

3. **Estimated sequence of lower bounds**, $\underline{r}^e$. The third representation for the lower bound is an estimated sequence of lower bounds. This sequence is obtained by the following procedure. The model introduced in Section 2 is modified such that it is a shadow rate

model with a constant lower bound where the lower bound is considered as a parameter to be estimated. Recursively estimating this model starting from January 2008 – when it is clear that the lower bound is not binding – gives $\underline{\hat{r}}_t^e$ for January 2008 onwards. Finally, for every t the lower bound is given by $\underline{r}_t^e = \min\{\underline{\hat{r}}_t^e, 0\}$.[19] This sequence of lower bounds is then used as an exogenous process when the shadow rate model with time-varying lower bound is estimated. In this specification the yield curve data are used to pin down the elements of the sequence when less judgment is involved.

4. **The deposit facility rate**, $\underline{r}^d$. The deposit facility rate of Eurosystem provides a risk-free rate for reserves at euro area banks. Hence, it should provide a floor for the euro area interest rate when it is a natural candidate for the lower bound of interest rates. The deposit facility rate is a monetary policy instrument in the euro area when the rate is not constant but varies in time. So, in this sequence the lower bound is zero or the negative value given by the deposit facility rate.

5. **A constant lower bound**, $\underline{r}^c$. For comparison, I also include a constant lower bound interest rate. I consider two reasonable values $\underline{r}_t^c = -0.3$ percent and $\underline{r}_t^c = -0.4$ percent $\forall t$.[20]

In total there are six different specifications for the lower bounds of interest rates. These paths are shown in Figure 3 for the time period where time-varying lower bounds differ from zero.

Time-varying lower bounds differ from each other somewhat when they can be used to evaluate the effects of different lower bound sequences on a shadow rate model. The maximum difference in lower bounds based on data ($\underline{r}^n$, $\underline{r}^m$, $\underline{r}^e$) is 10 basis points and the average difference is 6 basis points in the sample shown in Figure 3. Differences in the levels are larger when the deposit rate is taken into account. Turning to the dynamics of the lower bounds, it can be seen that sequences

[19] Before January 2008 the lower bound is set at zero. Moreover, the $\hat{r}^e$ sequence is below zero from November 2008 to December 2012 and gets negative values of around -0.1 and -0.2 percent. I also set these values at zero in order to make this sequence more comparable with other sequences. In any case, the results are almost identical if observations from 2008 to 2012 are not set to zero, and the conclusions given the paper are the same.

[20] Before December 2015 the deposit facility rate was -0.2 percent for a long period of time but the lowest yields have been around -0.5 in the sample. So, a reasonable lower bound would be between those values, and I set the constant lower bounds at -0.3 and -0.4 percent. Results in this paper do not depend on the choice of the constant lower bound: I also considered -0.2 and -0.5 percent, and the conclusions given in the paper are the same.

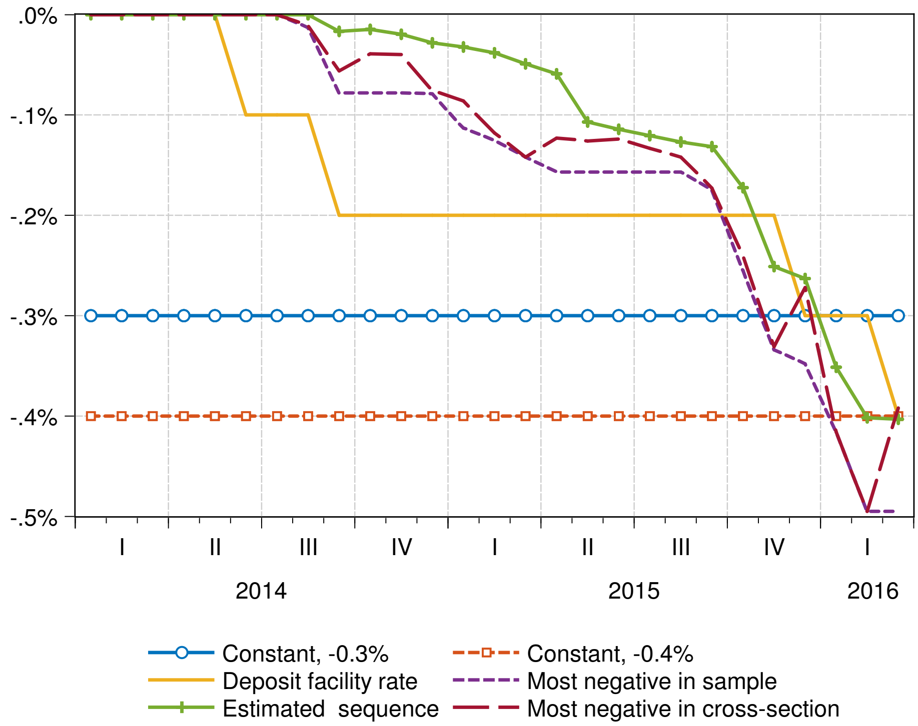


Figure 3: Lower bounds for interest used in the paper: The most negative in cross-section is $\underline{r}^n$, the most negative in sample is $\underline{r}^m$, estimated sequence is $\underline{r}^e$, deposit facility rate (announcement) is $\underline{r}^d$ and constant lower bounds are $\underline{r}^c$.

differ from each other in timing when a sequence goes into negative territory. Moreover, at the end of the sample $\underline{r}^n$ increases while the other sequences decrease. In general, however, correlations between data based measures ($\underline{r}^n$,$\underline{r}^m$,$\underline{r}^e$) are around 0.98. The high correlation between the estimated sequence, $\underline{r}^e$, and purely data-based lower bounds ($\underline{r}^n$ and $\underline{r}^m$) shows that the purely data based sequences are not completely arbitrary since they are close to the estimated sequence, which is based on the information provided by the yield curve.

The estimated sequence also provides strong evidence for the hypothesis of time-varying lower bound. Typically recursive estimation is used to test the stability of parameters, i.e. adding the data observation by observation should not change the parameter values significantly. Here recursive estimation quite clearly shows that the parameter for the lower bound is not a constant but instead declines. Hence, the recursively estimated shadow rate model where the lower bound is a parameter to be estimate provides evidence favoring the use of a time-varying lower bound.

5 Evaluation of the model with time-varying lower bounds

5.1 Estimation results

Table 1 gives ML estimates of the parameters and their standard errors for the model described in Section 2 when different sequences for the lower bound are applied. Standard errors are based on the inverse of the Hessian evaluated around estimated parameters (see, Krippner, 2015b, p. 74, for details).

The parameters imply that the long-run expected value for the short run interest rate varies around 0.2 - 1.3 percent. These values are low – but plausible – since long-term interest rates have also been historically at very low levels. Somewhat higher values for the long-run expected short run interest rates is given by the $\underline{r}^d$ sequence while the other estimates are between 0.2 and 0.8 percent. For all the sequences, the usual results are obtained: parameters under the $\mathbb{P}$ measure are quite imprecisely estimated whereas parameters under the $\mathbb{Q}$ measure are estimated with high accuracy. Moreover, under the $\mathbb{P}$ measure the factor dynamics are very persistent.

5.2 Cross-sectional fit and yield curves

I now describe the cross-sectional fit of the model, including a version introduced in Section 2 without the lower bound when it is a typical three factor arbitrage free dynamic Nelson and Siegel model. A comparison between shadow rate models and the same model without lower bound enables one to confirm that the use of a lower bound is appropriate for euro area data. Table 2 displays the results where the goodness of fit for yield curves is described by the root mean square error (RMSE) across maturities and the overall mean absolute error (MAE), both expressed in basis points. Table also gives the log-likelihood.

Table 1: Estimated parameters

Parameter and its std. in parenthesis	Sequence used as a lower bound					
	$\underline{r}^c$, -0.3%	$\underline{r}^c$, -0.4%	$\underline{r}^d$	$\underline{r}^m$	$\underline{r}^e$	$\underline{r}^n$
ϕ	0.497	0.492	0.517	0.553	0.528	0.561
	(0.014)	(0.008)	(0.001)	(0.002)	(0.003)	(0.001)
κ_{11}	0.030	0.053	-0.003	0.223	0.097	0.147
	(0.325)	(0.109)	(0.213)	(0.061)	(0.035)	(0.024)
κ_{12}	0.002	0.007	0.002	0.001	0.003	-0.002
	(0.141)	(0.033)	(0.225)	(0.108)	(0.002)	(0.098)
κ_{13}	-0.369	-0.251	-0.325	-0.113	-0.129	-0.140
	(0.079)	(0.104)	(0.142)	(0.058)	(0.038)	(0.030)
κ_{21}	0.556	0.579	0.657	0.567	0.540	0.618
	(0.113)	(0.183)	(0.324)	(0.130)	(0.079)	(0.120)
κ_{22}	0.973	1.041	0.835	1.675	1.009	1.745
	(0.417)	(0.139)	(0.328)	(0.261)	(0.001)	(0.360)
κ_{23}	-0.671	-0.734	-0.677	-0.477	-0.567	-0.824
	(0.216)	(0.147)	(0.213)	(0.001)	(0.053)	(0.176)
κ_{31}	0.301	0.374	0.305	0.073	0.174	-0.081
	(1.600)	(0.248)	(0.694)	(0.047)	(0.019)	(0.382)
κ_{32}	0.256	0.271	0.212	0.224	0.553	0.010
	(1.774)	(0.419)	(0.722)	(0.383)	(0.030)	(0.762)
κ_{33}	0.478	0.108	0.791	0.328	-0.074	0.462
	(0.284)	(0.231)	(0.439)	(0.326)	(0.030)	(0.353)
θ_1	0.113	0.425	0.060	0.621	0.248	0.621
	(2.130)	(1.520)	(2.901)	(0.342)	(1.344)	(0.532)
θ_2	0.089	0.364	1.275	-0.356	0.248	0.071
	(4.251)	(1.248)	(1.148)	(0.224)	(0.672)	(0.464)
θ_3	-2.944	-2.330	-2.737	-0.598	-2.144	-0.246
	(3.096)	(1.834)	(1.973)	(0.342)	(1.428)	(0.749)
σ_1	0.813	0.738	0.762	0.896	0.750	0.911
	(0.042)	(0.041)	(0.036)	(0.039)	(0.055)	(0.019)
σ_2	1.170	1.246	1.099	1.404	1.035	1.397
	(0.022)	(0.072)	(0.040)	(0.042)	(0.063)	(0.044)
σ_3	2.767	2.481	2.759	3.562	2.811	3.714
	(0.138)	(0.091)	(0.015)	(0.015)	(0.242)	(0.017)

Note: θ_1, θ_2, θ_3 and σ_1, σ_2, σ_3 are in percentage points. θ gives the long-run mean for factors and σ the variance of innovations for factors. κ describes the element of the matrix for the autoregressive parameters when $\Delta t = 1/12$

Table 2: Goodness of fit for yield curves (basis points)

Lower bound applied	RMSE for sample maturities in years 0.25	0.5	1	2	5	7	10	MAE	Log-likelihood
No lower bound	8.3	0.0	7.1	7.2	**1.3**	0.8	4.6	3.4	8136.8
Constant, -0.3 %, $\underline{r}^c$	7.5	1.0	6.1	5.5	2.0	0.2	4.2	2.9	8179.2
Constant, -0.4 %, $\underline{r}^c$	7.7	0.0	6.0	5.4	2.0	0.0	4.7	2.9	8210.3
Deposit rate, $\underline{r}^d$	7.2	1.2	5.7	5.1	2.2	0.3	**4.0**	2.8	8217.7
Min. in sample, $\underline{r}^m$	6.5	**0.0**	**4.0**	1.4	3.7	0.1	6.0	**2.3**	8316.8
Estimated, $\underline{r}^e$	7.0	1.1	5.4	4.4	2.2	0.1	4.3	2.7	8292.0
Min. in c-s., $\underline{r}^n$	**6.4**	1.1	4.0	**0.2**	4.0	**0.0**	6.4	2.4	**8329.1**

Note: RMSE is root mean square error and MAE is mean absolute error. The lowest RMSE and MAE as well as the highest value for log-likelihood are highlighted in each column.

First, it is clear that by including a lower bound in the model the fit is improved significantly. The log-likelihood improves by at least 42.2 log-points and the maximum improvement with $\underline{r}^n$ sequence by 192.3 log-points. These are statistically significant improvements by a large margin.[21] Generally, RMSE and MAE are smaller in models where a lower bound is imposed compared to the model without lower bound. Second, models with time-varying lower bound give better fits than models with constant lower bound, and the improvements in log-likelihood are large and statistically significant when $\underline{r}^d$ sequence is excluded.[22] Moreover, RMSE and MAE are lower in models where time-varying lower bound is applied than in models with constant lower bound. These results suggest that a time-varying lower bound should be preferred over a constant one. Third, it is hard to determine which time-varying lower bounds give the best fit. In any case, a model with $\underline{r}^n$ sequence has the highest log-likelihood.

The choice of lower bound for interest rates affects mostly the fit of short-term yields since short-term rates are typically closer to the lower bound than are long-term interest rates. To

[21]There is one more parameter in the constant lower model than in the model without lower bound. 99 percentage critical value from χ^2 distribution is 6.6 with one degree of freedom. Hence, 42.2 is statistically significant. $\underline{r}^n$ sequence can be described by 21 additional parameter relative to the no lower bound model when 99 percentage critical value from χ^2 distribution is 38.9.

[22]The maximum improvement in log-likelihood between models applying a constant lower bound and a time-varying lower bound is 118.9 log-points given by r^n sequence. This improvement is produced by 20 additional parameters. The minimum improvement is 81.7 log-points with 19 additional parameters given by r^e sequence. Both improvements in log-likelihood are statistically significant. However, the improvement in log-likelihood given by deposit facility rate ($\underline{r}^d$) is not statistically different from the value of log-likelihood given by the model with a constant lower at the level of -0.4 percent.

demonstrate how the choice of lower bound matters for the fit of yield curve in different circumstances a group of yield curves produced by different choices for the lower bound is shown in Figure 4. A fit for the yield curve provided by the shadow rate model with time-varying lower bound is demonstrated in Figure by the fit given by the $\underline{r}^e$ sequence.

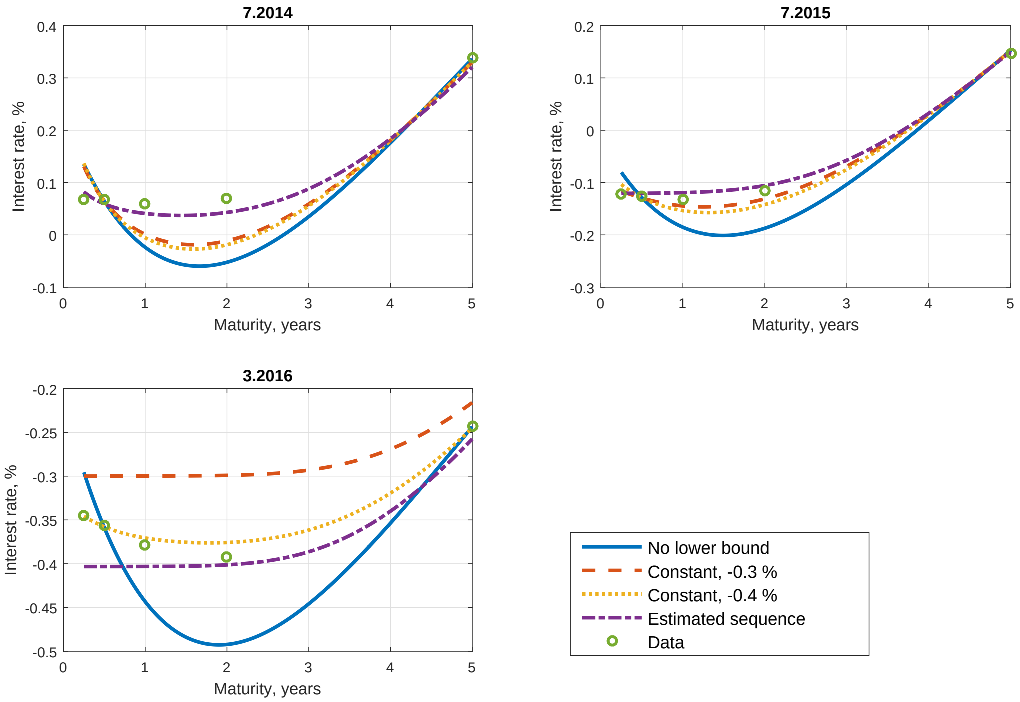


Figure 4: Yield curves for short-term interest rates.

Figure 4 shows that the model without lower bound produces a yield curve in which the fitted interest rates are too negative. So, by setting an appropriate lower bound on a term structure model the fit for the yield curve can be improved. However, finding an appropriate constant lower bound is difficult – as discussed in Section 4. On the one hand, if the lower bound is set "too" low, the fitted interest rates are too low as well (see panel 7.2014). On the another hand, setting the lower bound too high often causes observed yields to be lower than the lower bound (see, $\underline{r}^c$, -0.3 % in panel 3.2016). That is, when short-term interest rates seem to fluctuate significantly and are trending downwards, it is hard to set a constant lower bound for interest rates which would be suitable for a long period of time. Hence, a model with time-varying lower bound might be appropriate. In favourable circumstances the constant lower bound model may fit the data (see panel 7.2015), but the fit for time-varying lower bound model is equally satisfactory. Obviously, the fit is not perfect all the time, but the model seems to behave consistently and gives a reasonable fit for short-term interest rates. Thus, the model with time-varying lower bound seems to give the needed flexibility compared to the constant lower bound counter part and produces a robust fit for the yield curve in different circumstances.

5.3 Out of sample forecasts

A good in-sample fit could be a reflection of over-fitting when, dispute the good fit, the predictions given by the model are poor. Thus, to see the forecast performance of shadow rate models with different lower bounds in low-interest environment out of sample forecast are evaluated at Table 3. The forecasts are done as follows: model parameters are estimated with data up to the end of 2010. These parameters values are then used to generate forecasts for 6 and 12 month ahead at monthly frequency to the end of sample. The forecasts are evaluated for maturities 0.5, 1 and 10 year. I also included predictions from a random walk when the forecast is the same as the last observed value.

Table 3: Out-of-sample root mean square error (bsp)

Model or lower bound applied	Forecasting horizon					
	6 months			12 months		
	Yield in years			Yield in years		
	0.5	1	10	0.5	1	10
Random walk	29	34	52	43	49	80
No lower bound	48	52	52	61	66	80
Constant, -0.3 %	41	44	52	53	57	76
Constant, -0.4 %	43	45	51	56	60	77
Deposit rate, $\underline{r}^d$	**37**	**42**	51	**52**	**57**	76
Min. in sample, $\underline{r}^m$	37	42	51	52	58	75
Estimated, $\underline{r}^e$	38	42	**50**	53	58	**75**
Min. in cross-section, $\underline{r}^n$	38	42	51	52	58	76

Note: the lowest RMSE is highlighted in each column.

A similar pattern is found in Table 3 in terms of in-sample fit. The models with a lower bound outperform the model without a lower bound. Moreover, models with the time-varying lower bound seem to be somewhat better than models with a constant lower bound. The lowest RMSE for the out of sample forecast is now obtained when the estimated sequence, $\underline{r}^e$, or deposit facility rate, $\underline{r}^d$, is used as a time-varying lower bound. However, the differences between the models are quite small. So, it is not clear that one of the models outperform the others, but there is no evidence that shadow rate models with time-varying lower bound forecast less well than any other model in the same class of models. The shadow rate models could do better when more data be comes available since the lower bounds start to vary only at the end of the sample. All the models lose to predictions provided by the random walk model expect when the 10 year interest rate is forecasted. However, the forecasting performance of the models could be improved if the parameters were updated after each forecasting round.

In summary, a shadow rate model with time-varying bound fits the euro area data relatively well. It gives reasonable estimates of the parameters, the in-sample fit is better than for a model with constant lower bound, and it produces reasonable yield curves in a robust manner. Finally, its out of sample performance is similar or better than the constant lower bound model. So, the results can be interpreted to favor the use of shadow rate models with time-varying lower bound for euro area data.

6 The stance of monetary policy at the lower bound

The one of main purposes of arbitrage free dynamic Nelson and Siegel models is to help explain the yield curve dynamics via level, slope and curvature factors and to provide forecasts of interest rates. These issues were discussed in Section 5. For shadow rates models there is an additional useful feature since these models provide measures for the stance of monetary policy at the lower bound of interest rates. The stance of monetary policy is not straightforward to measure at the lower bound of interest rates since typical measures – like policy rate or some other short-term interest rate – are restricted by the lower bound and thus do not represent the stance of monetary policy. However, typically non-standard monetary policy measures are aimed at affecting the yield curve (see, for survey Krishnamurthy and Vissing-Jorgensen 2011) as the yield curve may still contain information about the stance of monetary policy.

Shadow rate models can be used to capture information on the stance of monetary policy embedded in the yield curve since such a model provides a shadow yield curve which is not restricted by the lower bound. Two measures can derived from the shadow yield curve: Liftoff horizon (LOH) and a short-term shadow rate. In Section 6.1 I compare LOH measures produced by different lower bounds and in Section 6.2 a similar study is done for the shadow rate.

6.1 Liftoff horizon (LOH)

The liftoff horizon measures (typically under $\mathbb{Q}$) the market-implied expectation of the horizon in which the short-term interest rate rises above a certain threshold. The threshold is chosen such that when the short-term interest passes the threshold one should look for the policy rate to lift off from the very low levels of interest rates where the lower bound significantly matters for monetary policy.[23] The LOH-measure can be interpreted such that higher values of LOH imply that the monetary policy stance is more accommodative since households and firms will face very low interest rates for a longer time. However, it should noted that LOH does not take into account the pace of tightening after liftoff and it is only available for the time-period during which the effective lower bound is binding. Hence, it is not a complete measure of monetary policy stance.

To derive the LOH I follow Bauer and Rudebusch (2016) who argue that LOH is the optimal forecast for liftoff with absolute-error loss function.[24] Starting from the current term structure I use Monte Carlo simulations to obtain K paths for $\tilde{\mathbb{E}}\left[r_t^j(\tau) \mid x_t\right]$ which gives a set of forecasts $\left\{\tilde{\mathbb{E}}_t r_t^j(\tau)\right\}_{j=1}^{K}$. Then, for every j I find the first $\bar{\tau}$ which solves $\left[\tilde{\mathbb{E}}_t r_t^j(\tau)\right]_{\tau}^{\tau+n} \geq \bar{r}$ where $\bar{r}$ is the threshold level. That is, I find the first moment when the expected path cross the threshold and stay above at least n moments of time. I choose n to be six months and $\bar{r} = 0$. This procedure,

[23] There is no well defined procedure to set the threshold, but Wu and Xia (2016) and Bauer and Rudebusch (2016) set the threshold at 25 basis points for the US.

[24] This measure is also used by Wu and Xia (2016) to estimate the liftoff horizon.

in turn, gives me a set of liftoff days $\left\{\bar{\tau}_t^j\right\}_{j=1}^K$ for every t. I define LOH as follows:

$$\text{LOH}_t = \text{median}\left(\left\{\bar{\tau}_t^j\right\}_{j=1}^K\right). \tag{18}$$

Figure 5 shows the paths for LOH when different lower bounds are used in the model. All LOH measures start to get positive values at the end 2011, but the values are almost the same since the values of lower bounds are the same (zero).[25] Hence, here I focus on time frame in which the lower bounds differ from each other.

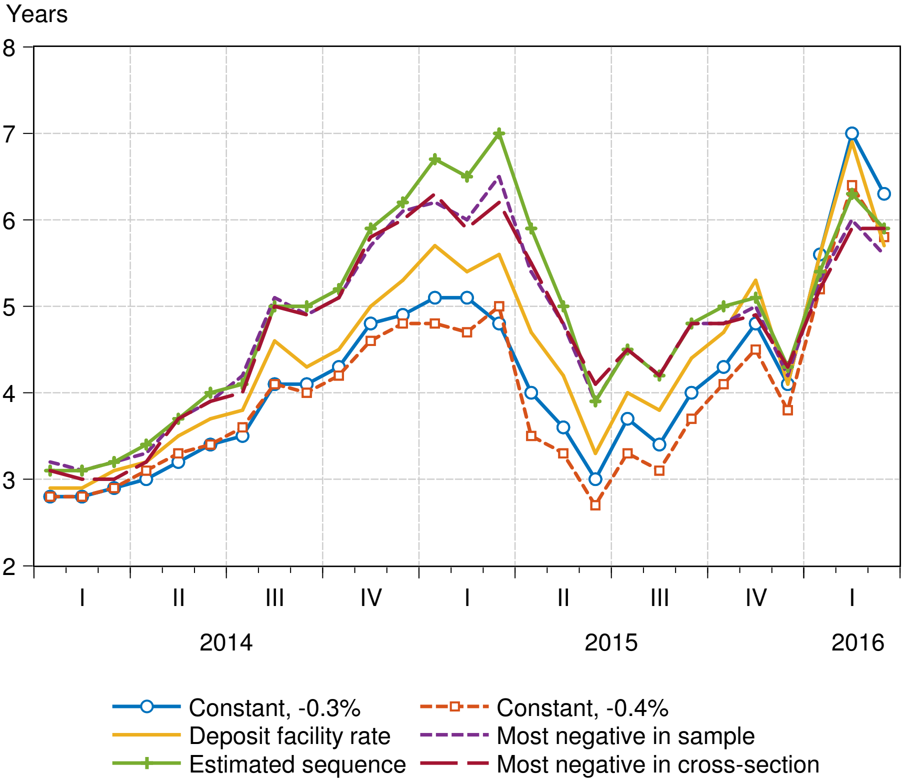


Figure 5: Estimates of liftoff horizon (LOH) by alternative lower bounds.

Figure 5 shows that LOH can be seen to reasonably describe the monetary policy stance in the euro

[25] Of course, the constant lower bounds get values different from zero.

area. After June 2014 there is a clear jump in LOH measures when a set of non-standards measures was announced by the Governing Council of ECB. Moreover, QE was announced in January 2015 when the LOH paths get their highest values. The subsequent decline illustrates the improved economic outlook after QE. For instance, it was then considered whether QE should ended earlier than announced.[26] After June 2015 the LOH measures generally follow communications of the General Council, and the decline at the end of 2015 reflects the fact that the market expected more monetary stimulus than was decided on by the General Council at December 2015. In 2016 LOH measures increased significantly since the outlook for inflation and economic growth deteriorated, which increased market expectations of new monetary policy measures. Additional monetary stimulus was decided on by the Governing Council at its March meeting, but LOH measures decreased slightly since the market expected more stimulus.[27] Finally, it is interesting to notice that the average correlation between LOH measures and 5-year, 5-year inflation expectations (one of the key inflation expectation measures for ECB) is -0.85.

A second feature in Figure 5 is that LOH is a relatively robust measure across different lower bound paths – especially when time-varying lower bounds are considered. There is some variation around the turn of the year 2014 - 2015 but still LOH can be considered a robust measure across models with different time-varying lower bound. However, it seems that the paths given by models with constant lower bound deviated downwards somewhat from the paths given by models with time-varying lower bound. The levels of constant lower bounds might be too low at the turn of the year 2014 - 2015 when these models come close to the model without lower bound. Typically term structure models without lower bound do forecast that liftoff from low levels of interest rates happens too quickly, which could explain this result.

An additional examination of robustness of the LOH measure is provided by Figure 6. In Figure 6, I consider the robustness of the LOH measure given by two leading candidates for time-varying lower bound: estimated sequence ($\underline{r}^e$) in panel a) and minimum in cross-section ($\underline{r}^n$) in panel b). In addition to the baseline estimate the robustness of LOH is examined by estimating the model

[26] See, for example, questions regarding the Introductory statement to the press conference (with Q&A) in June 2015. Available at homepage of ECB.

[27] The dynamics of LOH measures are in line with dynamics of interest rates in the current period (see Figure 2, panel (b)).

with data only up to January 2016, to see how additional observations affect the LOH measure. The model is also estimated with an extended sample which includes interest rates at maturities of 3 and 30 years in addition to the ones used in baseline estimation.[28] Using a longer interest rate may give a better estimate for the level factor since the 10 year interest rate is somewhat volatile.

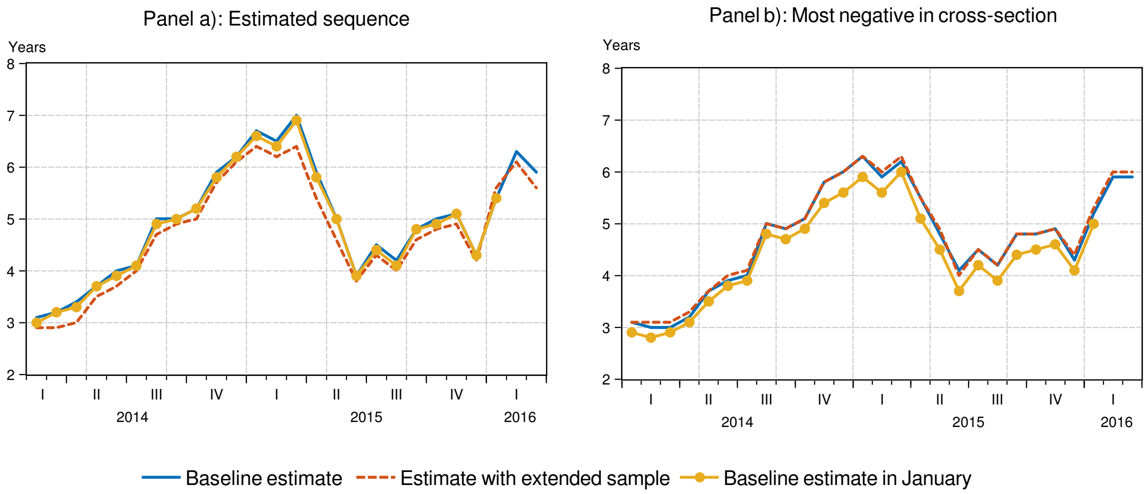


Figure 6: LOH measures given by two lower bound sequences when the estimation period and sample of interest rates are changed.

The LOH measure provided by a shadow rate model which uses $\underline{r}^e$ or $\underline{r}^n$ sequences for time-varying lower bound can be considered robust at least for small changes in the estimation period or in the sample of interest included in the estimation. Both sequences provide LOH measures which are basically the same. Krippner (2015b) finds that the liftoff measure is robust for changes in the lower bound, and this result seems to hold in shadow rate models with time-varying lower bounds.

In summary, these results confirm previous results in the literature. Krippner (2015b) and Bauer and Rudebusch (2016) also found that the liftoff measure can be used to guide the stance of monetary policy at the zero lower bound, and the finding is this section confirms that conclusion. Moreover, for the euro area Lemke and Vladu (2014) reported that for April 2014 the

[28] Maturities used in the baseline estimation are given in Section 3.

median expectation for the one month forward rate to pass 50 bp. level was somewhat over 3 years, and Pericoli and Taboga (2015) estimate that the liftoff horizon from zero at beginning of 2015 was around 5 years. Both of these estimates are relatively close to those presented here.

6.2 Shadow rate

Shadow rates can be considered as interest rates which would occur if cash did not exist. In such a case interest rates would not be restricted by a lower bound. The short-term shadow rate (hereafter, shadow rate) could then be considered an artificial policy rate which accounts for the effects of non-standard monetary policy and, thus, can be consider a measure of the monetary policy stance. That is, non-standard monetary policy measures provide stimulus by lowering long-term interest rates, but this easing action is not captured by short-term interest rates since the lower bound is binding. However, the shadow rate is is not restricted by the lower bound when it can capture the effects of non-standard measures embedded in the yield curve. Hence, the shadow rate can be considered a measure of monetary policy stance when short-term interest rates are restricted by the lower bound. Especially, in the euro area, many non-standard monetary policy measures are used when it is hard to evaluate the stance monetary policy, and so this type of measure could be useful. Moreover, Wu and Xia (2016) argued that a shadow rate can be used as a policy rate in quantitative macro models during the lower bound period. Then the effects of non-standard monetary policy measures can be analyzed if generally applied policy rate is replaced by the shadow rate during the period when the lower bound restricts the policy rate.

However, the problem associated with the use of shadow rate as a measure of monetary policy stance is that the level of the shadow rate is sensitive to the choice of the (constant) lower bound. Bauer and Rudebusch (2016), Christensen and Rudebusch (2016) and Krippner (2015b) have shown that quite small (around 5 bp.) changes in the level of the lower bound can cause large (around 50 bp.) movements in the shadow rate. Hence, the shadow rate will be a useful measure of monetary policy stance only if the lower bound can be set with high accuracy. Unfortunately, this seems not to be the case, but the levels of lower bounds used in studies (with US data) range from 0.25 to -0.25 percent. This has led many authors (for instance, Bauer and Rudebusch 2016; Christensen and Rudebusch 2016) to conclude that the shadow rate is not a useful measure of monetary policy stance.

The shadow rate is given by $s_t = x_t^1 + x_t^2$ and Figure 7 shows shadow rates for different lower bound sequences applied in the model. Again, I focus on the period when the lower bound paths differ from each other.[29]

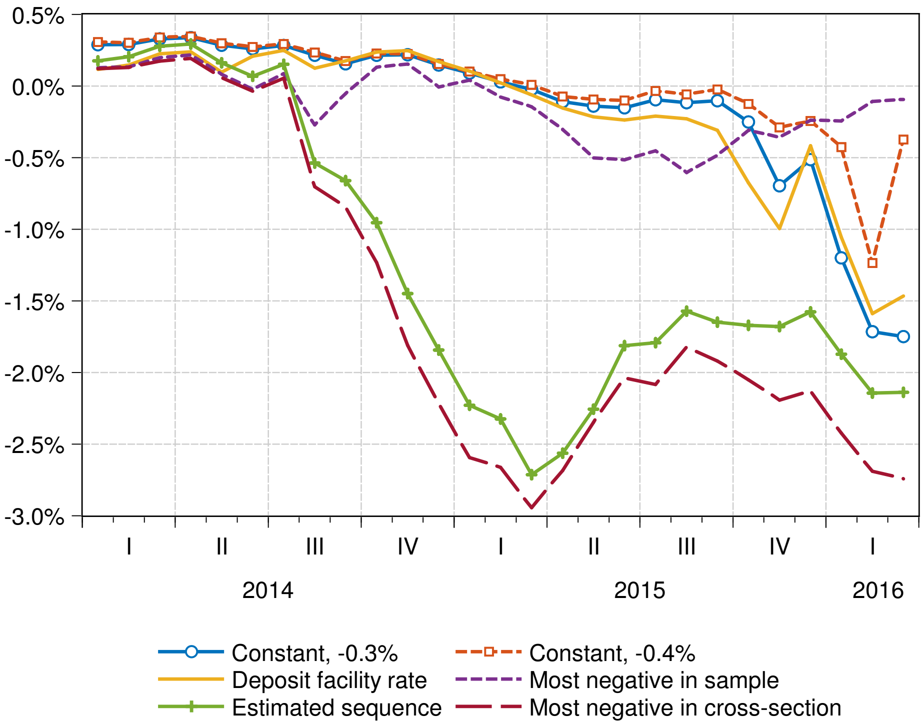


Figure 7: Shadow rates from models with different lower bound sequences.

Figure 7 reproduces the finding in the literature: different lower bounds produce different shadow rate paths. The level and dynamics of shadow rates differ from each other significantly. The lowest correlation between shadow rates is 0.43, which occurs between $\underline{r}^{c,-0.3}$ and $\underline{r}^{n}$, whereas for the LOH measures the lowest correlation is 0.85. Hence, the LOH measure seems to be quite

[29]The lower bound paths are described in Figure 3

robust over different lower bound sequences, but for shadow rates the choice of the lower bound matters significantly for the dynamics and for the level of shadow rate.

To assess how well the shadow rates describe the stance of monetary policy, I derive their correlations with LOH measures. LOH measures are generally considered reasonable measures of the stance of monetary policy (see Section 6.1) and therefore can be used as a benchmark. A high negative correlation between a shadow rate and an LOH measure should be considered as evidence that the shadow rate could be seen to reasonably describe the stance of monetary policy. Table 4 shows correlations between shadow rates and LOH measures when the sample is 1.2012 - 3.2016, a period when LOH measures start to get values different from zero.

Table 4: Correlation between shadow rates and LOH-measures 1.2012 - 3.2016

Shadow rate	Liftoff horizon (LOH)						Average
	$\underline{r}^c$, -0.3%	$\underline{r}^c$, -0.4%	$\underline{r}^d$	$\underline{r}^m$	$\underline{r}^e$	$\underline{r}^n$	
$\underline{r}^c$, -0.3%	-0.74	-0.67	-0.61	-0.52	-0.52	-0.54	-0.60
$\underline{r}^c$, -0.4%	-0.71	-0.65	-0.62	-0.55	-0.55	-0.56	-0.60
$\underline{r}^d$	-0.61	-0.56	-0.50	-0.40	-0.39	-0.42	-0.48
$\underline{r}^m$	-0.16	-0.18	-0.20	-0.27	-0.23	-0.29	-0.22
$\underline{r}^e$	-0.78	-0.71	-0.80	-0.86	**-0.87**	-0.87	-0.82
$\underline{r}^n$	**-0.81**	**-0.74**	**-0.81**	**-0.86**	-0.86	**-0.88**	**-0.83**

Note: Average is row-wise average and gives the average correlation between the shadow rate and all LOH measures. The lowest value for each column is highlighted.

Table 4 shows that when the estimated ($\underline{r}^e$) or the minimum for the cross-section ($\underline{r}^n$) sequence is used as a time-varying lower bound the shadow rate model produces a shadow rate which is highly negatively correlated with LOH measures. Other sequences for the lower bound seem not to be very highly correlated with LOH measures. Table 4 shows that a shadow rate model with time-varying lower bound can produce a shadow rate which can considered to reasonably describe the stance of monetary policy.

To test the robustness of shadow rates, a similar exercise is done with shadow rates as was done with LOH measures. Figure 8 considers the robustness of shadow rates provided by the $\underline{r}^e$ sequence (panel a) and $\underline{r}^n$ sequence (panel b). The figure shows the baseline estimate of the shadow rate by the same model, but the data running only up to January 2016, and a shadow rate estimate based on a sample that includes 3 and 30 year interest rates in addition to the ones used in baseline estimation.

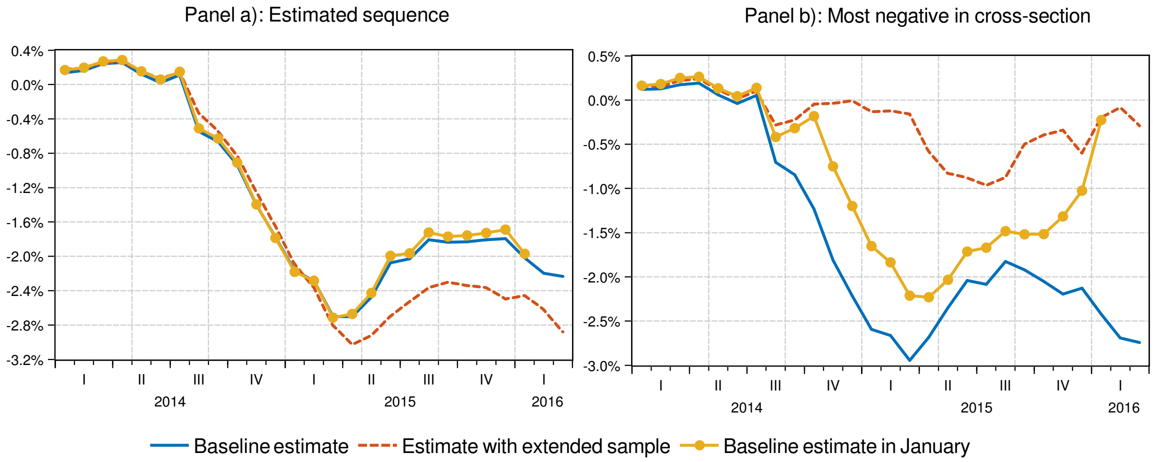


Figure 8: Shadow rates given by two sequences of lower bounds based on varying samples.

Figure 8 shows that the shadow rate given by the estimated sequence, $\underline{r}^e$, produces quite robust estimates: changing the estimation period by a few observations does not change the estimate. Moreover, including longer interest rates in the sample does not significantly change the level or dynamics of the shadow rate, although the level of the shadow rate is slightly lower. However, these conclusions do not hold when the $\underline{r}^n$ sequence is used in the estimation. With this sequence it can be seen that the level and dynamics of the shadow rate are sensitive to small changes in estimation period as well as changes in the maturities of interest rates used in the estimation. The changes in the shadow rate are large: in January 2016 the estimate of the shadow rate was slightly negative whereas two months later the same estimate was 2 percentage points lower. Further, including 3 and 30-year interest rates in the sample crucially changes the estimate of the shadow rate.

Krippner (2015a) explains why a 3 factor shadow rate model is prone to provide fluctuating estimates of the shadow rate. Krippner's detailed analysis of the model used by Wu and Xia (2016) shows that the source of fluctuations in the shadow rate is the level of lower bound relative to the short-term data. If the short-term data are above the lower bound, the value of the shadow rate is high and, if the data are below the lower bound, the value of shadow rate can be quite negative. This also explains why $\underline{r}^n$ sequences do not give robust estimates of the shadow rate.

In euro area data, short-term interest rates have been higher than, for example, the 2-year interest rate (see, Figure 2, panel (b)). Now, in $\underline{r}^n$ sequence the lower bound is set at the minimum of interest rates for the cross-section of yields for every t when at times short-term interest rates are above the lower bound. The three factor model is very non-linear as it may fit this "hook-shape" pattern, and if the lower bound is low enough it does not prevent this (see, Figure 4, panel 3.2016). The appearance of the "hook-shape" pattern in fit depends on small changes in the level of lower bound relative to the short-term interest rates when the fit may change significantly depending on the data and the lower bound. This "hook-shape" pattern is also transmitted to shadow rates, since shadow rates are based on the same parameters and factor loadings as the yield curve. So, the short-term shadow rate can be positive, but the 2-year shadow rate can be very negative and the shadow rate could fluctuate significantly depending on the data used in the estimation. Moreover, this "hook-shape" pattern in the shadow rates explains why the short-term shadow rate can be slightly positive even when the LOH measure still indicates strong stimulus via monetary policy.

In any case, it seems that when the estimated sequence is used as a time-varying lower bound the three factor shadow rate model is able to provide robust estimates of the shadow rate. The history of the shadow rate does not change when new data are added to the sample, and the estimate is robust for changes in the composition of interest rates used in estimation. Euro area short-term interest rates – and the effective lower as well – have fluctuated quite significantly since autumn 2015 when the market started to expect changes in the deposit facility rate. Despite this fluctuation, the shadow rate model with the estimated sequence as a time-varying lower bound has produced robust estimates for the shadow rate. Hence, this sequence seems to be an appropriate choice for the time-varying lower bound, at least when shadow rate modeling is considered.

The problem with use of the shadow rate as a measure of monetary policy stance is that the level and dynamics of the shadow rate depend on the level of the lower bound, and there is no consensus on what it should be. When a time-varying lower bound is used, the sequence of lower bounds is unambiguously defined *given* the procedure for setting the time-varying lower bound. Here I have shown that the procedure which gives the $\underline{r}^e$ sequence pins down a time-varying lower bound that gives a reasonable shadow rate. To further test the robustness of this result, the $\underline{r}^e$ sequence should be applied in different shadow rate models to see if the estimates of the shadow rate would differ across models. This type of systematic evaluation is beyond the scope of this paper.

To give some perceptive on the shadow rate given by the $\underline{r}^e$ sequence, I compare it to freely accessible shadow rates for the euro area given by Krippner (2015b) and Wu and Xia (2016).[30] Both of these shadow rates are estimated with a shadow rate model which applies a constant lower bound. Figure 9 shows these shadow rates, and the EONIA is added to the figure. The difference between the shadow rate and EONIA can be thought as a measure of the effects of non-standard monetary policy in terms of policy rate difference. That is, EONIA describes the stimulus which can be achieved by the standard policy rate based monetary policy when additional effects coming from non-standard measures are captured by shadow rates.

Figure 9 shows typical features of the shadow rate. When the lower bound does not restrict interest rates, shadow rates are equal to observed short-term interest rates. If, however, the lower bound starts to bind and long-term interest rates are affected by non-standard measures, the shadow rate is below the short-term rate. It is also clear that shadow rate estimates differ greatly between the models (as in Figure 7). The level of the shadow rate produced by the $\underline{r}^e$ sequence is more closely related to the shadow rate by Wu and Xia (2016) before 2013, but there is significant difference after that time period. However, after 2012 the correlation with the shadow rate estimate by Krippner (2015b) is 0.87 whereas the corresponding correlation by Wu and Xia is 0.63. Hence, the estimate of a shadow rate produced by time-varying lower bound $\underline{r}^e$ is correlated with existing measures, but it does avoid very low levels of shadow rate such as -6.9 percent.

[30]Krippner and Wu provide updates for their estimates at their homepage, which can be found here and here. Note that the estimate by Wu and Xia is not completely comparable to my or Krippner's estimates since they use data on Government bond yields.

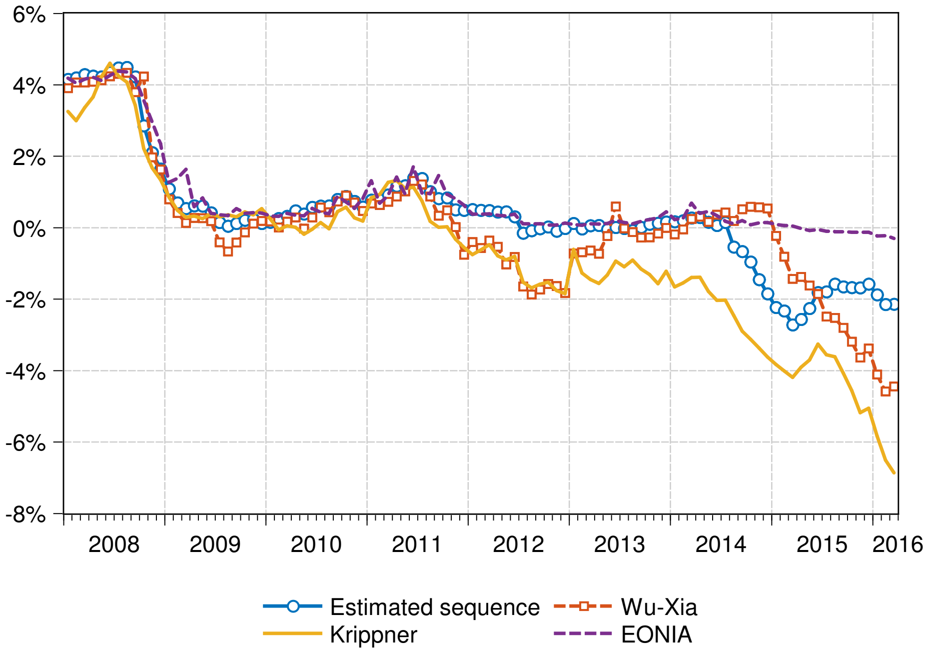


Figure 9: Shadow rates for euro area and EONIA.

7 Effects of monetary policy on the interest rate and time-varying lower bound

As argued in Section 4 the time-varying lower bound for the euro area can be seen as reflecting the expected path of the deposit facility rate. The reasoning is as follows: first, the deposit facility rate and its expected path provide a lower bound for the euro area interest rates since no bank would lend overnight at a rate lower than the rate at which it can deposit with the central bank. Second, the deposit facility rate and its expected path are not constants but rather a monetary policy instrument which motivates time-variation in the lower bound. Hence, the time-variation of the deposit facility rate and its expected path provide a basis for a time-varying lower bound for interest rates.

In the term structure model with constant or no lower bound the effects of monetary policy on the yield curve are captured by factors. A time-varying lower bound provides a new chancel through which monetary policy can affect interest rates: changes in the lower lower bound – which are caused by the actions of monetary policy authority – now also matter for interest rates of different maturities. The effects of these changes vary across maturities depending on how tightly the lower bound restricts the path of expected forward rate.

In Section 7.1, I derive an analytic expression which can be used to decompose changes in forward rates to contributions caused by time-varying lower bound and into contributions which result from variation in factors. In Section 7.2, I apply the decomposition to euro area data. Analysis in Sections 5 and 6 showed that the estimated sequence, $\underline{r}^e$, produces the most reasonable results in different aspects and, thus, I use that sequence to describe the time-varying lower bound.

7.1 Changes in forward rates and monetary policy

Under the $\mathbb{Q}$ measure the (instantaneous) forward rate for maturity τ at time t, $f_t(\tau)$, is given by equation (16). Differentiating equation (16) respect $\underline{r}_t$ and $f_t^s(\tau)$ gives (for details, see appendix B)

$$df_t(\tau) = \Phi d\underline{r}_t + (1 - \Phi) df_t^s(\tau). \tag{19}$$

Hence, a change in forward rate at maturity τ derives from a change in the lower bound or in the shadow forward rate. These changes are weighted by the probability mass associated with the censored part of the shadow rate distribution, Φ, and the mass associated with the uncensored part, $1 - \Phi$. If $\Phi(\tau) = 1$; when $f_t(\tau) = \underline{r}_t$, all changes in the forward rate are associated with changes in the lower bound. At the other extreme, when $\Phi(\tau) = 0$, the lower bound does not restrict the formation of the forward rate when changes in the forward rate come from changes in the factors.

There are two lessons for the monetary policy. First, higher values of Φ increase the effectiveness of changes in the lower bound on interest rates. That is, the level of interest rates can be effectively moved by monetary policy by changing the lower bound of interest rates. Note particularly that if long-term forward rates are restricted by the lower bound, then fluctuations in the lower bound also affect long-term interest rates quite strongly. Hence, by changing the lower bound the whole yield curve can be affected by monetary policy. Paradoxically, the more tightly the lower bound of interest rates restricts the yield curve, the more effectively can the yield curve be affected by monetary policy, if it can change the lower bound.

Second, the derivation above suggests that if the marginal effects of non-standard monetary policy on factors is constant then these policies have a diminishing marginal effect on interest rates per unit of investment. That is, the lower the levels of interest rates, the higher the value of Φ and thus the less the influence of factors on f_t. The effects of monetary policy on factors is not modeled here, as further research is needed. However, the discussion here provides a channel by which marginal returns on non-standard measures on interest rates may diminish. Appendix A.2 illustrates graphically how changes in factors affect the forward rate in the presence of a lower bound.

It should be noticed that in normal times, when $\Phi \approx 0$, changes in the lower bound do not matter for the yield curve. However, currently there are two conditions which may have change that. First, there is high abundant excess liquidity in the euro area when short-term rates are restricted by the deposit facility rate and its expected path. Second, long-term interest rates are at historically very low levels, where the lower bound might matter for the dynamics of these rates as well. However, raises an empirical question: how strongly does the lower bound restrict the euro are yield curve?

7.2 How much the lower bound restricts euro are yield curve?

The previous section introduced a theoretical basis for understanding how changes in the lower bound matter for the yield curve. However, the relevance of the lower bound for interest rates depends on $\Phi(\tau)$, which defines how tightly the lower bound restricts forward rates. Here I introduce two different measures which describe $\Phi(\tau)$. First, I directly measure $\Phi(\tau)$ when restrictions by the lower bound on forward rates can be studied. Second, I measure the difference between fitted yield and its shadow rate counterpart, which Bauer and Rudebusch (2016) called the ZLB wedge. This measure describes how tightly the lower bound restricts interest rates.

Finally, I use equation (19) to decompose changes in the interest rates into changes which originate from variation in the lower bound and which come from changes in factors. It should emphasized that the decomposition is derived under the $\mathbb{Q}$ measure.

Figure 10 shows the value of $\Phi\left[(f_t^s(\tau) - \underline{r}_t^e)/\omega(\tau)\right]$ for 2-year and 10-year shadow forward rates after 2009. Higher values of Φ indicate that a larger part of the distribution of the shadow forward rate is below the lower bound. The value of Φ can be interpret as the probability implied by markets that a shadow forward rate randomly selected from the shadow rate distribution is below the lower bound.

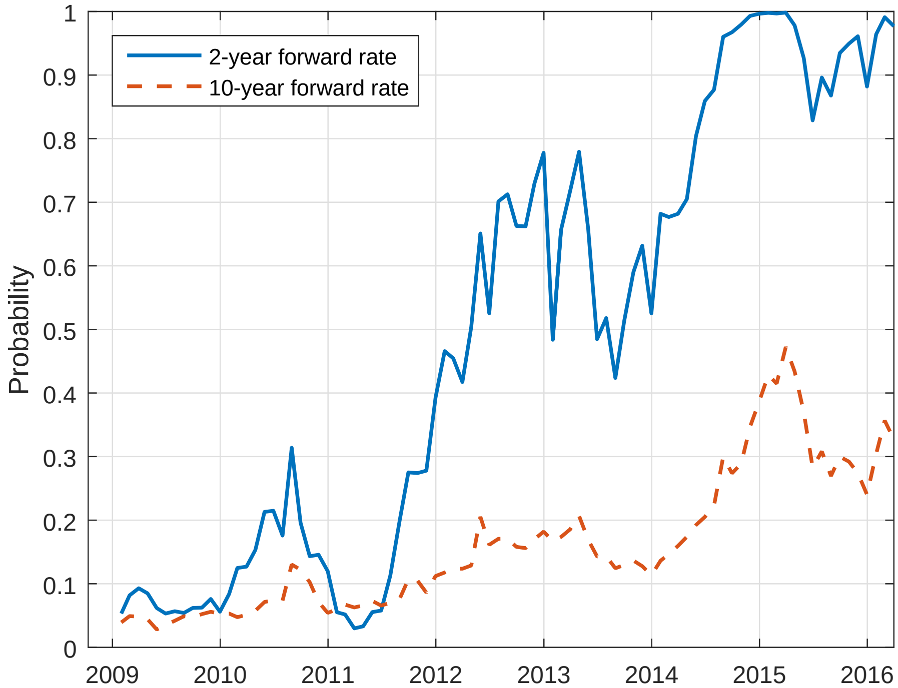


Figure 10: Evolution of probability implied by markets for a shadow forward rate being below the lower bound.

Figure 10 shows that the lower bound did not restrict the dynamics of forward rates significantly before 2011. In the second half of 2011 the forward rate 2 years ahead seems to be significantly restricted by the lower bound. Note that, if interest rates are expected to decrease, then forward rates can be significantly restricted by the lower bound even if the current short-term rates are well above the lower bound. At the turn of the year 2014 - 2015 fluctuations in the 2-year forward rate are completely determined by changes in the lower bound, and the influence of lower bound has been strong ever since. This implies that short-term interest rates should respond heavily to changes in the deposit facility rate or its expected path, which seems to be the case as shown by Figure 2. Similar developments can be seen for the 10-year forward rate but to a smaller extend. The implementation of QE and the following rise of the yield curve significantly reduced the tightness of the lower bound for the forward rate at 10 year maturity. Recently, the significance of the lower bound for the long-term forward rate have increased since interest rates has declined. Generally, it can be concluded that the lower bound for interest rates significantly affects the short- and longer-term forward rates.

The second measure is the so-called ZLB wedge by Bauer and Rudebusch (2016).[31] ZLB wedge is the difference between fitted and shadow interest rates. Equation (17) defines interest rates using the standard term-structure relationship when interest rates are averages of forwards rates over the relevant horizon. In a shadow rate model, interest rate $R_t(\tau)$ is the sum of the shadow rate, $S_t(\tau)$, and the option effect, $Z_t(\tau)$, when the ZLB wedge is defined as

$$Z_t(\tau) = R_t(\tau) - S_t(\tau). \tag{20}$$

That is, ZLB wedge describes the average option effect of cash over maturities from 0 to τ. Put differently, ZLB wedge shows how much the lower bound restricts interest rates at different maturities. Figure 11 shows fitted the 10-year interest rate, corresponding shadow interest rate and ZLB wedge.

The ZLB wedge confirms the results from Figure 10. In the second half of 2011, ZLB wedge started to increase, and the highest value is obtained before the announcement of QE in January 2015. Then, the 10-year interest rate would evidently have been negative if the lower bound did not exist. QE and the rise of long-term interest in May 2015 decreased ZLB wedge to around 1 percentage point, but in year 2016 the ZLB-wedge increased to around 1.5 percentage points.

[31] Here the term ZLB is not appropriate since the lower bound is not zero. However, I use the name in any case to emphasize that this is the same measure as in Bauer and Rudebusch (2016).

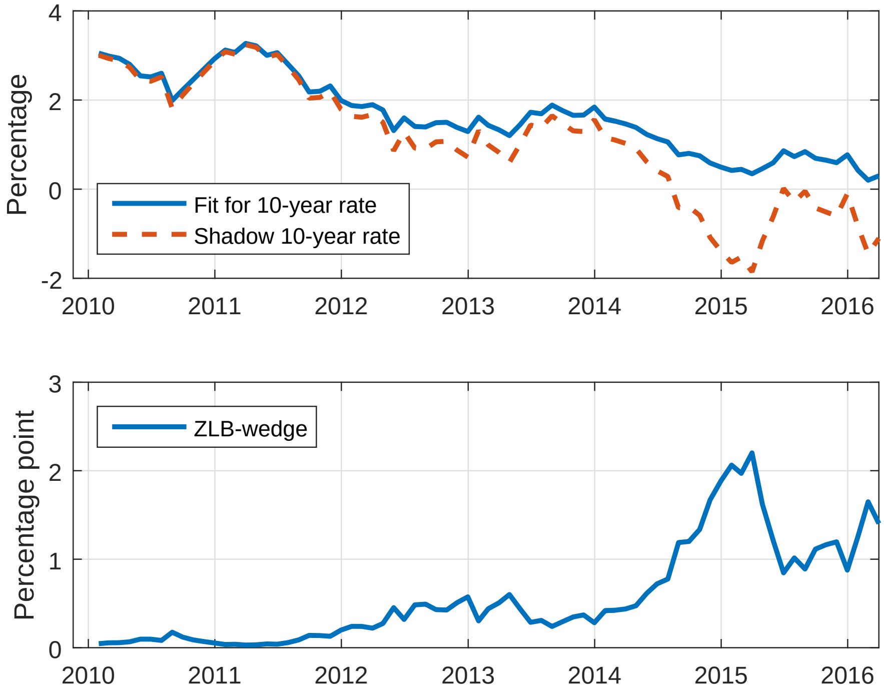


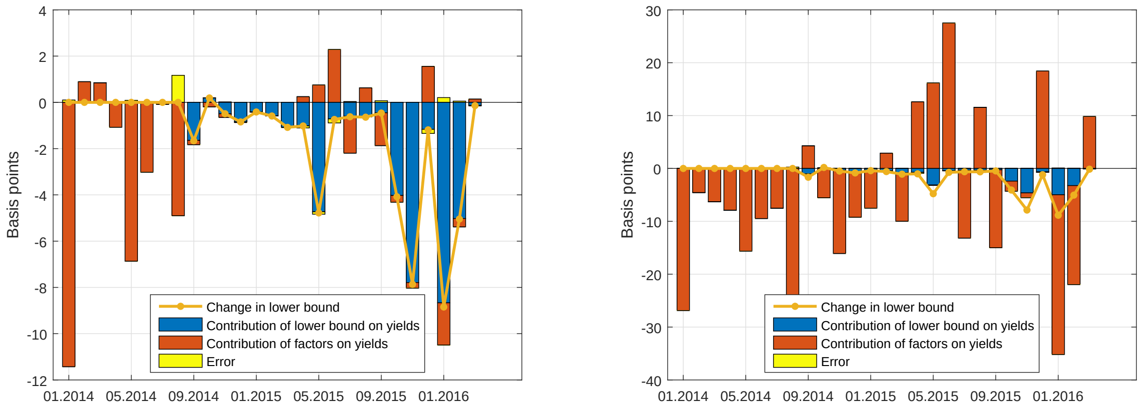
Figure 11: The top panel shows fitted and shadow interest rate and the bottom panel shows their difference which is called as ZLB wedge.

This implies that, without the lower bound, the 10-year interest rate would be highly negative in the euro area. The value of ZLB-wedge is high since the highest ZLB wedge value for the US estimated by Bauer and Rudebusch (2016, Figure 7) was around 0.7 percentage points.

Finally, I derived a decomposition for monthly changes in interest rates based on equation (19).[32] Changes in fitted the interest rate are decomposed into two components: i) changes which originate from changes in the lower bound and, ii) changes due to variation in factors. The size of the first component depends on Φ and $d\underline{r}_t$, for which the change in the lower bound, $d\underline{r}_t$, is also shown in Figure (line with dots). Moreover, the residual of the decomposition is called the error, which represents an approximation error since equation (19) is only accurate around the point

[32]Note that the decomposition is based on the fit provided by the model when changes in yields in Figure 19 are not necessarily equal to the changes in the data. Moreover, it should noticed that results depend on the choice of lower bound sequence. Different paths for the lower bound give somewhat different the results since the dynamics of the lower bounds are different.

of evaluation. Again, it should emphasized that this decomposition is done under measure $\mathbb{Q}$.[33] Figure 12 shows decompositions for 2-year and 10-year interest rates.



(a) 2-year yield (b) 10-year yield

Figure 12: Decomposition (under $\mathbb{Q}$) for monthly changes of interest rates into changes in lower bound and changes in factors (level, slope and curvature).

In the first half of 2014 the effective lower bound was constant when its contribution on changes in interest rates is naturally zero. The first notable change in the 2-year interest rate caused by a change in the lower bound happened in September 2014 when the deposit facility rate was cut by 0.10 bp. The lower bound used here, i.e $\underline{r}^e$, decreased 2 bp. The change in 2-year interest rates implies that the lower bound was quite strongly affecting short-term interest rates. However, there was a smaller effect in the 10-year interest rate, which implies that long-term interest rates were less restricted by the lower bound. In autumn 2015 economic developments and communications by the Governing Council increased expectations of additional monetary stimulus, and the deposit facility rate was expected to be cut further: $\underline{r}^e$ decreased in October and November by 4 and 8 bp. Now clearly the dynamics of short-term yields were driven by these expectations, which also seem to be a major driver of long-term interest rates. That is, the lower bound is now more binding for both short and long-term interest rates than it was in September 2014. Moreover, in the beginning of year 2016 the Council noted that risks in inflation were downwards and that they would evaluate the stance of monetary policy at the next meeting. Short-term interest rates

[33] Especially, changes in long-term yields might not represent changes in factors but rather changes in term-premia. However, many estimates of term-premia suggest that they have been quite low during 2015.

decreased below the deposit facility rate, indicating again that the market expected another cut in the deposit facility rate. As a result, the lower bound measured by the $\underline{r}^e$ sequence decreased 9 bp. This decline in the lower bound had a significant effect on short- and long-term interest rates. Overall, it can be concluded that in the euro area the lower bound significantly restricts short-term interest rates and somewhat also long-term rates. For monetary policy, this implies that cuts – or expectations of future cuts – in the deposit facility rate not only affect short-term rates but these cuts can also lower the long-term rates.

8 Conclusion

Typically a constant lower bound for nominal interest rates is applied in shadow rate models. This might be appropriate for models applied, for example, to US data but might not be for data on other economies. For euro area data it is hard to set a constant lower bound since below the zero lower bound the short-term interest rates were still trending downwards. Moreover, short-term interest rates seem to be anchoring to the expected path of the deposit facility rate when the expected path of the deposit facility rate provides a lower bound for interest rates. This rate is a monetary policy instrument when it is natural to think that the lower bound is time-varying.

Evidence presented in this paper favors the use of a time-varying lower bound in a shadow rate model applied to euro area data. First, recursive estimation of a shadow rate model with a constant lower bound, where the lower bound is a parameter to be estimated, showed that the value of the lower bound parameter decreases after the first half of 2014. This suggest that the parameter is not stable and might be time-varying. Second, the fit and forecasting performance of the shadow rate model with time-varying lower bound outperformed its constant lower bound counterpart. Third, the constant lower bound model with a reasonable choice for the lower bound does not produce an acceptable shadow rate, but the model with time-varying lower bound does. To summarize, there is good reason to assume that the lower bound for nominal interest rates is time-varying in the euro area and the evidence quite unambiguously favors the time-varying lower bound model over the constant lower bound counterpart.

When the lower bound is controlled by monetary policy via changes in the deposit facility rate, this gives a new channel for monetary policy to affect the yield curve. The intensity of this channel depends on the tightness of the lower bound over interest rates. The more restrictive the lower bound, the more effective cuts in the deposit facility rates are for lowering the interest rates. If long-term interest rates are restricted by the lower bound, these rates can be affected by monetary policy very efficiently when the policy rate controls the lower bound for nominal interest rates. Medium- and long-term interest rates are generally considered to be important drivers of investments and consumption when – paradoxically – the binding lower bound for nominal interest rates does not mean that the ability of monetary policy to lower interest rates and stimulate consumption and investments is decreased.

Interest rates in the euro area are at historically very low level, which suggests that the lower bound might be relevant for long-term interest rates. Measures applied in this paper confirm that: long-term interest rates have been somewhat restricted by the lower bound after the first half of 2014. This then implies that fluctuations in the expected path of the deposit facility rate should influence long-term rates. This indeed seems to be the case: at the turn of the year 2015 - 2016 a significant driver of the ten year interest rate was changes in the lower bound, which originated from expected and actual changes in the deposit facility rate. However, these results should be interpreted with care for two reasons. First, it is unclear how negative interest rates are transmitted by the banking sector to firms and households. Therefore, the effects of cuts in the deposit facility rate on consumption and investment are uncertain. Second, the result does not mean that interest rates could be taken to any negative number, as there might be a level at which arbitrage with cash is operational.

There are many avenues for further research on shadow rate term structure models with time-varying lower bound. This is the (one of) first paper(s) to introduce a time-varying lower bound for a shadow rate model with the time-varying lower bound modeled in the simplest manner, i.e. exogenous in the model. This is obviously an unsatisfying feature of the model; the dynamics of the lower bound should be made endogenous. Time-varying lower bound should also be applied in different types of dynamic term structure models to see how these models behave with time-varying lower bound.

A Graphical analysis of censored distributions

A.1 Forward rate, shadow forward rate and lower bound

In Figure A1 the blue line describes the density distribution of the forward shadow rate. This distribution is not censored since the shadow rate can take negative values when the expected value of forward shadow rate is given by the vertical red line. The black line is the lower bound, $\underline{r}_t$, when lower values for the forward rate are not possible. The shaded area describes Φ, which is the probability mass of the censored part of the distribution. The expected forward rate is given by the dashed red line. It is higher than the forward shadow rate because some negative values for forward rate are not possible because of the lower bound and the probability mass associated with these values now being assigned a higher value, which is the lower bound.

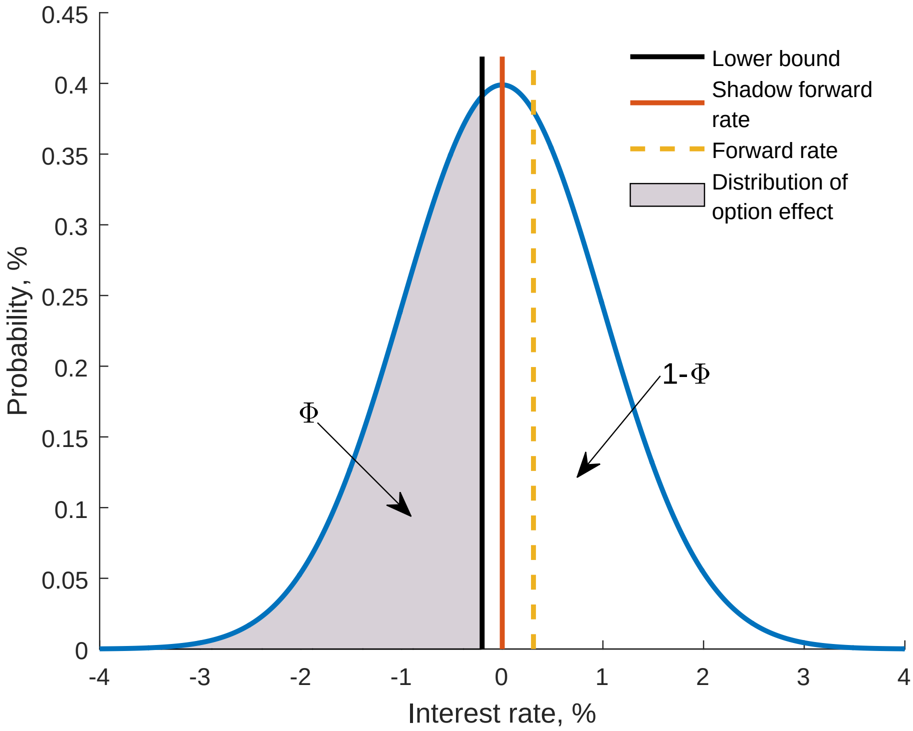


Figure A1: An illustration of the determination of forward rate, f_t, and the shadow forward rate, f_t^s in the shadow rate model for some τ. Shaded area describes the censored part of the distribution where probability mass equals Φ and the white area under density describes the non-censored part.

A.2 The shift of forward rate

The initial shadow rate and its distribution as well as the observed forward rate are described by dashed lines in Figure A2. The lower bound is fixed in this example. Now consider that factors change such that the expected shadow rate moves from 0 to -1 percent. Without a lower bound, the change in the forward rate would be equal to the change in the shadow rate, but now the lower bound restricts the effects of factors on the observed forward rate, which falls by just 40 basis points. Note also that now the value of Φ increases; to further reduce the forward rate a larger change in factors is now needed. Finally, the lower bound is also the limit on how low the forward rate can go.

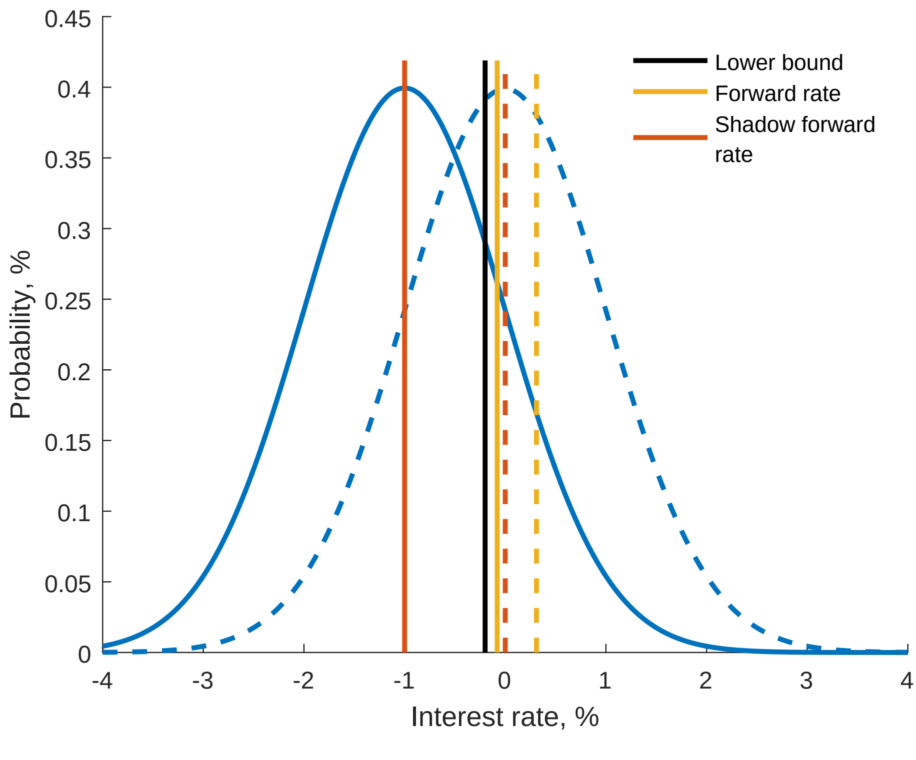


Figure A2: Illustration of how changes in shadow rate affect the observed forward rate. Dashed lines describe the intial state and solid lines describe the final state of variables.

B Detailed derivation of equation (19)

Rewrite equation (16) as

$$f_t(\tau) = \Phi\left[\frac{\underline{r}_t - f_t^s(\tau)}{\omega(\tau)}\right]\underline{r}_t + \left(1 - \Phi\left[\frac{\underline{r}_t - f_t^s(\tau)}{\omega(\tau)}\right]\right) f_t^s(\tau) + \omega(\tau)\phi\left[\frac{\underline{r}_t - f_t^s(\tau)}{\omega(\tau)}\right] \quad \text{(A1)}$$

where $\Phi(\cdot)$ is the cumulative distribution function for unit normal distribution and $\phi(\cdot)$ is its density function. The differentiation of equation (A1) is given by

$$df_t(\tau) = \frac{\partial f_t(\tau)}{\partial \underline{r}_t} d\underline{r}_t + \frac{\partial f_t(\tau)}{\partial f_t^s(\tau)} df_t^s(\tau). \quad \text{(A2)}$$

The first partial derivative is as follows:

$$\begin{aligned}\frac{\partial f_t(\tau)}{\partial \underline{r}_t} &= \Phi + \underline{r}_t\phi\frac{1}{\omega(\tau)} + \phi f_t^s(\tau)\frac{1}{\omega(\tau)} + \omega(\tau)(-\phi)\left(\frac{\underline{r}_t - f_t^s(\tau)}{\omega(\tau)^2}\right) \\ &= \Phi \quad \text{(A3)}\end{aligned}$$

where I have denoted $\Phi(\cdot) = \Phi$ and $\phi(\cdot) = \phi$. The second partial derivative is given by

$$\begin{aligned}\frac{\partial f_t(\tau)}{\partial f_t^s(\tau)} &= -\underline{r}_t\phi\frac{1}{\omega(\tau)} + 1 - \Phi + f_t^s(\tau)\phi\frac{1}{\omega(\tau)} + \omega(\tau)(\phi)\left(\frac{\underline{r}_t - f_t^s(\tau)}{\omega(\tau)^2}\right) \\ &= 1 - \Phi \quad \text{(A4)}\end{aligned}$$

Substituting equations (A3) and (A4) into equation (A2) yields equation (19) in the text.

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