Patrick M. Crowley – David Hudgins

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EURO AREA MONETARY AND FISCAL POLICY TRACKING DESIGN IN THE TIME-FREQUENCY DOMAIN

PATRICK M. CROWLEY[*] and DAVID HUDGINS[†]

[*] *Corresponding author: Department of Decision Sciences and Economics, College of Business, OCNR 373, 6300 Ocean Drive, Texas A & M University – Corpus Christi, Corpus Christi, TX, 78412 U.S.A., E-mail: patrick.crowley@tamucc.edu, Phone: 361-825-6011*

[†] *Department of Decision Sciences and Economics, College of Business, OCNR 314, 6300 Ocean Drive, Texas A & M University – Corpus Christi, Corpus Christi, TX, 78412 U.S.A., E-mail: david.hudgins@tamucc.edu, Phone: 361-825-5574*

Abstract. This paper first applies the *MODWT* (Maximal Overlap Discrete Wavelet Transform) to Euro Area quarterly GDP data from 1995 – 2014 to obtain the underlying cyclical structure of the GDP components. We then design optimal fiscal and monetary policy within a large state-space *LQ*-tracking wavelet decomposition model. Our study builds a MATLAB program that simulates optimal policy thrusts at each frequency range where: (1) both fiscal and monetary policy are emphasized, (2) only fiscal policy is relatively active, and (3) when only monetary policy is relatively active. The results show that the monetary authorities should utilize a strategy that influences the short-term market interest rate to undulate based on the cyclical wavelet decomposition in order to compute the optimal timing and levels for the aggregate interest rate adjustments. We also find that modest emphasis on active interest rate movements can alleviate much of the volatility in optimal government spending, while rendering similarly favorable levels of aggregate consumption and investment. This research is the first to construct joint fiscal and monetary policies in an applied optimal control model based on the short and long cyclical lag structures obtained from wavelet analysis.

Keywords: Discrete Wavelet Analysis, Euro area, Fiscal Policy, LQ Tracking, Monetary Policy, Optimal Control

JEL classifications C49 . C61 . C63 . C88 . E52 . E61

1 Introduction

This study combines discrete wavelet analysis, the macroeconomic accelerator model, and optimal control theory to generate fiscal and monetary policy interactions in an LQ tracking econometric model. The accelerator model has proved to be a useful theoretical and empirical tool over many decades since Samuelson (1939). Chow (1967), Kendrick (1981), Kendrick and Amman (2010), Kendrick and Shoukry (2013), Hudgins and Na (2014), and Crowley and Hudgins (2014) have all modeled quarterly fiscal policy within an applied macroeconomic optimal control *LQ* (Linear-Quadratic) tracking framework. Using U.S. data, Kendrick and Shoukry (2013) simulate the tracking performance and debt structure of the quarterly and annual models within a closed-economy macroeconomic model that allows for monetary policy through interest rate components, but restricts the interest rate so that monetary policy is extremely passive. Hudgins and Na (2014) examine optimal robust policies using a similar model for the U.S. economy, but without money and interest rates.

All of the papers quoted above note that policymakers must consider different lag structures in the consumption and investment equations in order to simulate the optimal policies under various parameters. Instead of estimating various time series models with alternating lags, another approach utilizes wavelet analysis in order to gain insight into the lag structure through the time-frequency domain. Crowley and Hughes Hallett (2014), Crowley and Hudgins (2014), and Crowley and Hughes Hallett (2015) use an *MODWT* (Maximal Overlap Discrete Wavelet Transform) to obtain the time-frequency domain cyclical decomposition of U.S. GDP component data for the period 1947 – 2012. Using Euro Area data, our analysis will build a control model using the results obtained from a similar estimation strategy using wavelets, which will also allow the estimation of the optimal timing of the quarterly fiscal and monetary policy impulses. In this paper, we therefore show that the cyclical timing and lag structure can be determined with greater precision by using the quarterly lag structure gathered through the application of wavelet filters in the time-frequency domain.

Using the longer cycles obtained from wavelet analysis also addresses the findings of recent neoclassical research. This is consistent with Leeper et al. (2010), for example, who find that the speed of the fiscal adjustment impacts the policy effectiveness. In addition, Leeper et al. (2010) find that government investment with comparatively weak productivity can induce government investment policy to be contractionary for growth in the long-run. Crowley and Hughes Hallett (2014 and 2015) use an *MODWT* wavelet decomposition and find that U.S. fiscal policy has not been destabilizing or procyclical over the various business cycle frequencies; however, they also found that it has not been effective as a countercyclical stabilizer either. Further, Crowley and Hudgins (2014) show that designing quarterly fiscal policy rules that are built upon the full range of short and long term cyclical wavelet components avoids the bias that might otherwise be introduced through inadequate recognition of the interplay of the short-term lags with the long term cyclical components.

Wavelet analysis does not directly address the foresight components; the optimal control analysis that uses the components from wavelet analysis, however, is based partly on the future expectations of government policy changes. Indeed, Leeper et al. (2010) find that the agents' fiscal foresight has some impact on the policy effectiveness.

Kriwoluzky (2012), using a *VMA* (Vector Moving Average) model, shows that consumption initially reacts negatively to a pre-announcement period of a government spending shock, but reacts positively after the realization of increased spending. Karantounias (2013) projects optimal taxes when the government authorities rely upon an exogenous government spending probability model, but the public has pessimistic expectations. The result is that a paternalistic planner will employ distortionary taxation by exploiting household mispricing and shifting household expectations, which leads to higher tax rates during favorable shocks and lower tax rates for adverse shocks.

Svec (2012) models an altruistic government that optimally sets labor taxes and one-period debt in an economy, where uncertain consumers believe that the true approximating probability model lies within a range of probabilities. The results show that the political government that maximizes consumer welfare under the consumers' own subjective utility functions finances a smaller portion of a government spending shock from taxes than it would if consumers did not face model uncertainty. Kendrick and Amman (2006) discuss optimal control design in stochastic macroeconomic systems with forward-looking variables. Although the time-frequency domain does not give any information on agents' potential foresight structure, it does indirectly address this issue since it captures all of the underlying cyclical components that the agents are using to build their expectations in an attempt to develop foresight.

1.1 Purpose and Scope

The purpose of this paper is to construct optimal fiscal and monetary policy that utilizes an optimal *LQ* (Linear-Quadratic) tracking control model that is formulated within the time-frequency domain based on an *MODWT* wavelet decomposition. This is the first research to design macroeconomic policy by integrating both monetary and fiscal policy into and optimal control framework based on discrete wavelet analysis. Section 2 examines the *MODWT* wavelet methods and the decomposition of data in the time-frequency domain, using Euro Area data over the period 1995 – 2014. Section 3 builds a macroeconomic time-frequency accelerator model that is used within an optimal control system to determine optimal control feedback rules for monetary and fiscal policy. We convert the *LQ* tracking design into an *LQ* regulator design using the method employed by Crowley and Hudgins (2014) and Hudgins and Na (2014), and develop a MATLAB software program to compute the optimal joint fiscal and monetary policy. This framework allows the policymaker to render deterministic, stochastic *LQG* (Linear-Quadratic Gaussian) and robust controller designs, but the research presented in this paper only presents simulations for the deterministic *LQ* tracking control design.

Section 4 estimates and simulates the time-frequency model developed in section 3. The simulations explore an *FHEC* (Frequency Harmonizing Emphasis Control) political targeting strategy. This approach allows the policymakers to place larger weights on the tracking errors for government spending, consumption, investment and the interest rate, at frequency ranges between 1 and 8 years. This is not possible under an aggregate model without time-frequency decomposition. This renders an entirely new operational procedure for constructing both optimal fiscal and monetary policy, and for determining the likely effects over each frequency range.

We simulate the model under three different policymaker scenarios: (1) dual emphasis on active use of fiscal and monetary policy, (2) emphasis on active use of fiscal policy with relatively passive monetary policy, and (3) active use monetary policy with relatively passive fiscal policy. We find that dual active policy with a modest emphasis on active short-term market interest rate movements can alleviate much of the volatility in optimal government spending levels, while still generating favorable balance between the aggregate consumption and investment trajectories.

The results also suggest that the monetary authorities should utilize a strategy that influences the aggregate short-term market interest rate to follow an undulating trajectory that adjusts according to a wavelet-based cyclical decomposition. The optimal timing and levels of the aggregate adjustments are determined based on the components of the wavelet decomposition, rather than being restricted to an arbitrarily determined steady growth rate, or a slow sequence of *ad hoc* small discrete jumps in the operating target interest rates (on the marginal lending facility and the deposit facility) with arbitrary timing and/or magnitude. When the monetary authorities follow a passive policy that utilizes its instruments to restrict the market interest rate such that it closely tracks its intermediate target rate, aggregate government spending becomes much more volatile. Conversely, when the fiscal authorities passively keep government spending close to its target level, the optimal interest rate does not experience a large increase in volatility, but the resulting performance of investment deteriorates. This finding suggests that monetary authorities should not be overly restrictive in tracking their intermediate target too closely, but should instead allow enough flexibility to accommodate adjustments in government purchases, consumption, and investment.

2 *MODWT* Wavelet Analysis

Here we provide a brief overview of the mathematical background of time-frequency analysis and the discrete wavelet methodology. Time-domain analysis cannot provide a proper basis for analysis when frequencies are changing; in other words, time-domain methods generally cannot reveal valuable and helpful information which are hidden in different frequencies. So in order to shed some light on what is occurring in the frequency domain, we need to employ a frequency domain approach.

The most well-known frequency domain method is the Fourier Transform (*FT*). This method transforms time series data from the time domain to the frequency domain, but the problem with the *FT* approach is that the data is transformed into just the frequency domain, so there is no ability to simultaneously analyze relationships in both the time and frequency domains. Indeed, with the *FT* method, the time series under consideration should be locally and globally stationary; but unfortunately, this also imposes a limitation since a significant number of economic and financial time series are locally and globally non-stationary. This limitation results from the trade-off between the frequency resolution and time resolution (Weedon, 2003) and even when employing a windowed *FT* approach, the *FT* cannot properly capture dynamics.

2.1 Wavelet Analysis

Wavelet analysis has its roots in multi-scale decomposition, the so-called multiscale analysis or multiresolutional analysis, which was developed by Meyer (1986), Mallat (1989a,b), Strang (1989), and Daubechies (1986). Technically speaking, multiscale analysis is an approximation operation through a dense vector space (Hilbert space) with empty intersects from coarsest to less detailed information.

Following Mallat's explanation for the pyramid algorithm and multiresolutional analysis, the value of a variable *x* at time instant *k*, x_k, can be written as follows:

$$x_k \approx S_{J,k} + d_{J,k} + d_{J-1,k} + ... + d_{1,k} \tag{1}$$

where $d_{j,k}$ are detail components (wavelet "crystals"), j = 1,…, J; $S_{J,k}$ is a trend component (the wavelet "smooth"); and *J* stands for the number of scales (frequency bands). Equations (1) – (3) summarize the *DWT* (Discrete Wavelet Transform) process. A variable x_k is filtered by a (low-pass) filter, *l*, and a wavelet (high-pass) filter at each step. In other words, we filter out information at a different set of frequencies in each step until we reach an approximated variable which contains only the trend. In this regard, in the first step x_k is decomposed into $d_{1,k}$ (the high frequency part) and $S_{1,k}$ (the low frequency part); consequently, the decomposed signal at scale 1 (J = 1) can be written as follows:

$$x_k \approx S_{1,k} + d_{1,k} \tag{2}$$

Then, this same process is performed on $S_{1,k}$ while the signal will be subsampled by 2, so that:

$$S_{1,k} \approx S_{2,k} + d_{2,k} \tag{3}$$

This recursive procedure is continued until we reach scale *J*. So finally, we have a set of detailed variables (high frequency components or "crystals") and a smoothed trend component (or the "smooth") as in equation (1). This approach then enables us to simultaneously investigate the data in both the time and frequency domain.

It should be noted that there are many different wavelet filter functions that are used in discrete wavelet analysis, such as the Symlet, Coiflet, Haar, Discrete Meyer, Biorthogonal, Daubechies and so on, which can be approximated for use in the filtering process as pairs of low pass and high pass filters. In this paper, we employ a Daubechies 4-tap (D4) wavelet as the wavelet function[1], which is an asymmetric wavelet, and we employ the *MODWT* (Maximum Overlap Discrete Wavelet Transform) as our method of time-frequency decomposition. Since the *DWT* suffers from two shortcomings, namely (1) "dyadic data requirements" and (2) the fact that the *DWT* is "non-shift invariant" (see Crowley (2007)), the *MODWT* is used as an alternative method which addresses the aforementioned drawbacks and provides some other advantages.[2]

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[1] We employ a periodic boundary condition for the GDP data, but for the interest rate data, we employ a reflection boundary assumption.

[2] For more information see Crowley (2007).

2.2 MODWT Wavelet Decomposition Results

The empirical analysis uses Eurostat Real GDP data in millions of constant euros for the Euro Area 18, for the period 1995 quarter 1 to 2014 quarter 2. We also use 3 month money market euro area interest rates, also taken from the Eurostat database, but here we decompose from the beginning of the series in 1991 and then just use the data corresponding to the same time span of the GDP data[3].

The *MODWT* is applied to the data using a two-step procedure in order to obtain the crystals and the smooth trend at frequencies $j = 1,\ldots, 5$. One of the problems in using the *MODWT* filter approach is that the crystals do not always aggregate up to the original series due to the loss of orthogonality as the higher order (lower frequency) wavelets are passed, observation by observation, through the series, and are thereby "smoothed". As most cyclical activity can be most easily identified in the first differenced series, we employed a two-step procedure to derive the frequency cycles. First, the simple difference series for the GDP components was decomposed, and second, the level series by crystal were then reconstructed. Lastly, a *modified smooth* (*S*) is created by taking the original series and subtracting out the reconstructed frequency cycles, so that the sum of the summed crystals and the *modified smooth* is equal to the actual observation. In other words the *modified smooth* contains both the smooth and any corrections entailed due to the approximation that stems from using the MODWT.

Given that we use quarterly data in this study, the interpretation of the frequency cycles or crystals in terms of the frequency ranges that they contain are given in Table 1, and the crystals obtained, along with the original data are shown in Figures 1 – 4 below:

Table 1

The time intervals associated with each of the frequencies

j	*Time interval*
1	6 months to 1 year
2	1 – 2 years
3	2 – 4 years
4	4 – 8 years
5	8 – 16 years

In Figure 1, each crystal is displayed as series $d_{Y,1,k}$ to $d_{Y,5,k}$, which refers to the reconstructed fluctuations in the level data for *Y* at frequencies up to a 16 year cycle. The actual series (which is measured relative to the right hand side axis), refers to the actual level *real GDP* series (*Y*), and the *modified smooth* residual trend (*S*) referred to above. As can be seen, the modified smooth and the actual data series are quite closely aligned except when there are sharp turning points in the data, such as in the beginning of the "great recession" around 2008, when there is some divergence, implying that the *MODWT* decomposition does not model the data too well over these specific data ranges.

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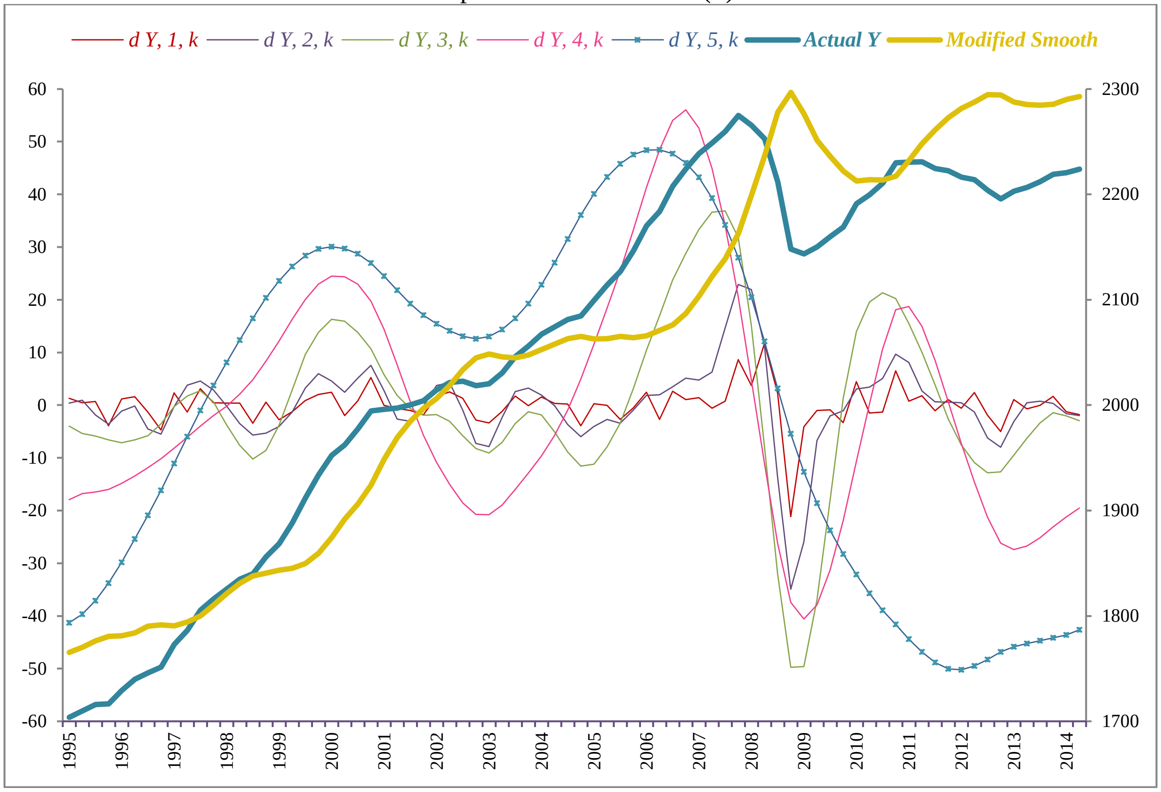
[3] This permits better resolution of the longer cycles in the interest rate data.

Figure 1
MODWT Decomposition for *real GDP* (*Y*): 1995 – 2014



The crystals exhibit a wide degree of variation in the type of fluctuations that are embodied in the original series. The higher frequency variations appear to be almost noise, but the lowest frequency variations captured by $d_{Y,5,k}$ capture the business cycle and appear to be highly irregular cycles, as one might expect. Interestingly there is a more volatile higher frequency cycle of 2 – 4 years (corresponding to $d_{Y,3,k}$) apparent in the data, corroborating the findings of Crowley and Hughes Hallett (2014, 2015). The longer lower frequency cycle of 8 – 16 years begins decreasing around 2007 and does not recover until after 2012.

The *MODWT* filter results for Government Purchases, Consumption, and Private Investment are illustrated in figures 2, 3, and 4, respectively. Figure 2 shows the government expenditure decomposition. There is clearly a smaller degree of fluctuation in any cycles detected in government expenditures at higher frequencies. Much of the variation in government expenditures is contained in longer cycles displayed in the $d_{G,4,k}$ and $d_{G,5,k}$ crystals. The limited euro area government stimulus is correctly identified mostly in the $d_{G,4,k}$ crystal (4-8 year cycles), with some of the stimulus bleeding over into the $d_{G,5,k}$ crystal. There is a sharp slowdown in government expenditure detected though in 2010 through 2013, which stems from the austerity measures put in place in the latter stages of the European sovereign debt crisis.

Figure 2

MODWT Decomposition for *Government Spending* (***G***): 1995 – 2014

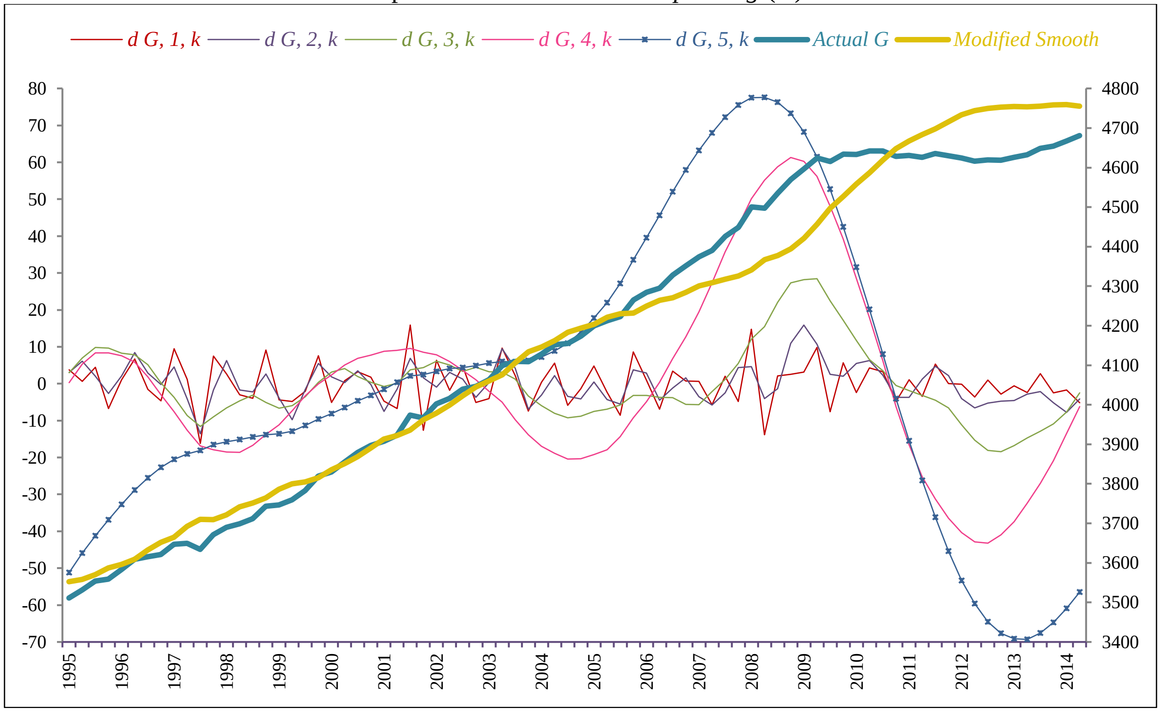
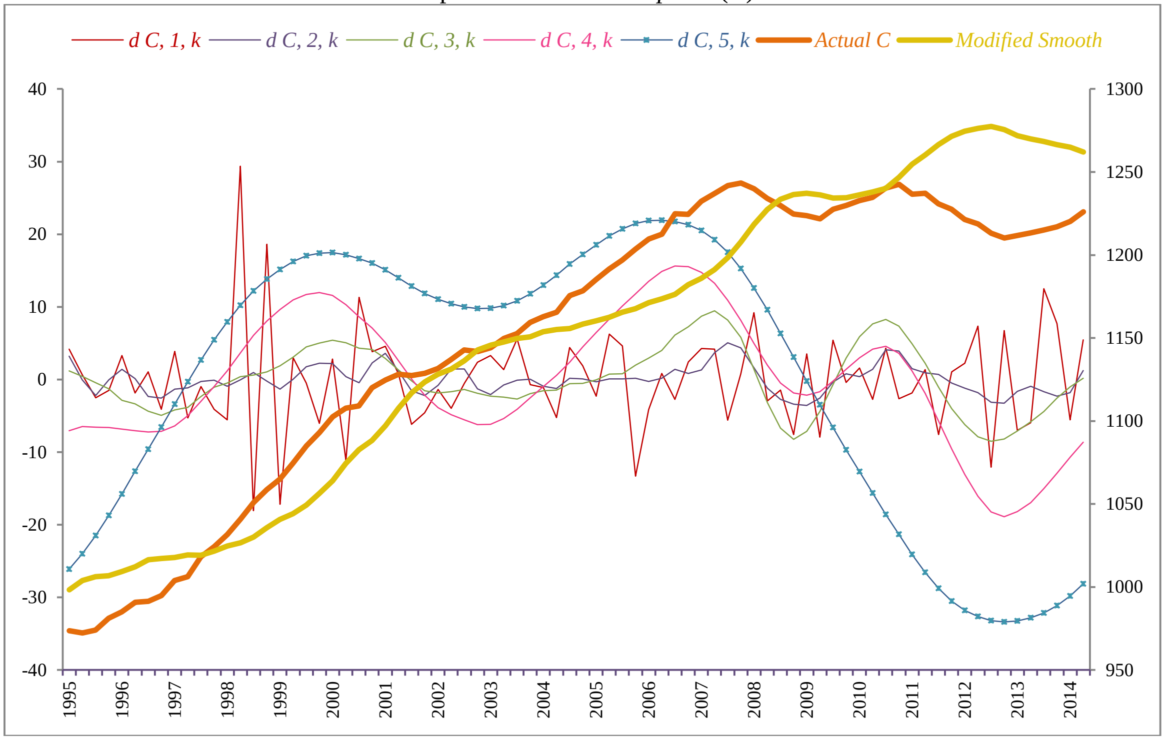


Figure 3

MODWT Decomposition for *Consumption* (***C***): 1995 – 2014



In figure 3, the consumption expenditure decomposition is shown. As might be expected, consumption expenditure cycles and the modified smooth are very similar to those observed for real GDP (Y), but with some differences. For example, the $d_{C,3,k}$ and $d_{C,4,k}$ consumption crystals fall to relatively lower values in the 2013 trough than in the 2008 – 2009 trough, as contrasted with real GDP, where the $d_{Y,3,k}$ and $d_{Y,4,k}$ crystals are relatively lower in the 2008 – 2009 trough than their values in 2013. The cyclical activity embodied in the detail crystals once again shows a large downturn during the great recession, and then a leveling rebound with another drop in consumption and output before resuming the upward trend after 2013.

Figure 4 shows the wavelet decomposition for investment. The actual data and the modified smooth are much more volatile than for the other components of output, while the fall in investment during the great recession is large at all frequencies. Also, it is clear that the level of investment has still not yet recovered to its pre-recession levels. At the end of the horizon, investment is the smallest at lower frequency ranges 4 and 5.

Figure 4

MODWT Decomposition for *Private Investment* (I): 1995 – 2014

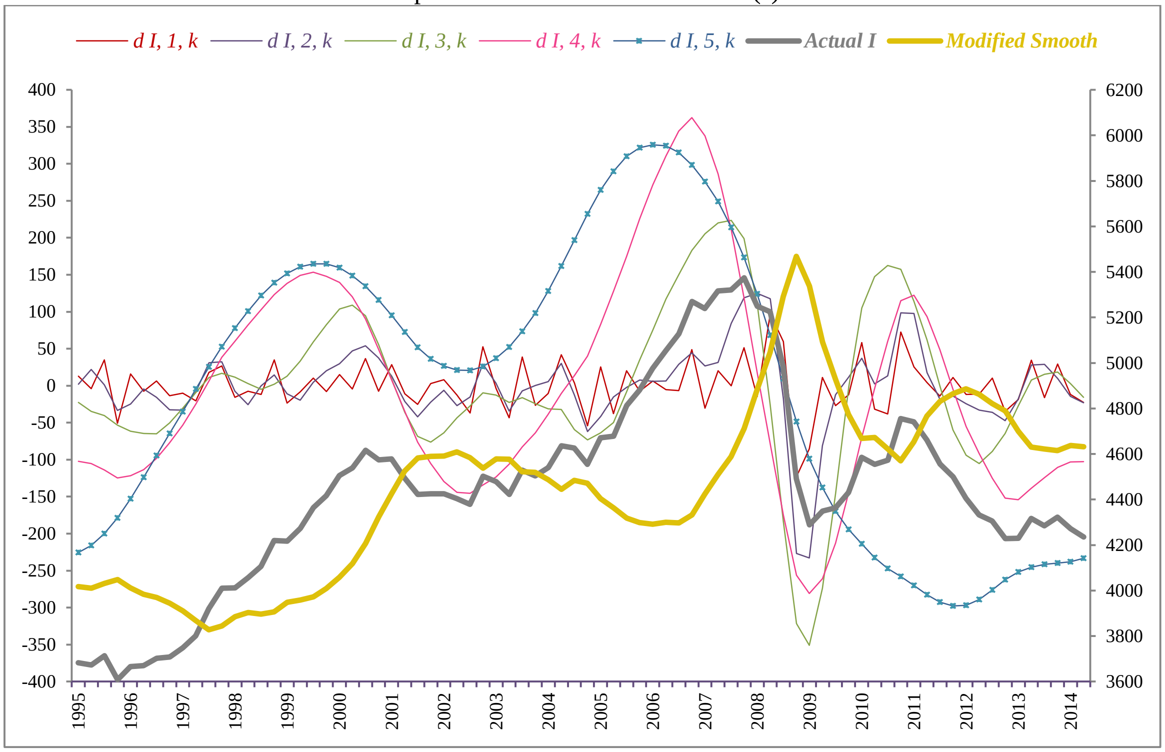
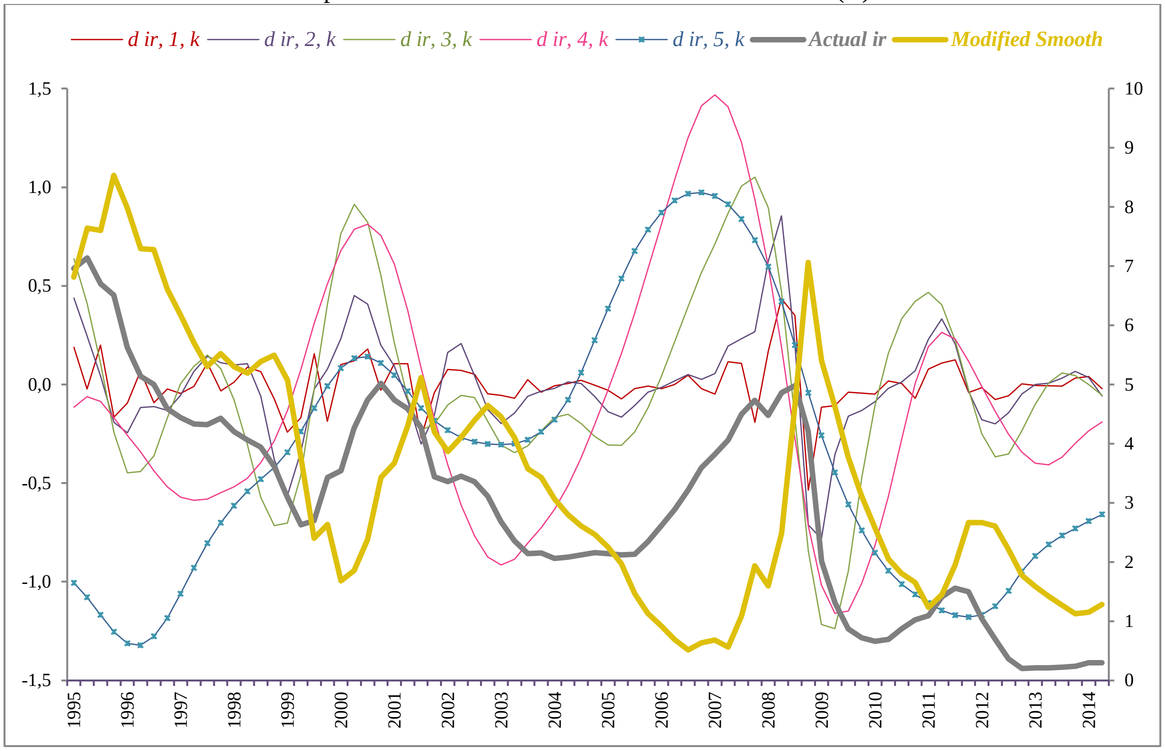


Figure 5 shows the wavelet decomposition for the 3-month nominal euro interest rate. There is a downturn in the aggregated interest rate and the modified smooth in the early 2000s, and in the period after from 2008 through 2011, before leveling out in the final quarters.

Figure 5

MODWT Decomposition for *3-month nominal euro interest rate* (***ir***): 1995 – 2014



3 Macroeconomic Model Derivation

This section uses the *MODWT* crystals to develop a macroeconomic accelerator model, where the foundation is based on the partial accelerator model of Crowley and Hudgins (2014). Let the variables be defined as follows:

C_k	=	total real personal consumption expenditures in period k
C_k^*	=	desired consumption, or target consumption, in period k
I_k	=	real gross private domestic investment in period k
I_k^*	=	desired investment, or target investment, in period k
Y_k	=	real gross national product in period k
G_k	=	government purchases of goods and services in period k
G_k^*	=	desired government purchases in period k
NX_k	=	real net exports in period k
$d_{C,j,k}$	=	the value of the consumption expenditure crystal for frequency j in quarter k, where $j = 1, \ldots, 5$

$d_{I,j,k}$ = the value of the private domestic investment crystal for frequency *j* in quarter *k*, where $j = 1, \ldots, 5$

$d_{G,j,k}$ = the value of the government purchases crystal for frequency *j* in quarter *k*, where $j = 1, \ldots, 5$

$S_{C5,k}$ = the value of the consumption modified smooth in quarter *k*, where $J = 5$

$S^*_{C5,k}$ = target value for the modified smooth trend consumption, in period *k*

$S_{I5,k}$ = the value of the investment modified smooth in quarter *k*, where $J = 5$

$S^*_{I5,k}$ = target value for the modified smooth trend investment, in period *k*

$S_{G5,k}$ = the value of the government purchases modified smooth in quarter *k*, where $J = 5$

$S^*_{G5,k}$ = target value for the modified smooth trend in government spending, in period *k*

$C_{j,k}$ = the prevailing consumption expenditure at frequency *j* in quarter *k*, which includes the sum of the consumption crystal and the consumption modified smooth, where $j = 1, \ldots, 5$

$C^*_{j,k}$ = the target consumption expenditure at frequency range *j* in quarter *k*

$I_{j,k}$ = the prevailing private domestic investment at frequency *j* in quarter *k*, which includes the sum of the investment crystal and the investment modified smooth, where $j = 1, \ldots, 5$

$I^*_{j,k}$ = the target investment expenditure at frequency range *j* in quarter *k*

$G_{j,k}$ = the prevailing government purchases at frequency range *j* in quarter *k*, which includes the sum of the government purchases crystal and the government purchases modified smooth, where $j = 1, \ldots, 5$

$G^*_{j,k}$ = the target government expenditure at frequency range *j* in quarter *k*

$G^d_{j,k}$ = the current cycle trend government purchases at frequency range *j* in quarter *k*.

T_k = net government taxes and income in quarter *k*, which equals total government tax and income minus total government transfer payments.

DEF_k = total government budget deficit in quarter *k*, which equals government purchases of goods and services minus net government taxes

$DEBT_k$ = total government debt in quarter *k*

ig_k = average quarterly interest rate on government debt in quarter *k*

ir_k = 3-month (quarterly) nominal euro market interest rate in quarter *k*

$ir_{j,k}$ = the prevailing 3-month (quarterly) euro market interest rate at frequency *j* in quarter *k*, which includes the sum of the interest rate crystals and the interest rate modified smooth, where $j = 1, \dots, 5$

$I^*_{j,k}$ = the target investment expenditure at frequency range *j* in quarter *k*

τ_k = rate of net tax (tax minus transfers) collection in quarter *k*

The actual prevailing level of consumption at frequency range *j* in period *k* follows an accelerator framework, where it is determined by the previous period's consumption level at frequency range *j*, plus a fraction of the difference between the current targeted level and the last period level, as shown in equation (4). The target consumption level ($C^*_{j,k}$) for each frequency *j* in the wavelet frequency range during period *k*, is expressed in equation (5) as a linear function of the expected rate of output production, where $\beta_{j,k}$ is the *MPC* (marginal propensity to consume). The net *MPC*, $b_{j,k}$, is formulated based on the net disposable income, where τ_k is the rate of net (taxes minus transfers) government tax and revenue collection. Here we make the simplifying assumption that these government variables represent an aggregate Euro Area member state government tax and revenue collections.

$$C_{j,k} = C_{j,k-1} + \gamma_{j,k}(C^*_{j,k} - C_{j,k-1}) \qquad j = 1, \dots, 5 \qquad (4)$$

$$C^*_{j,k} = a_{j,k} + b_{j,k} Y^e_{j,k} \qquad b_{j,k} = (1-\tau_k)\beta_{j,k} \qquad j = 1, \dots, 5 \qquad (5)$$

Consumption at each frequency consists of the specific contribution to the consumption spending at that frequency, given by the consumption crystal $d_{C,j,k}$, plus the modified smooth $S_{C5,k}$, which reflects the base-level trend consumption. This prevailing level of consumption $C_{j,k}$ at frequency range *j* is given by equation (6), where the contributions from the other four frequency ranges have been removed. Equations (7) and (8) specify this same time-frequency contribution for the prevailing levels of private domestic investment $I_{j,k}$, and government spending $G_{j,k}$, respectively.

$$C_{j,k} = d_{C,j,k} + S_{C5,k} \qquad j = 1, \dots, 5 \qquad (6)$$

$$I_{j,k} = d_{I,j,k} + S_{I5,k} \qquad j = 1, \dots, 5 \qquad (7)$$

$$G_{j,k} = d_{G,j,k} + S_{G5,k} \qquad j = 1, \dots, 5 \qquad (8)$$

The expected rate of output production is given by equation (9). The expected contribution of government purchases of goods and services at frequency range *j* in period *k* is given by equation (10).

$$Y^e_{j,k} = C_{j,k} + I_{j,k} + G^e_{j,k} + NX^e_{j,k} \qquad j = 1, \dots, 5 \qquad (9)$$

$$G^e_{j,k} = \phi_{j,k}\left[G_{j,k-1} - \pi_k(DEBT_{k-1} - DEBT_0)\right] + (1-\phi_{j,k})\hat{G}^d_{j,k-1}; \; 0 < \phi < 1; \; j = 1, \dots, 5 \qquad (10)$$

$$NX^{e}_{j,k} = n_{0,j,k} \qquad j = 1, \ldots, 5 \qquad (11)$$

Equations (9) and (10) model rational expectations behavior, where the effective contribution of government spending toward national output production is both limited and crowded out by any stock of national debt that exceeds its initial value. Any new fiscal policy regime will pulse the current cycle, and the analysis below explores cases where government spending is determined through optimal control.

The variable $G^{d}_{j,k}$ defines the current time-frequency trend of government purchases. Equation (10) shows that the expected value of government purchases in any period *k* is determined based on a weighted average of the actual spending in the previous period, and the trend value in the corresponding wavelet frequency range from the previous period. Since government purchases only affect the economy through the expected national output term in equations (5) and (9), all government spending changes have a limited impact. The higher is the value of $\phi_{j,k}$, the greater will be the effectiveness of fiscal policy at any given frequency range. The expected value of net exports in equation (11) is expected to be constant in order to focus on the domestic part of the economy. Although the monetary authorities could influence net exports through the euro exchange rate, monetary policy only enters the current model through the interest rate, as in Kendrick and Shoukry (2013).

Equations (12) and (13) specify the investment functions at each frequency range. The targeted level of investment in equation (12) is a linear function of the crowding out effects of both the government spending level and the prevailing interest rate in the previous period. The current level of investment in equation (13) follows its own accelerator function, similar to that for consumption.

$$I^{*}_{j,k} = e_{j,k} + g_{j,k}\, G_{j,k-1} + f_{j,k}\, ir_{j,k-1} \qquad j = 1, \ldots, 5 \qquad (12)$$

$$I_{j,k} = I_{j,k-1} + \theta_{j,k}\,(I^{*}_{j,k} - I_{j,k-1}) \qquad j = 1, \ldots, 5 \qquad (13)$$

Substitute equation (12) into (13) and substitute equations (10), (11), and (13) into equation (9). Then, substitute equation (9) into equation (5), and (5) into equation (4). After rearranging and including the disturbance term variables $\omega_{1,j,k}$ and $\omega_{2,j,k}$, this yields equations (14) and (15).

$$C_{j,k} = \delta_{0,j} + \delta_{1,j}\, C_{j,k-1} + \delta_{2,j}\, I_{j,k-1} + \delta_{3,j}\, G_{j,k-1} + \delta_{4,j}\, ir_{k-1} \qquad (14)$$

$$+ \delta_{5,j}\, \hat{G}^{d}_{j,k-1} + \delta_{6,j}\, DEBT_{k-1} + \delta_{7,j}\omega_{1,j,k-1} \qquad j = 1, \ldots, 5$$

$$I_{j,k} = \lambda_{0,j} + \lambda_{1,j}\, I_{j,k-1} + \lambda_{2,j} G_{j,k-1} + \lambda_{3,j}\, ir_{j,k-1} + \lambda_{4,j}\omega_{2,j,k-1} \qquad j = 1, \ldots, 5 \qquad (15)$$

where

$$z_{j,k} = \frac{1}{1 - \gamma_{j,k} b_{j,k}} \qquad (16)$$

$$\delta_{0,j} = z_{j,k}[\gamma_{j,k}a_{j,k} + \gamma_{j,k}b_{j,k}(n_{0,j,k} + \theta_{j,k}e_{j,k} + \phi\,\pi_k DEBT_0)]$$

$$\delta_{1,j,k} = z_{j,k}(1-\gamma_{j,k}); \qquad \delta_{2,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}(1-\theta_{j,k})$$

$$\delta_{3,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}(\phi_{j,k} + \theta_{j,k}g_{j,k}); \qquad \delta_{4,j,k} = -z_{j,k}\gamma_{j,k}b_{j,k}\theta_{j,k}f_{j,k}$$

$$\delta_{5,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}(1-\phi_{j,k}); \qquad \delta_{6,j,k} = -z_{j,k}\gamma_{j,k}b_{j,k}\phi_{j,k}\pi_{j,k};$$

$$\lambda_{0,j,k} = \theta_{j,k}e_{j,k}; \ \lambda_1 = 1-\theta_{j,k}; \ \lambda_{2,j,k} = \theta_{j,k}g_{j,k}; \ \lambda_{3,j,k} = \theta_{j,k}f_{j,k}$$

Since both consumption and investment depend upon the interest rate, the monetary authorities exert policy influence by selecting the nominal interest rate as an operational target, as in Kendrick and Shoukry (2013). The trend process for government purchases, $\hat{G}^{d}_{j,k}$, at each frequency range is defined by inserting the actual current level of government purchases into the following state equation, where $\rho_{1,j}$ is the growth coefficient.

$$\hat{G}^{d}_{j,k} = \rho_{1,j}G_{j,k-1} + \rho_{2,j}\,\omega_{3,j,k-1} \qquad j = 1, \ldots, 5 \qquad (17)$$

The aggregate levels of consumption, investment, and government purchases in the model are recovered from the time-frequency decomposition as follows. Applying equation (1) shows that the modified smooth expresses the trend residual for the original data after all five of the crystals from each frequency range have been removed, so that

$$S_{C5,k} = C_k - d_{C,1,k} - d_{C,2,k} - d_{C,3,k} - d_{C,4,k} - d_{C,5,k} \qquad (18)$$

$$S_{I5,k} = I_k - d_{I,1,k} - d_{I,2,k} - d_{I,3,k} - d_{I,4,k} - d_{I,5,k} \qquad (19)$$

$$S_{G5,k} = G_k - d_{G,1,k} - d_{G,2,k} - d_{G,3,k} - d_{G,4,k} - d_{G,5,k} \qquad (20)$$

The model is closed by equations (21) through (29). Recall that the prevailing consumption $C_{j,k}$ at each frequency range (*j* = 1, …, 5) consists of both the crystal $d_{C,j,k}$ and the modified smooth, $S_{C5,k}$, where the contributions to consumption from the other frequency ranges have been removed. Equation (21) shows that aggregate consumption can be found by summing the consumption levels at each frequency range, and then subtracting the modified smooth out four times. The aggregate consumption level in equation (21) is an algebraically identical to the identity relationship given by equation (18), where actual aggregate consumption C_k during each period *k* is the sum of the five consumption crystals $d_{C,j,k}$ at each frequency plus the modified smooth, $S_{C5,k}$. This construction of the aggregate variables follows the same procedure for private investment (I_k) and government purchases (G_k), as shown in equation (21).

$$C_k = \sum_{j=1}^{5} C_{j,k} - 4S_{C5,k} \qquad I_k = \sum_{j=1}^{5} I_{j,k} - 4S_{I5,k} \qquad G_k = \sum_{j=1}^{5} G_{j,k} - 4S_{G5,k} \qquad (21)$$

The modified smooth trend processes for consumption, investment, government purchases, and the interest rate are modeled as first-order difference equations in (22) – (25).

$$S_{C5,k} = s_{C,1} S_{C5,k-1} + s_{C,2} C_{k-1} + s_{C,3} \omega_{4,k-1} \tag{22}$$

$$S_{I5,k} = s_{I,1} S_{I5,k-1} + s_{I,2} I_{k-1} + s_{I,3} \omega_{5,k-1} \tag{23}$$

$$S_{G5,k} = s_{G,1} S_{G5,k-1} + s_{G,2} G_{k-1} + s_{G,3} \omega_{6,k-1} \tag{24}$$

$$S_{ir5,k} = s_{ir,1} S_{ir5,k-1} + s_{ir,2} ir_{k-1} + s_{ir,3} \omega_{7,k-1} \tag{25}$$

In equations (22) – (25), the coefficients on the lagged modified smooth trend variable, and the coefficients on the lagged aggregate consumption, investment, government spending, and the interest rate produce a weighted average growth contribution toward the current trend values of each series.

$$Y_k = C_k + I_k + G_k + NX_k \tag{26}$$

$$NX_k = n_0 \qquad \text{for all } k = 1, \ldots, K \tag{27}$$

The national income identity is given by equation (26). For the domestic model, equation (27) assumes that the value of net exports NX_k is constant. Net taxes in quarter k are defined as the total government tax and income minus total government transfer payments. Net taxes are assumed to be generated as a constant percentage τ of national output, as shown in equation (28). Government tax income and transfer payments could each be modeled as separate fiscal policy variables. However, as in Kendrick and Shoukry (2013), this analysis will treat them as passively determined variables for simplicity. This is also consistent with findings of Kliem and Kriwoluzky (2014), who show that that there is little evidence in the U.S. for the typical simple fiscal policy rules derived in *DSGE* (Dynamic Stochastic General Equilibrium) models where tax rates respond to output. Thus, the only actively determined fiscal policy variables are government spending over each frequency range, $G_{j,k}$, $j = 1, \ldots, 5$.

$$T_k = \tau Y_k \tag{28}$$

$$DEF_k = G_k - T_k \tag{29}$$

$$DEBT_k = DEF_k + (1 + ig_k) DEBT_{k-1} \tag{30}$$

The resulting government budget deficit (or surplus, if it is negative) in quarter k is given by DEF_k in equation (29). Equation (30) defines the euro area debt, $DEBT_k$, as the sum of the current quarterly budget deficit and the previous period debt stock, which is growing at the quarterly interest rate on government debt, ig_k.

This model in equations (14) through (30) has several advantages: it can be specified with either constant coefficients as is the case in this paper, or with time varying coefficients. It is also derived from the macroeconomic accelerator framework which has been widely used, such as in Chow (1967), Kendrick (1981), Kendrick and Shoukry (2013), Hudgins and Na (2014), and Crowley and Hudgins (2015). It also includes a rational expectation component where the level of government debt affects the impact of fiscal policy. Equations (14) and (15) also do not explicitly require the exact background model specification of the constituent equations that led to their final reduced form, since other theoretical underpinnings could potentially also lead to these same two equations.

This model is not meant to be directly applied as a complete econometric forecasting model. Its purpose is to simulate the optimal tracking control policy in the time-frequency domain, and thereby show how the technique could be employed within a larger model, such as the 135-equation model in Taylor (1993). Since our model will require a state-space with 80 equations, partly due to the modeling of variables based on their frequency decomposition, the addition of more equations would result in a large-scale model. Since the deterministic, stochastic, and robust optimal feedback control can all be simulated within this *MODWT* wavelet-based accelerator framework, our model offers a unique representation.

3.1. Optimal Control

The *LQ* tracking problem can be stated as follows. The objective is for the fiscal and monetary policymakers to choose the level of government purchases at each of the five frequencies so that it will minimize the quadratic performance index given in (31) subject to the linear state equations given by (14) – (30).

$$
\begin{aligned}
\min_{G_{j,k}} J = {} & \frac{1}{2}\left[q_{1,f}(C_{K+1}-C^*_{K+1})^2+q_{2,f}(I_{K+1}-I^*_{K+1})^2\right] \\
& + \frac{1}{2}\left[q_{S,C5,f}(S_{C5,K+1}-S^*_{C5,K+1})^2+q_{S,I5,f}(S_{I5,K+1}-S^*_{I5,K+1})^2\right] \\
& + \frac{1}{2}\left[\sum_{j=1}^{5} q_{3,j,f}(C_{j,K+1}-C^*_{j,K+1})^2+\sum_{j=1}^{5} q_{4,j,f}(I_{j,K+1}-I^*_{j,K+1})^2\right] \\
& + \frac{1}{2}\sum_{k=1}^{K}\left[q_{1,k}(C_k-C^*_k)^2+q_{2,k}(I_k-I^*_k)^2\right] \\
& + \frac{1}{2}\sum_{k=1}^{K}\left[q_{S,C5,k}(S_{C5,k}-S^*_{C5,k})^2+q_{S,I5,k}(S_{I5,k}-S^*_{I5,k})^2\right] \\
& + \frac{1}{2}\sum_{k=1}^{K}\left[\sum_{j=1}^{5} q_{3,j,k}(C_{j,k}-C^*_{j,k})^2+\sum_{j=1}^{5} q_{4,j,k}(I_{j,k}-I^*_{j,k})^2\right. \\
& + q_{5,k}(DEF_k-DEF^*_k)^2+q_{6,k}(DEBT_k-DEBT^*_k)^2 \\
& + \sum_{j=1}^{5} q_{7,j,k}\left[(G_{j,k}-G_{j,k-1})-(G_{j,k-1}-G_{j,k-1})^*\right]^2 \\
& + q_{8,k}(G_k-G^*_k)^2+q_{S,G5,k}(S_{G5,k}-S^*_{G5,k})^2
\end{aligned}
\qquad (31)
$$

$$+ \; q_{9,\,k}\,(ir_k - ir_k^*)^2 + \; q_{S,\,ir5,\,k}\,(S_{ir5,\,k} - S_{ir5,\,k}^*)^2$$

$$+ \; \sum_{j=1}^{5} q_{10,\,j,\,k}\,(ir_{j,k} - ir_{j,\,k-1})^2 \; + \; \sum_{j=1}^{5} q_{11,\,k}\,(ir_k - ir_{k-1})^2$$

$$\left. + \; \sum_{j=1}^{5} r_{G,\,j,\,k}\,(G_{j,k} - G_{j,k}^*)^2 + \; \sum_{j=1}^{5} r_{ir,\,j,\,k}\,(ir_{j,k} - ir_k^*)^2 \right]$$

If the coefficients on the error terms in equations (14), (15), (17), (22), (23), and (24) are set equal to 1, then the model can be simulated as a stochastic *LQG* design, as in Chow (1975), Kendrick (1981), and Kendrick and Shoukry (2013), Crowley and Hudgins (2014), or as a robust design as in Basar and Bernhard (1991) and Hudgins and Na (2014), or as a mixed H^{∞} /stochastic *LQG* design (Hudgins and Na, 2014). If the disturbance coefficients are zero, then the model becomes deterministic.

This *LQ* tracking problem has five fiscal policy control variables ($G_{j,k}$, j = 1, …, 5), five monetary policy control variables ($ir_{j,k}$, j = 1, …, 5), and eighty state variables. The benefits and drawbacks of the symmetric quadratic performance index for economic and engineering applications need not be discussed here, since they are well known, and have been discussed in Kendrick (1981). The first two terms in (31) penalize the tracking errors for aggregate consumption and investment, respectively, in the final period at the end of the planning horizon. The third and fourth terms specify penalties for the final period tracking errors of the modified smooth trends for consumption and investment. The fifth and sixth terms in (31) penalize the final period tracking errors over each frequency range for consumption and investment. Policymakers will assign higher weights to frequency ranges where that time cycle interval is emphasized. For example, the weighting parameters $q_{3,2,\,f}$ and $q_{4,2,\,f}$ on the final respective tracking errors for consumption and investment at frequency 2 will be assigned large values if policymakers are primarily concerned with 1 to 2 year cycles.

The seventh and eighth terms assign penalties for the aggregate consumption and investment tracking errors in each period, and the ninth and tenth terms provide a penalty for the tracking errors for the modified smooth trends for consumption and investment in each period. The eleventh and twelfth set of terms provide penalties for the consumption and investment tracking errors over each individual frequency range for each period. The thirteenth and fourteenth terms penalize the tracking errors associated with the current period budget deficit, and the current period national debt, respectively.

The fifteenth term penalizes the policymaker for large changes in government spending between periods. Hudgins and Na (2014) and Crowley and Hudgins (2014) employ this term as a pragmatic consideration, because government policymakers will prefer more stable spending patterns in the ongoing budget appropriation process, and will not desire large fluctuations from the previous budget. This term reflects the fact that most new budgets are largely designed by adjusting the prevailing budget on a line-by-line basis, which also supports a constant frequency range structure where the resistance to change across each frequency range can be decoupled and penalized separately. Hudgins and Na (2014) demonstrate the substantial additional cost savings in control effort that result when including this term under robust modeling designs, versus the case where government is not penalized for changing its spending above or below the stated quarterly target growth, which our analysis specifies as a rate of $g_{G,j,k}$ = 0.5% (which is about 2% annualized growth) between periods for each of the 5 frequency ranges, $j = 1, \ldots, 5$.

The sixteenth term assigns a penalty for deviations of aggregate government purchases from its target value, and the seventeenth term assigns a penalty for tracking error of the modified smooth trend level of government purchases. The eighteenth term penalizes for deviations of the short-term market interest rate from its target value, and the nineteenth term penalizes the tracking error for the smooth trend in the short-term market interest rate. The twentieth and twenty-first terms penalize the monetary authorities for changes in the interest rate between periods, since larger changes reflect an erratic, unstable monetary policy. The twenty-second term represents the fiscal policy control variables, and it provides a penalty for the government purchases tracking error at each of the 5 frequencies. The last term represents the monetary policy control variables, and penalizes for the market interest rate tracking error at each frequency.

We transform the *LQ*-tracking problem into a *LQ*-regulator problem using the procedure in Crowley and Hudgins (2014), thus creating a state-space with 80 state variables, 80 state equations, and 10 control variables. Although this transformation creates a higher dimensional state-space, it greatly simplifies the subsequent solution procedures for deterministic, stochastic, and H^{∞}- optimal control problems. This conversion method is similar to that used in Hudgins and Na (2014). Although this is a large scale system, the state-space construction procedures and the accompanying MATLAB program that we have developed have proven to be efficient and feasible to employ. This wavelet-based system framework can easily be adopted within the context of larger base models, such as the model developed by Taylor (1993), although the inclusion of the different frequency ranges would substantially increase the size of the larger econometric models.

The present transformation uses the Hudgins and Na (2014) and Crowley and Hudgins (2014) approach of embedding the constant terms within the state equations by defining a variable that is a sequences of recurring ones, for all $k = 1, \ldots, K$.

$$c_{k+1} = 1 c_k \quad ; \qquad c_1 = 1 \tag{32}$$

Based on equation (32) and its initial value of 1, $\boldsymbol{c_k} = 1$ for all $k = 1, \ldots, K$. This new state variable has the value of 1 in each period, and thus serves as a placeholder in each state equation, where the coefficient of this $\boldsymbol{c_k}$ variable in each individual equation is the constant. So, the constant terms in equations (14) and (15) become $\delta_{0,j} c_k$ and $\lambda_{0,j} c_k$, respectively, in the state space equations.

The model allows for optimal aggregate consumption, investment, government purchases, the interest rate, and the optimal modified smooth values of each of these to grow at quarterly target rates of $\boldsymbol{g_{C,k}}$, $\boldsymbol{g_{I,k}}$, $\boldsymbol{g_{G,k}}$, $g_{ir,k}$, $g_{S,C5,k}$, $g_{S,I5,k}$, $g_{S,G,k}$, $g_{S,ir,k}$, respectively, that are specified by the fiscal policymaker, which results in an annual growth rate of [(1 + $g_{(.),k}$)^4 – 1] per year. The quarterly consumption, investment, and government purchases tracking equations for each frequency range, and the aggregate consumption, investment, and government purchases, along with the modified smooth for each of these can thus be written, respectively, as

$$C^*_{j,k+1} = (1+g_{C,j,k})C^*_{j,k}; \quad I^*_{j,k+1} = (1+g_{I,j,k})I^*_{j,k}; \quad G^*_{j,k+1} = (1+g_{G,j,k})G^*_{j,k} \tag{33}$$
$$C^*_{k+1} = (1+g_{C,k})C^*_k; \quad I^*_{k+1} = (1+g_{I,k})I^*_k; \quad G^*_{k+1} = (1+g_{G,k})G^*_k$$
$$ir^*_{k+1} = (1+g_{ir,k})\,ir^*_k$$
$$S^*_{C5,k+1} = (1+g_{S,C5,k})S^*_{C5,k}; \quad S^*_{I5,k+1} = (1+g_{S,I5,k})S^*_{I5,k};$$
$$S^*_{G5,k+1} = (1+g_{S,G5,k})S^*_{G5,k}; \quad S^*_{ir5,k+1} = (1+g_{S,ir5,k})S^*_{ir5,k}$$

The 80-dimensional state vector is defined as follows:

$$x_k = \left[x_{1,k}\,;\, x_{2,k}\,;\, \ldots\,;\, x_{80,k}\right]^T \tag{34}$$

where

$$x_k = [\,C_{1,k}; C_{2,k}; C_{3,k}; C_{4,k}; C_{5,k}; S_{C5,k} \,|\, I_{1,k}; I_{2,k}; I_{3,k}; I_{4,k}; I_{5,k}; S_{I5,k} \,|\, c_k \,|$$
$$C^*_{1,k}; C^*_{2,k}; C^*_{3,k}; C^*_{4,k}; C^*_{5,k} \,|\, I^*_{1,k}; I^*_{2,k}; I^*_{3,k}; I^*_{4,k}; I^*_{5,k} \,|\, G^*_{1,k}; G^*_{2,k}; G^*_{3,k}; G^*_{4,k}; G^*_{5,k} \,|$$
$$\hat{G}^d_{1,k}; \hat{G}^d_{2,k}; \hat{G}^d_{3,k}; \hat{G}^d_{4,k}; \hat{G}^d_{5,k}; S_{G5,k} \,|\, C_k; I_k; G_k \,|\, C^*_k; I^*_k; G^*_k \,|\, NX_k \,|\, Y_k \,|$$
$$T_k; DEF_k; DEBT_k \,|\, G_{1,k-1}; G_{2,k-1}; G_{3,k-1}; G_{4,k-1}; G_{5,k-1} \,|$$
$$G_{1,k-1}-G_{1,k-2}; G_{2,k-1}-G_{2,k-2}; G_{3,k-1}-G_{3,k-2}; G_{4,k-1}-G_{4,k-2}; G_{5,k-1}-G_{5,k-2} \,|$$
$$(G_{1,k-1}-G_{1,k-2})^*; (G_{2,k-1}-G_{2,k-2})^*; (G_{3,k-1}-G_{3,k-2})^*; (G_{4,k-1}-G_{4,k-2})^*;$$
$$(G_{5,k-1}-G_{5,k-2})^* \,|\, DEF^*_k; DEBT^*_k \,|\, S^*_{C5,k}; S^*_{I5,k}; S^*_{G5,k} \,|$$
$$S^*_{ir5,k} \,|\, S_{ir5,k} \,|\, ir^*_k \,|\, ir_{k-1} \,|\, ir_{1,k-1}; ir_{2,k-1}; ir_{3,k-1}; ir_{4,k-1}; ir_{5,k-1} \,|$$
$$ir_{1,k-1}-ir_{1,k-2}; ir_{2,k-1}-ir_{2,k-2}; ir_{3,k-1}-ir_{3,k-2}; ir_{4,k-1}-ir_{4,k-2}; ir_{5,k-1}-ir_{5,k-2} \,|$$
$$ir_{k-2}]^T$$

Define the control vector so that the first five elements are difference between the actual and targeted level of government purchases and the last five elements are the tracking errors for the short-term market interest rate at each frequency:

$$u_k = \left[u_{G1,k}; u_{G2,k}; u_{G3,k}; u_{G4,k}; u_{G5,k} \,|\, u_{ir1,k}; u_{ir2,k}; u_{ir3,k}; u_{ir4,k}; u_{ir5,k}\right]^T \tag{35}$$

$u_{G\,j,k} = G_{j,k} - G^{*}_{j,k} \qquad u_{ir\,j,k} = ir_{j,k} - ir^{*}_{k}$

The disturbance vector for stochastic and robust design cases is defined by (36), where the vector is 0 for the deterministic case.

$$\omega_k = \Big[\omega_{1,1,k}\,;\,\omega_{1,2,k}\,;\,\omega_{1,3,k}\,;\,\omega_{1,4,k}\,;\,\omega_{1,5,k}\,|\,\omega_{2,1,k}\,;\,\omega_{2,2,k}\,;\,\omega_{2,3,k}\,; \qquad (36)$$
$$\omega_{2,4,k}\,;\,\omega_{2,5,k}\,|\,\omega_{3,1,k}\,;\,\omega_{3,2,k}\,;\,\omega_{3,3,k}\,;\,\omega_{3,4,k}\,;\,\omega_{3,5,k}\,;\,\omega_{4,k}\,;\,\omega_{5,k}\,;\,\omega_{6,k}\,;\,\omega_{7,k}\Big]^{T}$$

Since each of the first five control variables in the vector u_k include the negative of the targeted levels of government purchases, and each of the last 5 variables include the negative of the targeted interest rate at each frequency, these target variables are added to the 5 state equations for the individual frequencies of consumption and the 5 state equations for the individual frequencies for investment. The net effect of adding and subtracting the same variable is 0, but this allows the problem to be written in standard *LQ*-regulator format. Once the optimal control has been simulated to produce the values for $u_{G\,j,k}$ and $u_{ir\,j,k}$ over each frequency range, the target level of government purchases and the interest rate, $G^{*}_{j,k}$ and $ir^{*}_{j,k}$, will have to be added to $u_{G\,j,k}$ and $u_{ir\,j,k}$, respectively, in order to recover the values for government purchases, $G_{j,k}$, and the interest rate, $ir_{j,k}$, over each frequency range. However, these values are also automatically recovered with one lag in state equations 46 – 50 and 70 – 74, respectively, by adding the target values to the state values of government purchases and the interest.

The matrix state-space equation system can be written as follows, where the 80 state equations are given in Appendix 2.

$$x_{k+1} = A_k x_k + B_k u_k + D_k \omega_k \qquad (37)$$

dim x = (80, 1) *dim u* = (10, 1) *dim* ω = (19, 1)
dim A = (80, 80) *dim B* = (80, 10) *dim D* = (80, 19)

3.2. Transformed Deterministic Regulator Design

Consider the deterministic *LQ*-regulator problem where the disturbance vector is zero, or ω_k = 0, or alternatively, where the disturbance coefficient vector is D_k = 0. After rewriting expression (31) based on the state space system in (37), the objective is to minimize the performance index

$$\min_{u} J(u) = x_{K+1}^T Q_f \, x_{K+1} + \sum_{k=1}^{K} \left[x_k^T Q_k x_k + u_k^T R_k u_k \right] \tag{38}$$

subject to

$$x_{k+1} = A_k x_k + B_k u_k \quad ; \qquad x(1) = x_1 \tag{39}$$

where the size of the penalty weighting matrices are

dim Q_f = (80, 80) *dim* Q_k = (80, 80) *dim* R_k = (10, 10)

The solution to the *LQ* regulator problem is found by first computing the recursive equations (40) and (41) offline in retrograde time.

$$F_k = \left(B_k^T P_{k+1} B_k + R_k \right)^{-1} B_k^T P_{k+1} A_k \tag{40}$$

$$P_k = Q_k + A_k^T P_{k+1} \left(A_k - B_k F_k \right); \qquad P_{k+1} = Q_f \tag{41}$$

These recursive equations are much simpler to compute than the longer recursive equations employed by Chow (1975), Kendrick (1981), Amman (1996), and others that arise when solving the *LQ*-tracking problem. Using the values computed in (40) and (41), the unique optimal feedback control policy is computed in forward time by

$$u_k^* = -F_k x_k \tag{42}$$

The optimal closed-loop state trajectory is given by

$$x_{k+1} = \left(A_k - B_k F_k \right) x_k \quad ; \qquad x(1) = x_1 \tag{43}$$

The control equations in (40) and (41) are the same for the stochastic *LQG* form of the model with perfect state information. The state variable trajectory, however, would be calculated by equation (37), rather than (43). The control vector in equation (42) would then be computed by using equations (40) and (37).

4 Estimation and Simulation

The crystals and smoothed trend values from the *MODWT* decomposition were used to run *OLS* regressions in order to obtain the estimated coefficients for consumption, investment, and government purchases in equations (14), (15), and (17), respectively. The regression results with *t-statistics* are given in Appendix 1 tables A1 – A4. The equations obtained a good fit, and almost all of the coefficients have the expected sign.

In the investment equations, the coefficients on the lagged government purchases and the interest rate are all negative, thus demonstrating fiscal crowding-out and marginal efficiency of investment curves with an inverse relationship with the market interest rate. In the consumption equations, coefficients on the interest rate are negative at each frequency, demonstrating and inverse relationship between consumption and the interest rate. The coefficients on government spending are positive at frequency ranges 2 and 3, and negative for the other frequencies. This suggests a government crowding-in effect for 1 to 4 year cycles, and a crowding-out effect for the other frequency ranges.

The simulations define the initial values for the state variables in period 1 to correspond to the Euro area quarterly data in 2014, quarter 3, measured in billions of euros. Net exports are set at a constant value of n_0 = 108.57. The stock of government debt is set at $DEBT_0$ = 8,192.9902, which is 92.1% of the initial real GDP value of Y_0= 2,220.4352. Since the Euro area member states had an average budget deficit of 3% of output, the initial budget deficit is set at DEF_0 = 66.61306. Given the initial government purchases value of G_0= 466.745, the government purchases minus the deficit yields an initial value for net taxes of T_0 = 400.13204. This amount is 18% of total output, so the net tax rate is set at τ_0 = 0.18. The quarterly interest rate on the debt is set at ig_0 = .005, which is 2% per year. In equation (10), the weight for the current level of government purchases in the expectation formation equation is set at $\phi_{j,k}$ = 0.90 and the parameter weight for the adjustment for the national debt differential in the expectation is set at $\pi_{j,k}$ = 0.0005 for all frequency ranges in all periods.

The simulations fix the state variables at their initial values at period k = 1. Recall from equations (36) – (42), that the government chooses the optimal level of government spending at each frequency range, and the central bank chooses the optimal quarterly interest rate, j = 1, …, 5, starting in period k = 1. So, the level of government spending in period k = 1 and the interest rate in period k = 1 have their first effects on the state variables in period k = 2. In the 4-year simulations, optimal government spending and the optimal interest rate in quarter K = 16 determines the level of consumption, investment, and the other state variables in period K + 1 = 17. The simulations assume a desired fiscal austerity position, and thus set the initial target for each frequency of government purchases in period 1 to be 1% below the level of actual aggregate government purchases. The quarterly target growth rate for government purchases and its modified smooth are set at $g_{G,j,k}$ = $g_{S,G5,k}$ = 0.005, which is about 2% per year.

The consumption and investment values at each frequency and the aggregate levels are initially set at their 2014 quarter 3 levels. The initial targets for consumption and investment are set to be 1% above the aggregate consumption and investment levels, respectively. These same initial targets are also respectively assigned to the decomposed series for consumption and investment at each frequency, j = 1, …, 5. The quarterly target rates for aggregate and decomposed consumption and investment at all frequency ranges j = 1, …, 5, are all fixed, respectively at $g_{C,k}$ = $g_{I,k}$ = $g_{C,j,k}$ = $g_{I,j,k}$ = $g_{S,C5,k}$ = $g_{S,I5,k}$ = 0.0075, which is about 3% per year. These growth rates for consumption and investment are used for U.S. data in Kendrick (1981) and Hudgins and Na (2014), and Crowley and Hudgins (2014), whereas Kendrick and Shoukry (2013) use a higher target quarterly growth rate for national output of 0.015, as it progresses from a deep recession in 2008 through a recovery phase.

The target value of the quarterly interest rate is set at the (annualized) rate of 3% for all periods, so that the optimal growth rate for the interest rate is $g_{ir,\,k} = 0$. This 3% rate is consistent with a monetary policy that would accommodate the consumption and investment growth targets while generating little or no inflation.

The scenario presents a tradeoff for the policymakers. The targeted levels and growth rates for consumption and investment are above their current values due to the lack of full recovery of the growth path from the previous recession. Although the government desires an expansionary policy to achieve its consumption and investment targets, the desired government target spending level is lower than current spending, and the target growth rate for government spending is lower than that for consumption and investment (perhaps because of the constraints imposed by the "fiscal compact"). The monetary authorities desire to return to the interest rate that is consistent with a long-term annual economy growth rate of 3%, but are currently maintaining extremely low interest rates in order to encourage stimulation to curb sub-par economic performance. Additionally, the government is penalized for large changes in spending between periods, and the monetary authorities are penalized for large changes in the interest rate between periods. The simulated trajectories reflect the optimal balance over each period of the planning horizon.

Crowley and Hudgins (2015) use a wavelet-based accelerator model to simulate optimal fiscal policy under 3 cases of *FHEC* (Frequency Harmonizing Emphasis Control): long –term business cycle targeting, political cycle targeting, and short-term stabilization targeting. All of the simulations in this analysis assume a political cycle approach, where policymakers primarily emphasize the frequency ranges $j = 3$ and 4 (2 to 4 years, and 4 to 8 years), which covers roughly the average length in office of a typical Euro area member state government. The shortest and longest frequency ranges are assumed to be a lesser priority, and each of these carry equal emphasis. Although the MATLAB program that we have developed allows the user to input any number of periods for the planning horizon, all simulations set the number at $K = 16$ quarters, so that the horizon runs from the initial period $k = 1$ to the final period $K + 1 = 17$.

We simulate the model for three cases of policy approaches: (1) dual emphasis on active use of fiscal and monetary policy, (2) emphasis on active use of fiscal policy with relatively passive monetary policy, and (3) active use monetary policy with relatively passive fiscal policy. These are achieved by varying the weights in the performance index in equation (31). The full set of parameter values for each case is shown in appendix table A5. Only the relative values of the parameter weights affect the optimal control policies, whereas the absolute numbers of the parameters are irrelevant.

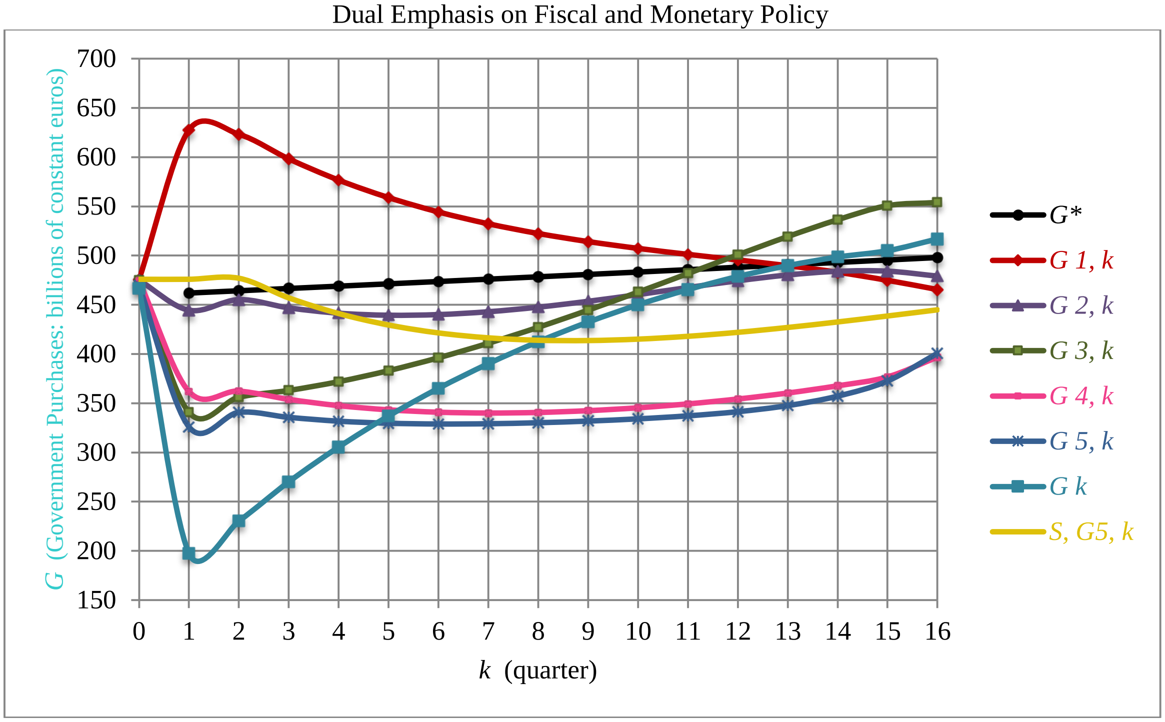
4.1 Dual Emphasis on Fiscal and Monetary Policy

Case (1) assumes that both fiscal and monetary policymakers take an active stance to track consumption and investment. Figures 6 – 9 show trajectories for government purchases, the short-term interest rate, consumption, and investment.

Figure 6

Government Purchases Optimal Forecast Trajectories

Dual Emphasis on Fiscal and Monetary Policy



In figure 6, government purchases at the lower frequency ranges 3 – 5 (from 2 to 16 years) immediately fall below the modified smooth in the first policy period (quarter 1). Whereas spending at frequency ranges 4 and 5 remain on a lower plateau throughout the horizon, however, the spending at frequency 3 (1 to 2 years) rises above the trend after period 7, and ends the horizon above the target. This is largely due to fact that government spending has a crowding-in effect on consumption at frequencies 2 and 3, and a crowding-out effect at all other frequency ranges. The spending in frequency range 1 takes the opposite trajectory, where a large initial level consistently falls thereafter. Spending at frequency range 2 closely tracks the target over all periods. The modified smooth falls below the trend after period 2, and remains below thereafter, although it begins to increase after period 9.

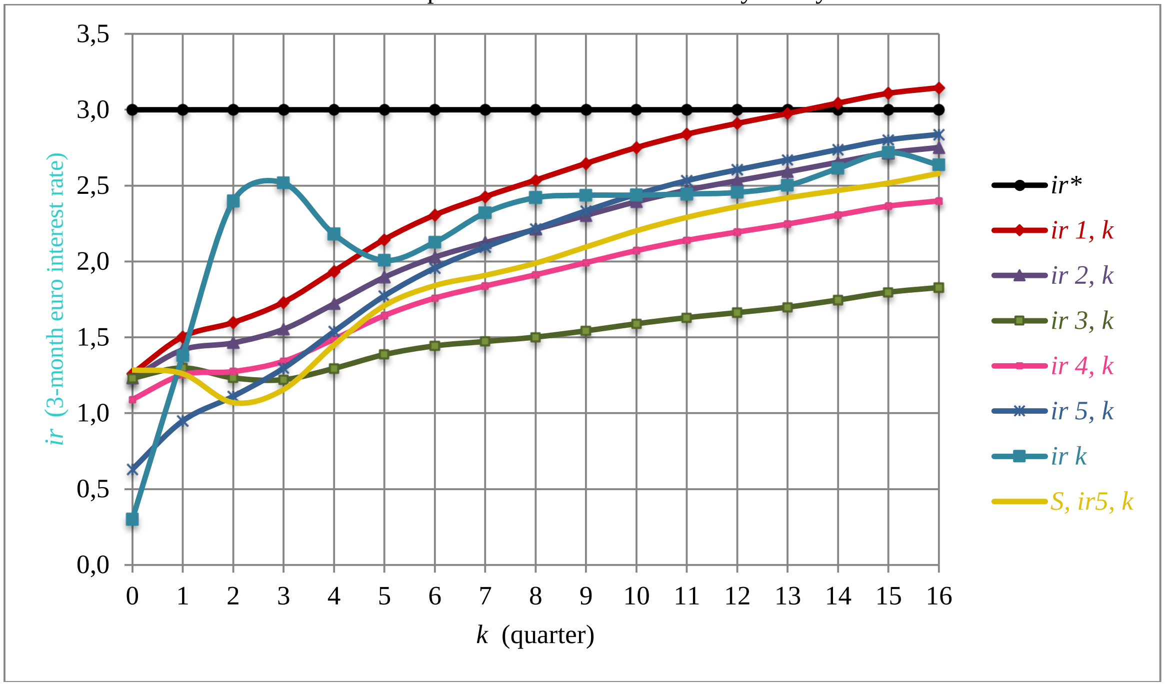
Since spending at all frequency ranges, other than frequency 1, is well below the smooth in the initial policy period (quarter 1), the optimal level aggregate government spending encounters a sizable initial drop. However, it steadily increases over the remainder of the horizon, until it finishes slightly above the target in quarter 17. Since government spending has a crowding-out effect on investment at all frequency ranges, the aggregate spending level remains mostly below the target, driven by the heavily weighted spending at the longer cycle frequency ranges 4 and 5. Figure 6 suggests that the fiscal authorities should concentrate a large percentage of spending on cycles with impacts less than 4 years, and dampening stabilization efforts for the longer cycles. Note however, that this conclusion depends upon the assumption of emphasizing the political targeting objective, which is a likely operative scenario for many governments.

Figure 7 shows the optimal short-term market interest rate forecast trajectories. The interest rate trajectories initially increase at all frequency ranges, calling for a slightly contractionary monetary stance in quarter 1. Since this causes positive crystals where the interest rate is above its modified smooth trend at each frequency, the aggregate interest rate increases substantially by 1% in quarter 1 and by another 1% in quarter 2. The interest rate momentum then levels off at all frequencies in the next period, before increasing at slightly larger rate in the middle of the horizon. The slow growth rate at each frequency range then begins to decline until the last quarter.

Figure 7

Short-term Market Interest Rate Optimal Forecast Trajectories

Dual Emphasis on Fiscal and Monetary Policy



The trajectory at frequency range 3 (2 to 4 years) decreases in quarters 2 and 3, and then grows very slowly. Since the policymakers have a strong weight on consumption and investment at frequencies 3 and 4, there is pressure to adopt an expansionary policy stance, and optimally keep the interest rate depressed at those frequency ranges; hence, these interest rate trajectories are consistently the lowest.

Interestingly, the aggregate interest rate follows undulating pattern where the rippling effect in its trajectory is more pronounced than for any of its component wavelets. The optimal policy increases the interest rate for the first three periods until it reaches 2.5%, then evenly decreases it over the next two quarters until it reaches 2%. Then, there is a slow and uneven contractionary growth throughout the remainder of the horizon, where the interest rate reaches a high of 2.7% in quarter 16, and falls to 2.6% in the last quarter. This is a crucial finding, since it suggests that monetary authorities should optimally adopt a policy of setting its operating interest rate targets so that it influences the short-term market interest rate to achieve an undulating pattern of alternating increases and decreases, rather than a slow, steady increase, or slow sequence of small discrete jumps, across the horizon. Moreover, the only way to ascertain the timing of the alternating interest rate levels is through the frequency range decomposition. This result is likely due to the growth cycles found at higher frequencies in the major components of GDP.

Whereas fiscal policy can actually allocate funds to projects spanning a given frequency range based on the optimal control trajectories for each frequency range, the monetary authorities, such as the *ECB* (European Central Bank) that adopt a *SID* (Short-term Interest Rate Doctrine) focus, rather than a *RPD* (Reserve Position Doctrine) focus, can only choose the aggregate interest rates on the marginal lending facility and the deposit facility as its operating targets, and choose aggregate market interest rates as intermediate targets (Bindseil, 2004). However, this finding suggests that using the wavelet decomposed model to determine optimal interest levels at each frequency range is equally important for defining optimal monetary policy. The wavelet decomposition provides the necessary information for the monetary authorities to optimally influence the aggregate market interest rate by using its standing facilities to adjust the operating targets so that the market interest rate moves according the optimally determined underlying frequency range structure, as computed in the control model.

Figures 8 and 9, respectively, show the consumption and investment trajectories. Consumption is consistently above the target and the modified smooth trend at the lower frequency ranges 4 and 5, and it is below the target and the trend at the other frequencies. The higher consumption at the longer cycles partly results from the lower government spending at those frequency ranges, resulting in less crowding-out. The modified smooth trend consistently tracks the target closely, whereas aggregate consumption is always above the target.

The investment trajectories in figure 9 show a wide range of behavior. Investment at frequency 1 is crowded-out by both the higher government spending and higher interest rate levels at that frequency range. Investment at frequency ranges 2 and 3, as well as aggregate investment, increase over the middle of the planning horizon, and then begin to decrease. Investment at frequency range 3 ends at the target, due to its high relative weighting. At longer cycles of frequency ranges 4 and 5, investment steadily increases over the entire horizon, and is above the target from period 5 onward. The final level of aggregate investment is below the target and the smooth, partly as the result of the crowding-out due to the higher final levels of the interest rate and government spending.

Figure 8

Consumption Optimal Forecast Trajectories

Dual Emphasis on Fiscal and Monetary Policy

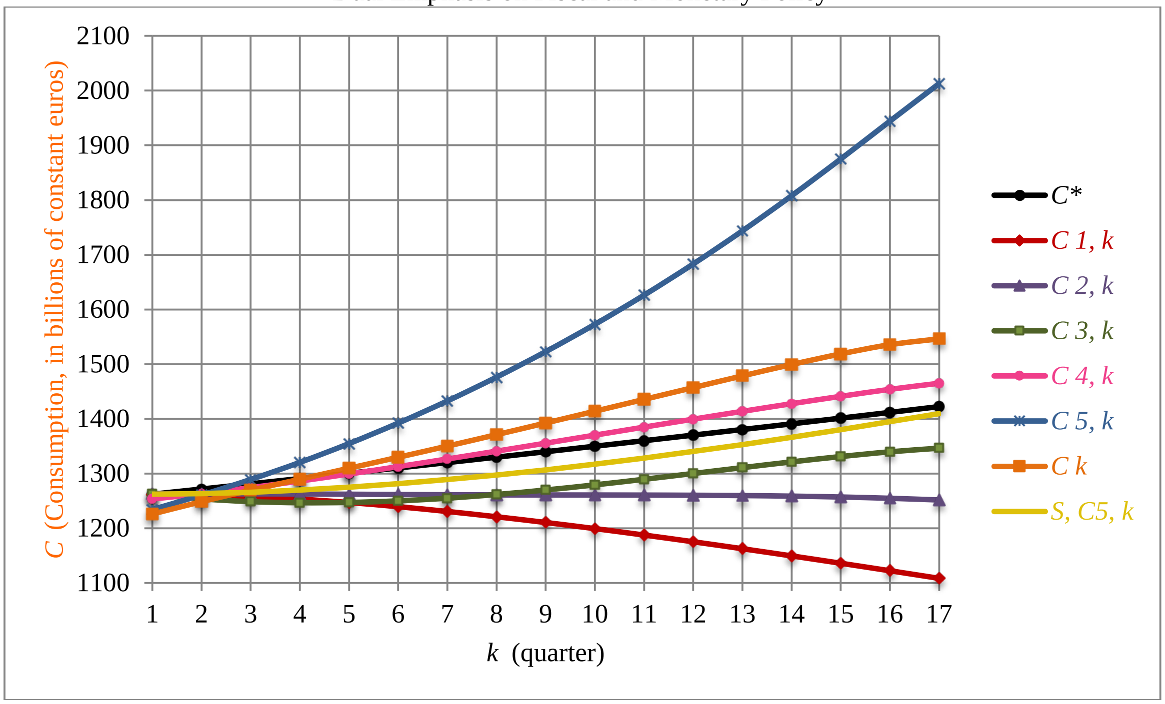
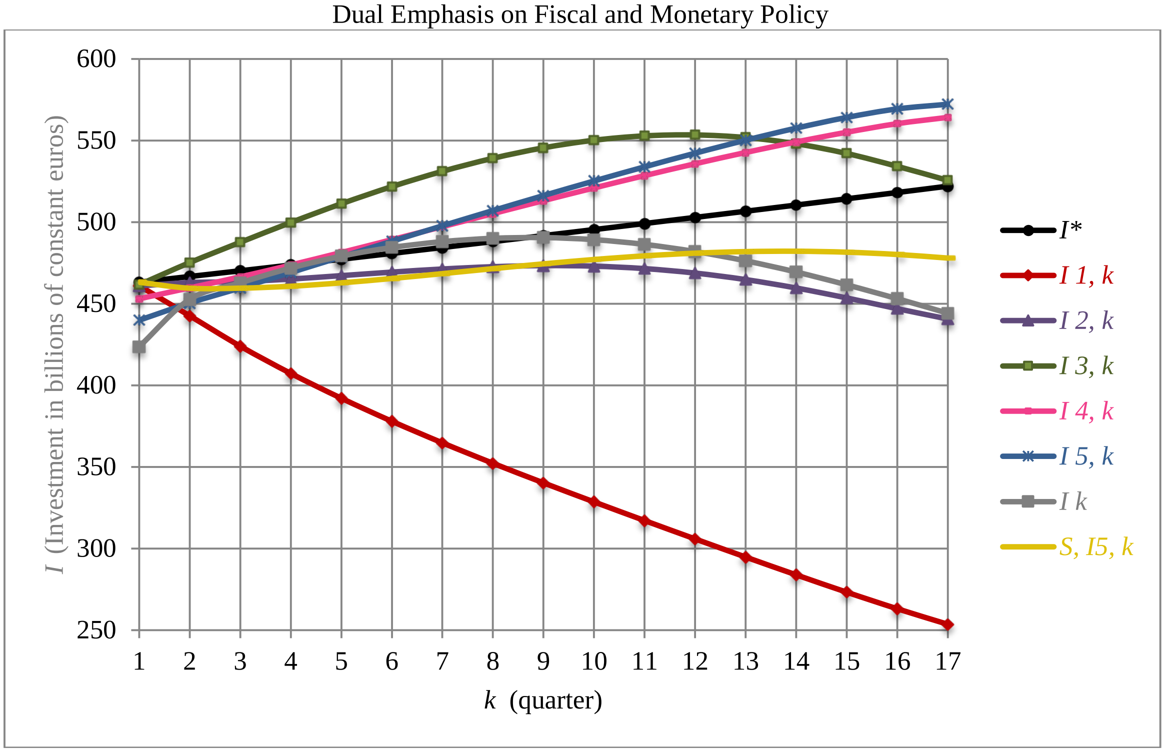


Figure 9

Investment Optimal Forecast Trajectories

Dual Emphasis on Fiscal and Monetary Policy



4.2 Fiscal Policy Emphasis with Relatively Passive Monetary Policy

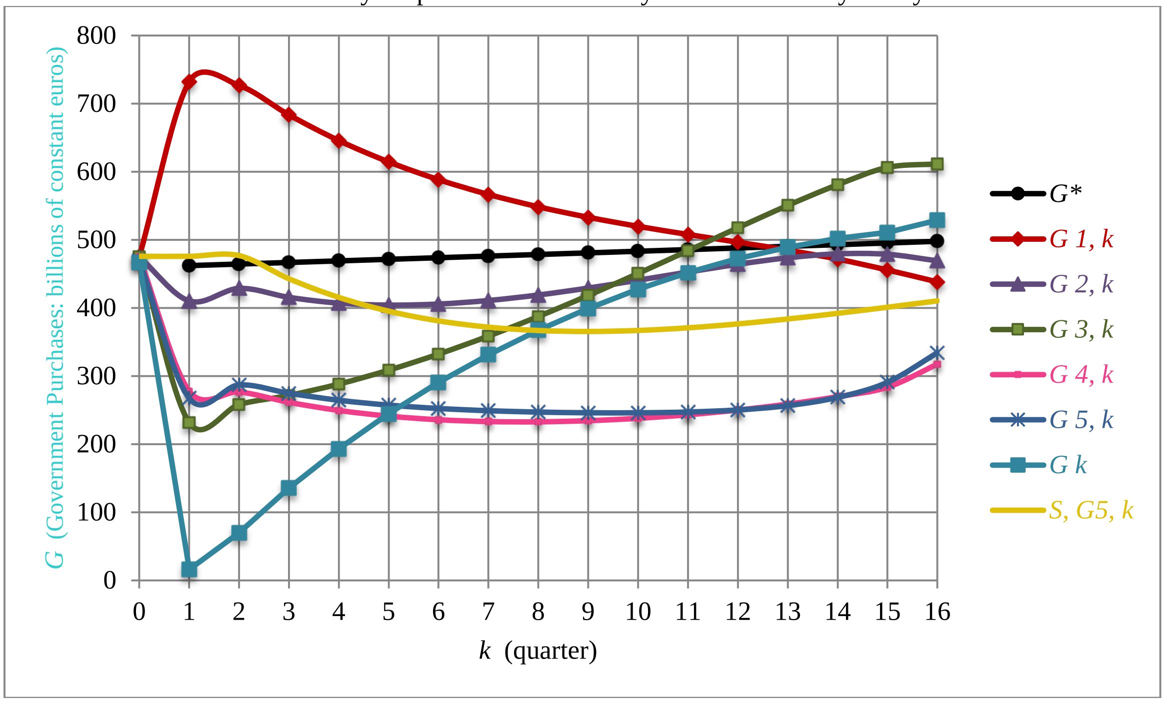
Case (2) simulates the model when the monetary authorities choose to keep the market interest rate very close its target rate of 3%. This results in a passive monetary policy that defers any active optimal policy adjustments to the fiscal policymakers. This case is more relevant for countries such as the United States, where fiscal policy is characterized by a high degree of flexibility in taxing and spending. Case (2) is the least relevant case for the Euro area, since the "Fiscal Compact" restricts the ability to use large fiscal stabilization measures.

Figures 10 – 13 show the results for this case of fiscal policy emphasis with passive monetary policy. The government purchases trajectories in figure 10 are in the locations relative to each other as they were in Figure 6. However, the trajectories in figure 10 are considerably more volatile. As a result, aggregate government purchases follow a much larger decline in periods 1 through 5 in case (2), given by figure 10. This occurs because the passively contractionary monetary policy is putting downward pressure on investment. This is counteracted by initially tight fiscal policy, which crowds-in investment.

Figure 10

Government Purchases Optimal Forecast Trajectories

Fiscal Policy Emphasis with Relatively Passive Monetary Policy

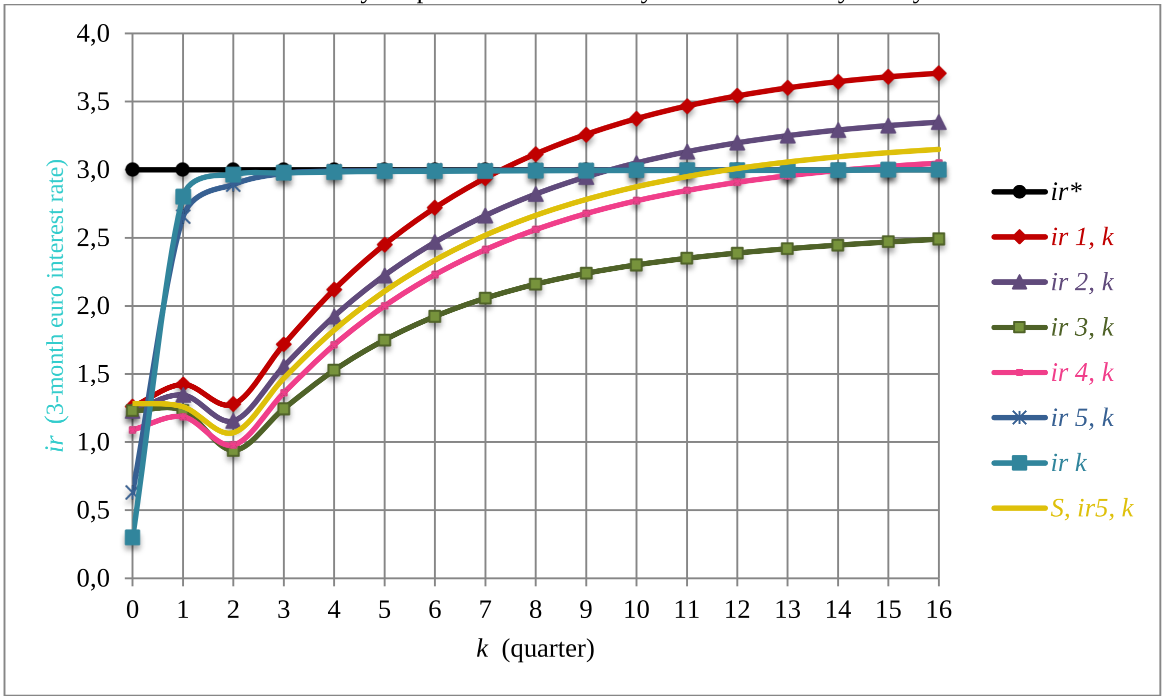


The short-term market interest rate trajectories at frequency ranges 1 through 4, and the modified smooth, experience a small initial increase in period 1, then a small decrease in period 2, and then follow upward growth paths that level off at the end of the horizon. The interest rate at frequencies 3 and 4, where consumption and investment tracking errors are more heavily weighted, are consistently below the trend and the target, while the interest rate at frequency ranges 1 and 2 are consistently above the trend, and also growth above the target in the middle of the horizon. However, the interest rate at frequency 5 immediately jumps to 2.7%, followed by 2.9%, and remains at 3% thereafter. Thus, the aggregate interest rate jumps to 2.8% in period 1, then to 2.96% in period 2, and then closely tracks the 3% target thereafter.

Figure 11

Short-term Market Interest Rate Optimal Forecast Trajectories

Fiscal Policy Emphasis with Relatively Passive Monetary Policy



This restrictive interest rate policy would cause crowding-out in both consumption and investment, but it is counteracted by the more active fiscal policy. The consumption trajectories at frequency ranges 3 – 5, as well as the aggregate consumption and the trend grow faster and end up at higher levels in figure 12 than in figure 8, while consumption at frequencies 1 and 2 is lower in figure 12. This partly occurs because the much lower level of government spending in the early periods has crowded in investment, and has crowded in consumption at the lowest two frequencies.

The investment trajectories in figure 13 follow similar patterns to those in figure 9, where except for frequency 1, investment at all trajectories increases above the target throughout the middle of the horizon, and then falls. However, aggregate investment tracks the target more closely in figure 13 than in figure 8. When the interest rate is restricted to tracking its 3% target closely, it is expected that investment would also have a small tracking error. However, this is achieved at the very high cost of much more volatile fiscal policy and larger ranges of variation in consumption.

Figure 12

Consumption Optimal Forecast Trajectories

Fiscal Policy Emphasis with Relatively Passive Monetary Policy

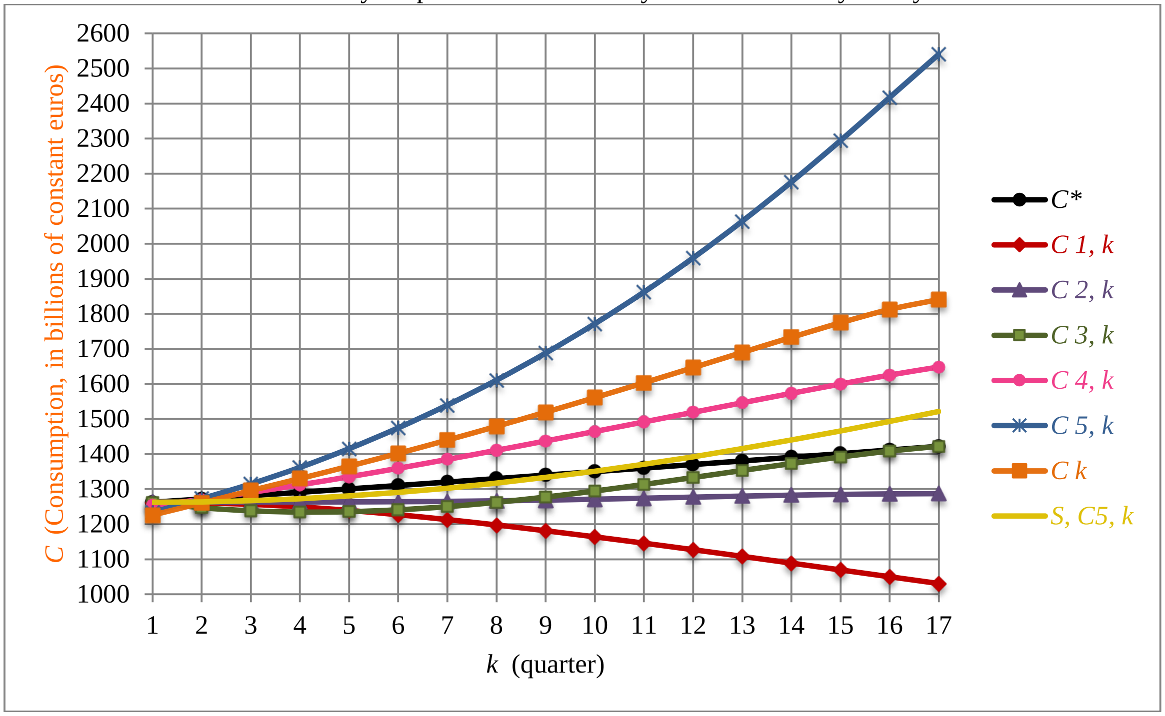
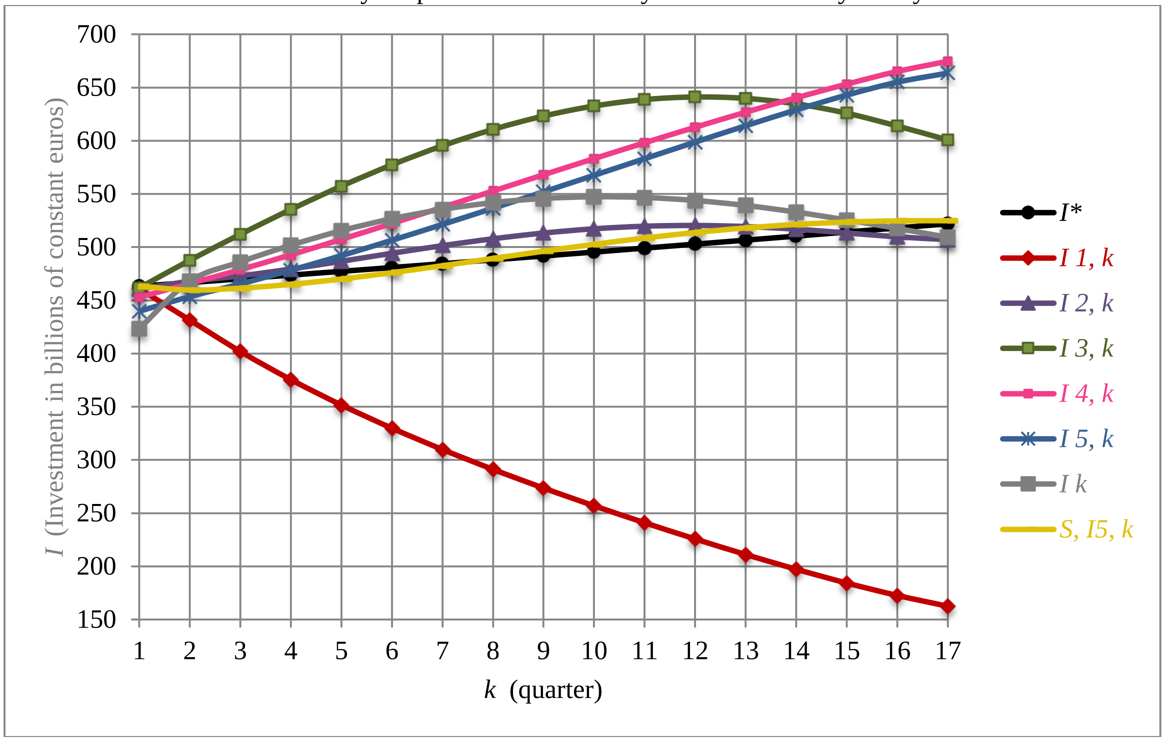


Figure 13

Investment Optimal Forecast Trajectories

Fiscal Policy Emphasis with Relatively Passive Monetary Policy



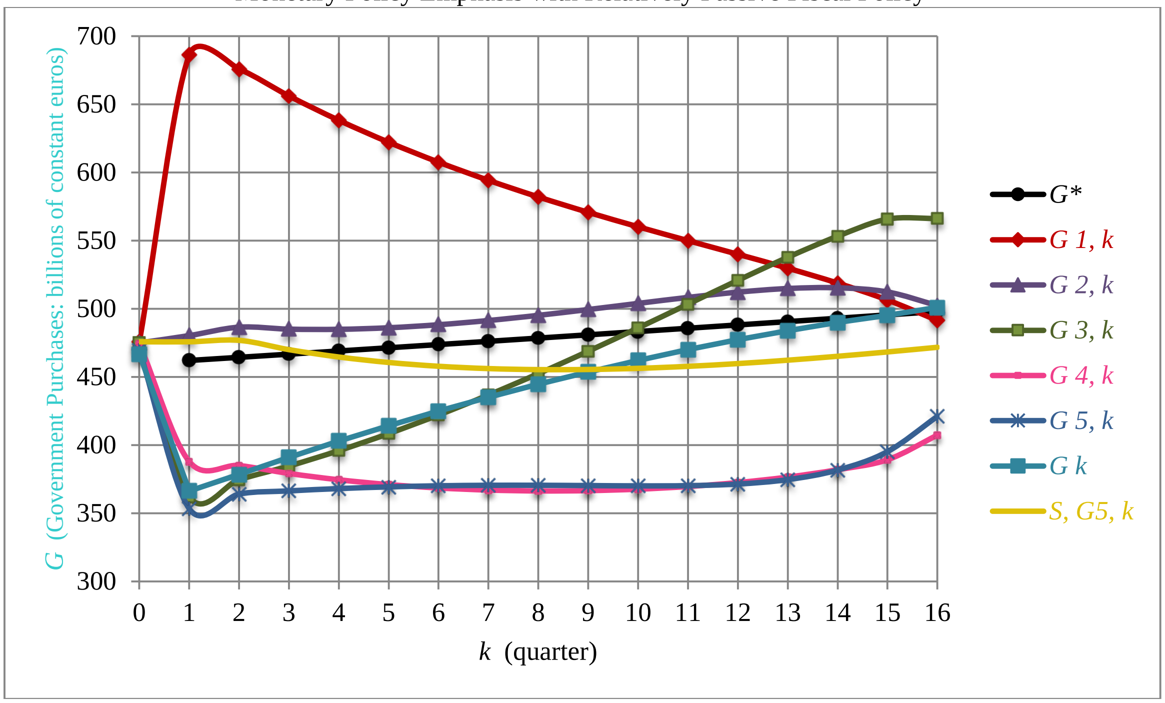
4.3 Monetary Policy Emphasis with Relatively Passive Fiscal Policy

Figures 14 – 17 considers case (3), the scenario when fiscal policy is restricted so that aggregate government expenditure more closely tracks its target. This is probably most relevant to the Euro area, given the constraints imposed by the "Fiscal Compact" and the recent moves by the ECB to institute a QE (Quantitative Easing) program. Figure 14 shows that the relative path of government spending trajectories follows a similar pattern to the previous cases. However, aggregate expenditure only encounters a small initial decline, and then tracks its target closely thereafter.

Figure 14

Government Purchases Optimal Forecast Trajectories

Monetary Policy Emphasis with Relatively Passive Fiscal Policy



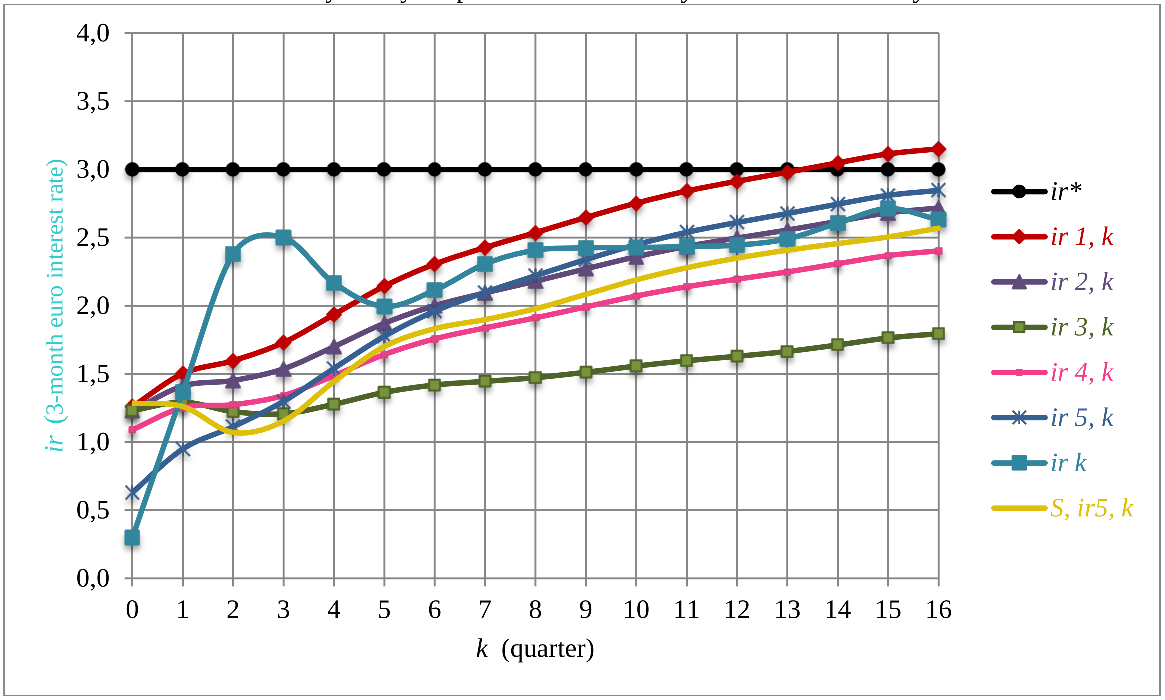
The short-term market interest trajectories in figure 15 follow a very similar pattern to their trajectories in figure 7, except that each curve (other than the target) is slightly lower in figure 15. This shows that restricting fiscal policy has only a small impact on optimal monetary policy. Due to the wavelet decomposition, the aggregate interest rate trajectory continues to follow an undulating up and down pattern, where the determination of the optimal thrust and timing of market interest rate changes requires the computation of the decomposed series. This pattern also holds when fiscal policy is even further restricted, although the simulations are not shown here.

Although the monetary authorities at the Governing Council of the ECB may follow a strategy that generally changes its benchmark operating target interest rates on the marginal lending facilities and deposit facilities by small discrete increments, the results in figure 15 suggest that the timing and duration of these benchmark interest rates should be determined so that the market interest rate follows the optimal feedback trajectory from the wavelet model. The signals that are produced from benchmark rates drive the short-term market interest rate, which affects the consumption and investment. It is not surprising that the wavelet cycles appear in the optimal market interest rate trajectory, since it reflects the shorter and longer cycles that occur in the economy. However, the simulations demonstrate of the importance of incorporating these cycles when making policy announcements.

Figure 15

Short-term Market Interest Rate Optimal Forecast Trajectories

Monetary Policy Emphasis with Relatively Passive Fiscal Policy



The consumption and investment trajectories in figures 16 and 17 are also similar to those in figures 8 and 9. However, when government purchases are restricted, the aggregate consumption and aggregate trajectories are consistently lower, and end at lower values in figures 16 and 17. This demonstrates that restrictive fiscal policy causes a substantial decrease in the performance of consumption and investment, although it has little effect on the optimal interest rate strategy for the monetary authorities.

Figure 16

Consumption Optimal Forecast Trajectories

Monetary Policy Emphasis with Relatively Passive Fiscal Policy

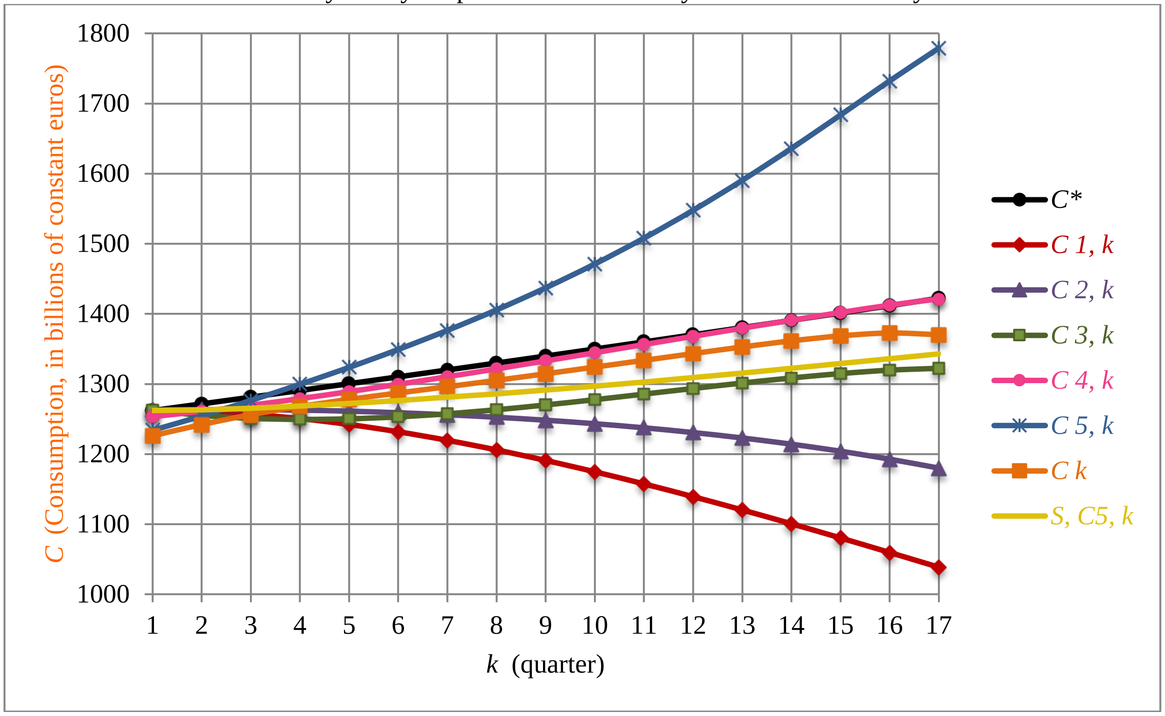
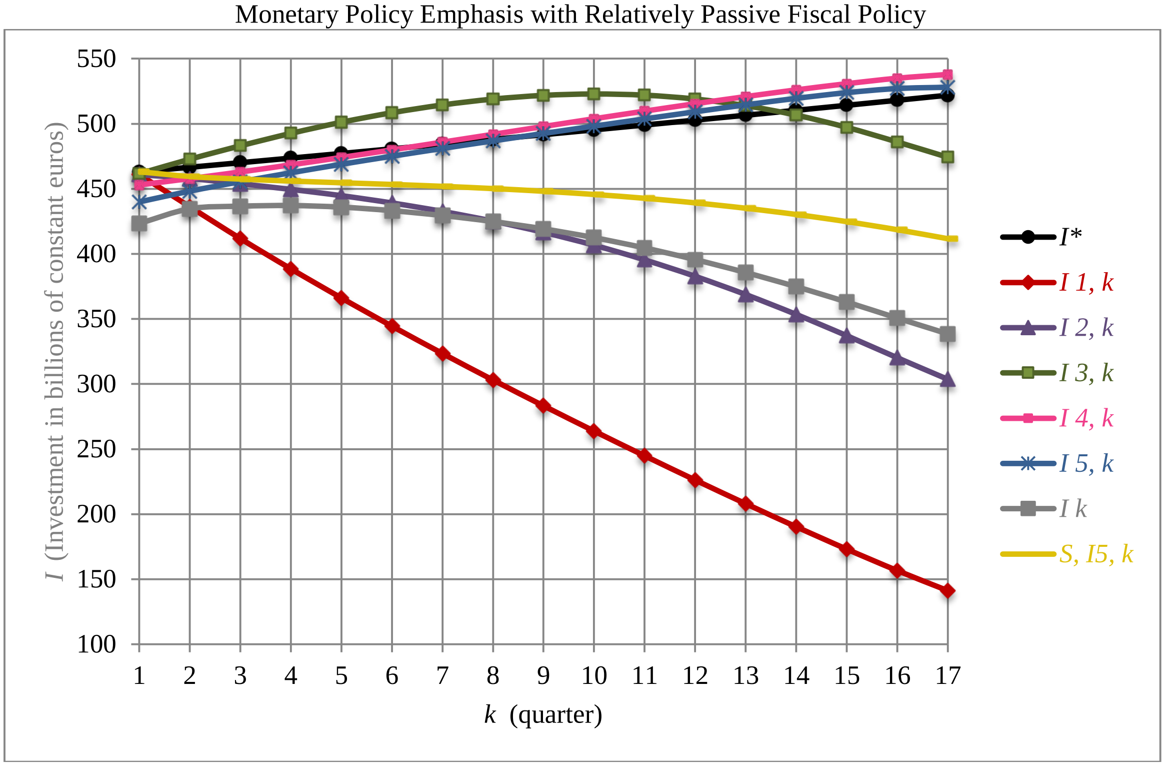


Figure 17

Investment Optimal Forecast Trajectories

Monetary Policy Emphasis with Relatively Passive Fiscal Policy



In all three cases, the objective is to design fiscal and monetary policies that are optimally determined through feedback control within the wavelet model. The primary emphasis must be given to monetary policy in the Euro area, due to the limited use of fiscal stabilization policy allowed by the "Fiscal Compact." Our approach is consistent with the ECB's two-pillar strategy that uses economic analysis and monetary analysis to achieve the final target of price stability, as measured by an annual inflation rate of less than 2% based on the HICP (Harmonized Index of Consumer Prices) for the Euro Area. Although the our model does not include an explicit determination of the aggregate price level, the targets for consumption, investment, government purchases, and the market interest rates were chosen to be consistent with their long-run natural full-employment growth rates. Thus, when policymakers track these targets, this promotes price stability.

The dual stabilization policy emphasis of case (1) provides the best option for achieving both output and price stability. Consumption is slightly above its target, investment is slightly below its target. Government expenditure is allowed some stabilizing flexibility, but mostly remains close to the target, although it still exhibits more volatility than that allowed by the "Fiscal Compact." Case (2) is not currently a viable option in the Euro Area, due to the additional volatility in government purchases; moreover, it would likely cause the most problems in price stability since the active fiscal policy causes investment to slightly exceed its target, while consumption exceeds its target by a considerably larger amount. The restrictive fiscal policy in case (3) still allows for a similar monetary policy to that in case (1), but this has a cost of less stability in output, since consumption and investment consistently achieve lower values in case (3). However, the restrictive fiscal policy with an active monetary policy in case (3) is more consistent with achieving the ECB's emphasis in the final target of price stability.

5 Conclusion

The wavelet-based design offers substantial insight into optimal monetary and fiscal policy. The analysis demonstrates that monetary authorities can optimally influence the aggregate short-term market interest rate based on wavelet decomposition, regardless of the restrictions on fiscal policy. The findings also demonstrate that following an overly restrictive fiscal policy can cause consumption and investment to diverge substantially from their targets. Thus, a dual approach where fiscal and monetary policy are at least moderately flexible can allow for improved economic performance. This is particularly relevant in the Euro Area case where the "Fiscal Compact" imposes a constraint on fiscal policy.

The model can be extended to analyze the wavelet-based model performance under the *LQG* stochastic error structure, and the robust design with a worst-case disturbance structure. The paper can also be extended to include an exploration of monetary policy through the exchange rate and the foreign sector. Adding this to the model would allow for a more complete analysis, but would also create a much larger state-space system. The authors intend to explore some of these extensions in future research. Lastly, the model could be expanded to explicitly include the price level. The current model is not meant to be a complete forecasting tool; however, it does demonstrate that considerable insight can be gained through the wavelet-based optimal control approach, and provides a framework for further extension.

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Appendix A1

The model procedure assumes that the economy would have some reaction to any announced, consistent government policy regime. Since no such control policy has yet been historically implemented, there was no past distinction between government spending under the optimal control policy, and the spending trajectory along the current cycle trajectory that reflects existing expectations. The rational reactions involving an adjustment for government debt under the lack of any announced consistent policy are zero, since there was no such policy against which to react. Thus, the lag of the current government spending trend variable and the lag of government debt variables are not included in equations (14) and (15). Instead, the values for these coefficients are assigned and evaluated under different scenarios in control system policy simulations.

Tables A1 – A3 show that the estimated equations for all of the consumption, investment, and government purchases equations over each frequency range have a good fit. All of the coefficients have the expected sign, and almost all of the coefficients are statistically significant. The consumption equation coefficients in table A1 show that both investment and government purchases have a crowding-in effect on consumption. The investment coefficients in table A2 show that consumption has a crowding-in effect, but government purchases has a crowing-out effect on investment. The government purchases coefficients in table 4 shows that the average quarterly growth rate is close to .005 per quarter (about 2% per year) at all frequencies.

Table A1

Estimated coefficients for $C_{j,k}$ at each frequency ($R^2 > 0.99$ for all equations, $j = 1, \ldots, 5$)

j	Constant_j	Coefficient $C_{j,k-1}$	Coefficient $I_{j,k-1}$	Coefficient $G_{j,k-1}$	Coefficient $ir_{j,k-1}$
1	42438.01	0.9322	0.1056	-0.0063	-1628.9254
t-statistic	*3.445*	*14.306*	*2.145*	*-0.057*	-3.903
2	45061.74	0.9140	0.1229	0.0199	-1730.7089
t-statistic	*4.485*	*16.841*	*2.945*	*0.221*	-5.126
3	48275.35	0.8815	0.1524	0.0704	-1744.7664
t-statistic	*5.919*	*19.893*	*4.365*	*0.963*	-6.472
4	30363.70	0.9943	0.0535	-0.0981	-975.7151
t-statistic	*3.374*	*21.349*	*1.427*	*-1.311*	-3.295
5	22607.80	1.0942	-0.0524	-0.2459	-459.4918
t-statistic	*2.131*	*23.808*	*-1.345*	*-3.616*	-1.210

Table A2

Estimated coefficients for $I_{j,k}$ at each frequency ($R^2 > 0.96$ for all equations, $j = 1, \ldots, 5$)

j	Constant $_j$	Coefficient $I_{j,k-1}$	Coefficient $G_{j,k-1}$	Coefficient $ir_{j,k-1}$
1	38221.18	1.0296	-0.1074	-1708.0304
t-statistic	*2.618*	*23.563*	*-1.863*	*-2.465*
2	41403.18	1.0556	-0.1401	-2028.6926
t-statistic	*3.718*	*31.213*	*-3.206*	*-3.851*
3	36408.31	1.0384	-0.1128	-1656.9915
t-statistic	*3.629*	*33.433*	*-3.005*	*-3.610*
4	30481.04	1.0071	-0.0708	-989.9166
t-statistic	*2.396*	*28.166*	*-1.701*	*-1.748*
5	37241.97	1.0035	-0.0836	-1011.3181
t-statistic	*2.371*	*29.703*	*-1.713*	*-1.282*

Table A3

Estimated coefficients for $G_{j,k}$ at each frequency

j	*Coefficient* $G_{j,k-1}$	R^2
1	1.0037	
t-statistic	*3295.88*	0.9999
2	1.0037	
t-statistic	*4507.06*	0.9999
3	1.0037	
t-statistic	*4519.06*	0.9999
4	1.0037	
t-statistic	*5082.43*	0.9999
5	1.0036	
t-statistic	*4283.56*	0.9999

Table A4 gives the paths for the modified smooth trends after extracting the crystals from all 5 frequency ranges for consumption, investment, government purchases, and the 3-month interest rate, as specified in equations (22), (23), (24), and (25), respectively. The summation of the two coefficients in each of the equations forms a weighted average trend growth rate. In consumption trend series equation, the coefficient on the lagged value of the series is $s_{C,1} = 0.89$, which is much larger than coefficient on the lagged value of aggregate consumption, given by $s_{C,2} = 0.11$. This pattern holds for the investment and government purchases modified smooth trend series, where the coefficients on the lagged value of both series is over 0.8, while the coefficients on the lagged aggregate investment and aggregate government purchases are less than 0.2. All three equations obtain a good fit, with statistically significant coefficients.

Table A4

Estimated coefficients for the *modified smooth* trend residuals for *Consumption*, *Investment*, and *Government Purchases* at each frequency range

	$s_{C,1}$	$s_{C,2}$	R^2
$S_{C5,k}$	0.8927	0.1133	
t-statistic	*48.8170*	*6.2208*	0.9999

	$s_{I,1}$	$s_{I,2}$	R^2
$S_{I5,k}$	0.8194	0.1861	
t-statistic	*37.5588*	*8.5257*	0.9967

	$s_{G,1}$	$s_{G,2}$	R^2
S_{G5}	0.8609	0.1441	
t-statistic	*46.0681*	*7.7119*	0.9997

	$s_{ir,1}$	$s_{ir,2}$	R^2
S_{ir5}	0.7966	0.2202	
t-statistic	*22.1258*	*0.0415*	0.9812

Table A5

Performance Index Coefficients

	Emphasis		
	Dual	Fiscal	Monetary
$q_{1,f}$ =	2.0	2.0	2.0
$q_{2,f}$ =	2.0	2.0	2.0
$q_{1,k}$ =	0.2	0.2	0.2
$q_{2,k}$ =	0.2	0.2	0.2
$q_{3,1,f}$ =	1.0	1.0	1.0
$q_{3,2,f}$ =	4.0	4.0	4.0
$q_{3,3,f}$ =	16.0	16.0	16.0
$q_{3,4,f}$ =	16.0	16.0	16.0
$q_{3,5,f}$ =	1.0	1.0	1.0
$q_{3,1,k}$ =	0.1	0.1	0.1
$q_{3,2,k}$ =	0.4	0.4	0.4
$q_{3,3,k}$ =	1.6	1.6	1.6
$q_{3,4,k}$ =	1.6	1.6	1.6
$q_{3,5,k}$ =	0.1	0.1	0.1
$q_{4,1,f}$ =	1.0	1.0	1.0
$q_{4,2,f}$ =	4.0	4.0	4.0
$q_{4,3,f}$ =	16.0	16.0	16.0
$q_{4,4,f}$ =	16.0	16.0	16.0
$q_{4,5,f}$ =	1.0	1.0	1.0
$q_{4,1,k}$ =	0.1	0.1	0.1
$q_{4,2,k}$ =	0.4	0.4	0.4
$q_{4,3,k}$ =	1.6	1.6	1.6
$q_{4,4,k}$ =	1.6	1.6	1.6
$q_{4,5,k}$ =	0.1	0.1	0.1
$q_{5,k}$ =	0.2	0.2	0.2
$q_{6,k}$ =	0.2	0.2	0.2

	Emphasis		
	Dual	Fiscal	Monetary
$q_{7,1,k}$ =	0.2	0.2	0.2
$q_{7,2,k}$ =	0.2	0.2	0.2
$q_{7,3,k}$ =	0.2	0.2	0.2
$q_{7,4,k}$ =	0.2	0.2	0.2
$q_{7,5,k}$ =	0.2	0.2	0.2
$q_{8,k}$ =	20.0	20.0	160.0
$q_{9,k}$ =	1,000,000,000,000	100,000,000,000,000	1,000,000,000,000
$q_{10,1,k}$ =	10,000,000,000,000	10,000,000,000,000	10,000,000,000,000
$q_{10,2,k}$ =	10,000,000,000,000	10,000,000,000,000	10,000,000,000,000
$q_{10,3,k}$ =	10,000,000,000,000	10,000,000,000,000	10,000,000,000,000
$q_{10,4,k}$ =	10,000,000,000,000	10,000,000,000,000	10,000,000,000,000
$q_{10,5,k}$ =	10,000,000,000,000	10,000,000,000,000	10,000,000,000,000
$q_{11,k}$ =	100,000,000,000	100,000,000,000,000	100,000,000,000
$q_{S,C5,f}$ =	2.0	2.0	2.0
$q_{S,I5,f}$ =	2.0	2.0	2.0
$q_{S,C5,k}$ =	0.2	0.2	0.2
$q_{S,I5,k}$ =	0.2	0.2	0.2
$q_{S,G5,k}$ =	0.2	0.2	0.2
$q_{S,ir5,k}$ =	100,000,000	100,000,000	100,000,000
$r_{G,1,k}$ =	10.0	10.0	10.0
$r_{G,2,k}$ =	10.0	10.0	10.0
$r_{G,3,k}$ =	20.0	20.0	20.0
$r_{G,4,k}$ =	20.0	20.0	20.0
$r_{G,5,k}$ =	10.0	10.0	10.0
$r_{ir,1,k}$ =	100,000	100,000	100,000
$r_{ir,2,k}$ =	100,000	100,000	100,000
$r_{ir,3,k}$ =	200,000	200,000	200,000
$r_{ir,4,k}$ =	200,000	200,000	200,000
$r_{ir,5,k}$ =	200,000	200,000	200,000

Appendix A2

The 79 state equations are listed as follows:

$$x_{1,k+1} = \delta_{1,1}x_{1,k} + \delta_{2,1}x_{7,k} + \delta_{0,1}x_{13,k} + \delta_{3,1}x_{24,k} + \delta_{5,1}x_{29,k} + \delta_{6,1}x_{45,k} + \delta_{4,1}x_{68,k} + \delta_{3,1}u_{G,1,k} + \delta_{4,1}u_{ir,1,k} + \delta_{7,1}\omega_{1,1,k}$$

$$x_{2,k+1} = \delta_{1,2}x_{2,k} + \delta_{2,2}x_{8,k} + \delta_{0,2}x_{13,k} + \delta_{3,2}x_{25,k} + \delta_{5,2}x_{30,k} + \delta_{6,2}x_{45,k} + \delta_{4,2}x_{68,k} + \delta_{3,2}u_{G,2,k} + \delta_{4,2}u_{ir,2,k} + \delta_{7,2}\omega_{1,2,k}$$

$$x_{3,k+1} = \delta_{1,3}x_{3,k} + \delta_{2,3}x_{9,k} + \delta_{0,3}x_{13,k} + \delta_{3,3}x_{26,k} + \delta_{5,3}x_{31,k} + \delta_{6,3}x_{45,k} + \delta_{4,3}x_{68,k} + \delta_{3,3}u_{G,3,k} + \delta_{4,3}u_{ir,3,k} + \delta_{7,3}\omega_{1,3,k}$$

$$x_{4,k+1} = \delta_{1,4}x_{4,k} + \delta_{2,4}x_{10,k} + \delta_{0,4}x_{13,k} + \delta_{3,4}x_{27,k} + \delta_{5,4}x_{32,k} + \delta_{6,4}x_{45,k} + \delta_{4,4}x_{68,k} + \delta_{3,4}u_{G,4,k} + \delta_{4,4}u_{ir,4,k} + \delta_{7,4}\omega_{1,4,k}$$

$$x_{5,k+1} = \delta_{1,5}x_{5,k} + \delta_{2,5}x_{11,k} + \delta_{0,5}x_{13,k} + \delta_{3,5}x_{28,k} + \delta_{5,5}x_{33,k} + \delta_{6,5}x_{45,k} + \delta_{4,5}x_{68,k} + \delta_{3,5}u_{G,5,k} + \delta_{4,5}u_{ir,5,k} + \delta_{7,5}\omega_{1,5,k}$$

$$x_{6,k+1} = s_{C,1}x_{6,k} + s_{C,2}x_{35,k} + s_{C,3}\omega_{4,k}$$

$$x_{7,k+1} = \lambda_{1,1}x_{7,k} + \lambda_{0,1}x_{13,k} + \lambda_{2,1}x_{24,k} + \lambda_{3,1}x_{68,k} + \lambda_{3,1}u_{ir,1,k} + \lambda_{4,1}\omega_{2,1,k}$$

$$x_{8,k+1} = \lambda_{1,2}x_{8,k} + \lambda_{0,2}x_{13,k} + \lambda_{2,2}x_{25,k} + \lambda_{3,2}x_{68,k} + \lambda_{3,2}u_{ir,2,k} + \lambda_{4,2}\omega_{2,2,k}$$

$$x_{9,k+1} = \lambda_{1,3}x_{9,k} + \lambda_{0,3}x_{13,k} + \lambda_{2,3}x_{26,k} + \lambda_{3,3}x_{68,k} + \lambda_{3,3}u_{ir,3,k} + \lambda_{4,3}\omega_{2,3,k}$$

$$x_{10,k+1} = \lambda_{1,4}x_{10,k} + \lambda_{0,4}x_{13,k} + \lambda_{2,4}x_{27,k} + \lambda_{3,4}x_{68,k} + \lambda_{3,4}u_{ir,4,k} + \lambda_{4,4}\omega_{2,4,k}$$

$$x_{11,k+1} = \lambda_{1,5}x_{11,k} + \lambda_{0,5}x_{13,k} + \lambda_{2,5}x_{28,k} + \lambda_{3,5}x_{68,k} + \lambda_{3,5}u_{ir,5,k} + \lambda_{4,5}\omega_{2,5,k}$$

$$x_{12,k+1} = s_{I,1}x_{12,k} + s_{I,2}x_{36,k} + s_{I,3}\omega_{5,k}$$

$$x_{13,k+1} = x_{13,k}$$

$$x_{14,k+1} = (1+g_{C,1,k})\,x_{14,k}$$

$$x_{15,k+1} = (1+g_{C,2,k})\,x_{15,k}$$

$$x_{16,k+1} = (1+g_{C,3,k})\,x_{16,k}$$

$$x_{17,k+1} = (1+g_{C,4,k})\,x_{17,k}$$

$$x_{18,k+1} = (1+g_{C,5,k})\,x_{18,k}$$

$$x_{19,k+1} = (1+g_{I,1,k})\,x_{19,k}$$

$$x_{20,k+1} = (1+g_{I,2,k})\,x_{20,k}$$

$$x_{21,k+1} = (1+g_{I,3,k})\,x_{21,k}$$

$$x_{22,k+1} = (1+g_{I,4,k})\,x_{22,k}$$

$$x_{23,k+1} = (1+g_{I,5,k})\,x_{23,k}$$
$$x_{24,k+1} = (1+g_{G,1,k})\,x_{24,k}$$
$$x_{25,k+1} = (1+g_{G,2,k})\,x_{25,k}$$
$$x_{26,k+1} = (1+g_{G,3,k})\,x_{26,k}$$
$$x_{27,k+1} = (1+g_{G,4,k})\,x_{27,k}$$
$$x_{28,k+1} = (1+g_{G,5,k})\,x_{28,k}$$
$$x_{29,k+1} = \rho_{1,1}x_{24,k} + \rho_{1,1}u_{G,1,k} + \rho_{2,1}\,\omega_{3,1,k-1}$$
$$x_{30,k+1} = \rho_{1,2}x_{25,k} + \rho_{1,2}u_{G,2,k} + \rho_{2,2}\,\omega_{3,2,k-1}$$
$$x_{31,k+1} = \rho_{1,3}x_{26,k} + \rho_{1,3}u_{G,3,k} + \rho_{2,3}\,\omega_{3,3,k-1}$$
$$x_{32,k+1} = \rho_{1,4}x_{27,k} + \rho_{1,4}u_{G,4,k} + \rho_{2,4}\,\omega_{3,4,k-1}$$
$$x_{33,k+1} = \rho_{1,5}x_{28,k} + \rho_{1,5}u_{G,5,k} + \rho_{2,5}\,\omega_{3,5,k-1}$$
$$x_{34,k+1} = s_{G,1}x_{34,k} + s_{G,2}\,x_{37,k} + s_{G,3}\,\omega_{6,k}$$
$$\begin{aligned} x_{35,k+1} = {} & \delta_{1,1}x_{1,k} + \delta_{1,2}x_{2,k} + \delta_{1,3}x_{3,k} + \delta_{1,4}x_{4,k} + \delta_{1,5}x_{5,k} - 4x_{6,k} + \delta_{2,1}x_{7,k} \\ & + \delta_{2,2}x_{8,k} + \delta_{2,3}x_{9,k} + \delta_{2,4}x_{10,k} + \delta_{2,5}x_{11,k} + \sum_{j=1}^{5}\delta_{0,j}x_{13,k} + \delta_{3,1}x_{24,k} \\ & + \delta_{3,2}x_{25,k} + \delta_{3,3}x_{26,k} + \delta_{3,4}x_{27,k} + \delta_{3,5}x_{28,k} + \delta_{5,1}x_{29,k} + \delta_{5,2}x_{30,k} \\ & + \delta_{5,3}x_{31,k} + \delta_{5,4}x_{32,k} + \delta_{5,5}x_{33,k} + \sum_{j=1}^{5}\delta_{6,j}x_{45,k} + \sum_{j=1}^{5}\delta_{4,j}x_{68,k} \\ & + \sum_{j=1}^{5}\delta_{3,j}u_{G,j,k} + \sum_{j=1}^{5}\delta_{4,j}u_{ir,j,k} + \sum_{j=1}^{5}\delta_{7,j}\,\omega_{1,j,k} \end{aligned}$$

$$\begin{aligned} x_{36,k+1} = {} & \lambda_{1,1}x_{7,k} + \lambda_{1,2}x_{8,k} + \lambda_{1,3}x_{9,k} + \lambda_{1,4}x_{10,k} + \lambda_{1,5}x_{11,k} - 4x_{12,k} \\ & + \sum_{j=1}^{5}\lambda_{0,j}x_{13,k} + \lambda_{2,1}x_{24,k} + \lambda_{2,2}x_{25,k} + \lambda_{2,3}x_{26,k} + \lambda_{2,4}x_{27,k} + \lambda_{2,5}x_{28,k} \\ & + \sum_{j=1}^{5}\lambda_{3,j}x_{68,k} + \sum_{j=1}^{5}\lambda_{2,j}u_{G,j,k} + \sum_{j=1}^{5}\lambda_{3,j}u_{ir,j,k} + \sum_{j=1}^{5}\lambda_{4,j}\,\omega_{2,j,k} \end{aligned}$$

$$x_{37,k+1} = x_{24,k}+x_{25,k}+x_{26,k}+x_{27,k}+x_{28,k} - 4\,x_{34,k} + \sum_{j=1}^{5}u_{G,j,k}$$
$$x_{38,k+1} = (1+g_{C,k})\,x_{38,k}$$
$$x_{39,k+1} = (1+g_{I,k})\,x_{39,k}$$
$$x_{40,k+1} = (1+g_{G,k})\,x_{40,k}$$

$$x_{41,k+1} = n_0 \, x_{13,k}$$
$$x_{42,k+1} = x_{35,k} + x_{36,k} + x_{37,k} + x_{41,k}$$
$$x_{43,k+1} = \tau \, x_{42,k}$$
$$x_{44,k+1} = x_{24,k} + x_{25,k} + x_{26,k} + x_{27,k} + x_{28,k} - 4\,x_{34,k} - x_{43,k} + \sum_{j=1}^{5} u_{G,j,k}$$
$$x_{45,k+1} = x_{44,k} + (1+i)\,x_{45,k}$$
$$x_{46,k+1} = x_{24,k} + u_{G,1,k}$$
$$x_{47,k+1} = x_{25,k} + u_{G,2,k}$$
$$x_{48,k+1} = x_{26,k} + u_{G,3,k}$$
$$x_{49,k+1} = x_{27,k} + u_{G,4,k}$$
$$x_{50,k+1} = x_{28,k} + u_{G,5,k}$$
$$x_{51,k+1} = x_{24,k} - x_{46,k} + u_{G,1,k}$$
$$x_{52,k+1} = x_{25,k} - x_{47,k} + u_{G,2,k}$$
$$x_{53,k+1} = x_{26,k} - x_{48,k} + u_{G,3,k}$$
$$x_{54,k+1} = x_{27,k} - x_{49,k} + u_{G,4,k}$$
$$x_{55,k+1} = x_{28,k} - x_{50,k} + u_{G,5,k}$$
$$x_{56,k+1} = g_{G,1,k} \, x_{24,k}$$
$$x_{57,k+1} = g_{G,2,k} \, x_{25,k}$$
$$x_{58,k+1} = g_{G,3,k} \, x_{26,k}$$
$$x_{59,k+1} = g_{G,4,k} \, x_{27,k}$$
$$x_{60,k+1} = g_{G,5,k} \, x_{28,k}$$
$$x_{61,k+1} = (1+g_{DEF,k})\,x_{61,k}$$
$$x_{62,k+1} = (1+g_{DEBT,k})\,x_{62,k}$$
$$x_{63,k+1} = (1+g_{S,C5,k})\,x_{63,k}$$
$$x_{64,k+1} = (1+g_{S,I5,k})\,x_{64,k}$$
$$x_{65,k+1} = (1+g_{S,G5,k})\,x_{65,k}$$
$$x_{66,k+1} = (1+g_{S,ir5,k})\,x_{66,k}$$
$$x_{67,k+1} = s_{ir,1} x_{67,k} + s_{ir,2} x_{69,k} + s_{ir,3} \omega_{7,k}$$
$$x_{68,k+1} = (1+g_{ir,k})\,x_{68,k}$$

$$x_{69,k+1} = 5x_{68,k} - 4x_{67,k} + \sum_{j=1}^{5} u_{ir,j,k}$$
$$x_{70,k+1} = x_{68,k} + u_{ir,1,k}$$
$$x_{71,k+1} = x_{68,k} + u_{ir,2,k}$$
$$x_{72,k+1} = x_{68,k} + u_{ir,3,k}$$
$$x_{73,k+1} = x_{68,k} + u_{ir,4,k}$$
$$x_{74,k+1} = x_{68,k} + u_{ir,5,k}$$
$$x_{75,k+1} = x_{68,k} - x_{70,k} + u_{ir,1,k}$$
$$x_{76,k+1} = x_{68,k} - x_{71,k} + u_{ir,2,k}$$
$$x_{77,k+1} = x_{68,k} - x_{72,k} + u_{ir,3,k}$$
$$x_{78,k+1} = x_{68,k} - x_{73,k} + u_{ir,4,k}$$
$$x_{79,k+1} = x_{68,k} - x_{74,k} + u_{ir,5,k}$$
$$x_{80,k+1} = x_{69,k}$$

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