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FISCAL POLICY TRACKING DESIGN IN THE TIME-FREQUENCY DOMAIN USING WAVELET ANALYSIS

PATRICK M. CROWLEY* and DAVID HUDGINS†

**Department of Decision Sciences and Economics, College of Business, OCNR 373, 6300 Ocean Drive, Texas A & M University – Corpus Christi, Corpus Christi, TX, 78412 U.S.A., E-mail: patrick.crowley@tamucc.edu, Phone: 361-825-6011*

†Corresponding author: Department of Decision Sciences and Economics, College of Business, OCNR 314, 6300 Ocean Drive, Texas A & M University – Corpus Christi, Corpus Christi, TX, 78412 U.S.A., E-mail: david.hudgins@tamucc.edu

Abstract. In this paper discrete wavelet filtering techniques are applied to decompose macroeconomic data so that they can be simultaneously analyzed in both the time and frequency domains. The *MODWT* (Maximal Overlap Discrete Wavelet Transform) is applied to U.S. quarterly GDP data from 1947 – 2012 to obtain the underlying cyclical structure of the GDP components. A MATLAB program is then used to design optimal fiscal policy within a *LQ*-tracking model with wavelet decomposition, and the results are compared with an aggregate model with no frequency decomposition. The results show that fiscal policy is more active under the wavelet-based model, and that the consumption and investment trajectories under the aggregate model are misaligned. We also simulate *FHEC* (Frequency Harmonizing Emphasis Control) strategies that allow policymakers to concentrate the policy thrust on tracking frequencies that are optimally aligned with policy goals under different targeting priorities. These strategies are only available by using time-frequency analysis. This research is the first to construct fiscal policy in an applied optimal control model based on the short and long cyclical lag structures obtained from wavelet analysis. Our wavelet-based optimal control procedure allows the policymaker to construct a pragmatic tracking policy, avoid suboptimal policies gleaned from an aggregate model, and reduce the potential for destabilization that might otherwise result due to improper thrust and timing.

Keywords: LQ Tracking, Macroeconomics, Optimal Control, Discrete Wavelet Analysis, Fiscal Policy

JEL classifications C49 . C61 . C63 . C88 . E61

1 Introduction

The macroeconomic accelerator model has proved to be a useful theoretical and empirical tool since Chow (1967). Kendrick (1981), Kendrick and Amman (2010), Kendrick and Shoukry (2013), and Hudgins and Na (2014) have all modeled quarterly fiscal policy within an applied macroeconomic optimal control *LQ* (Linear-Quadratic) tracking framework. Kendrick (1981) compared the deterministic model results to the closed-economy performance of the U.S. under stochastic and adaptive control. Kendrick and Amman (2010), Kendrick and Shoukry (2013), and Hudgins and Na (2014) all present strong cases for the improvements of implementing a quarterly fiscal policy rather than an annual policy. Kendrick and Shoukry (2013) simulate the tracking performance and debt structure of the quarterly and annual models within a closed-economy macroeconomic model that contains monetary and interest rate components. Hudgins and Na (2014) examine optimal robust policies using a similar model for the U.S. economy, but without money and interest rates in the model.

In each of these models, the particular lag structures in the policy variable design and in the consumption and investment functions could have destabilizing effects on the response dynamics. All of these analyses note that it is important for modelers to consider different lag structures in these equations in order to simulate the optimal policies under various parameters. Instead of estimating various time series models with alternating lags, another approach is to utilize wavelet analysis in order to gain insight into the lag structure through the time-frequency domain. Crowley and Hughes Hallett (2014) use a *MODWT* (Maximal Overlap Discrete Wavelet Transform) to find the frequency domain cyclical decomposition of U.S. GDP component data for the period 1947 – 2012. This analysis will build the control model using the results obtained from a similar wavelet estimation strategy used by Crowley and Hughes Hallett (2014).

This will also set the stage for the optimal timing of the quarterly fiscal policy impulses. Kendrick and Shoukry (2013) select the first quarter as the period when government appropriations are determined for the year, and then compare this annualized scenario to a counterfactual situation where appropriations can be made during each quarter. They note that selection of other quarters for annual fiscal policy change might result in somewhat different performance when comparing quarterly to annual fiscal policy scenarios. However, the cyclical timing and lag structure can be determined with greater precision by using the quarterly lag structure gathered through wavelet filters in the time-frequency domain.

Using the longer cycles obtained from wavelet analysis also addresses the findings of recent neoclassical research. This is consistent with Leeper et al. (2010), for example, who find that the speed of the fiscal adjustment impacts the policy effectiveness. In addition, Leeper et al. (2010) find that government investment with comparatively weak productivity can dictate a contractionary government investment policy in the long-run. Crowley and Hughes Hallett (2014) use a *MODWT* wavelet decomposition and find that U.S. fiscal policy has not been destabilizing or procyclical over the various business cycle frequencies; however, they also found that it has not been effective as a countercyclical stabilizer either. Designing quarterly fiscal policy rules that are built upon the full range of short and long term cyclical wavelet component avoids the bias that might otherwise be introduced through inadequate recognition of the interplay of the short-term lags with the long term cyclical components.

The wavelet analysis does not directly address the foresight components; however, the optimal control analysis that uses the components from wavelet analysis is based partly on the future expectations of government policy changes. Leeper et al. (2010) find that the agents' fiscal foresight has some impact on the policy effectiveness. Kriwoluzky (2012) uses a *VMA* (Vector Moving Average) model and shows that consumption initially reacts negatively to a pre-announcement period of a government spending shock, but reacts positively after the realization of increased spending. Karantounias (2013) projects optimal taxes when the government authorities rely upon an exogenous government spending probability model, but the public has pessimistic expectations. The result is that a paternalistic planner will employ distortionary taxation by exploiting household mispricing and shifting household expectations, which leads to higher tax rates during favorable shocks and lower tax rates for adverse shocks.

Svec (2012) models an altruistic government that optimally sets labor taxes and one-period debt in an economy, where uncertain consumers believe that the true approximating probability model lies within a range on probabilities. The results show that the political government that maximizes consumer welfare under the consumers' own subjective utility functions finances a smaller portion of a government spending shock from taxes than it would if consumers did not face model uncertainty. Kendrick and Amman (2006) discuss optimal control design in stochastic macroeconomic systems with forward-looking variables and find that although the frequency domain does not give any information on agents' potential foresight structure, it does indirectly address this issue since it captures all of the underlying cyclical components that the agents are using to build their expectations in an attempt to develop foresight.

1.1 Purpose and Scope

The purpose of this paper is to construct optimal fiscal policy within an optimal *LQ* (Linear-Quadratic) tracking control model that is formulated within the time-frequency domain based on the *MODWT* wavelet decomposition. This is the first research to integrate optimal control and discrete wavelet analysis in order to design macroeconomic policy. Section 2 examines the *MODWT* wavelet methods and the decomposition of data in the time-frequency domain, using data from the period 1947 – 2012. Section 3 builds a macroeconomic time-frequency accelerator model that is used within an optimal control system to determine optimal control feedback rules for fiscal policy. We convert the *LQ* tracking design into a *LQ* regulator design using the method employed by Hudgins and Na (2014), and develop a MATLAB software program to compute the optimal fiscal policy. This framework allows the policymaker to render deterministic, stochastic *LQG* (Linear-Quadratic Gaussian) and robust controller designs, but the research presented in this paper only presents simulations for the deterministic *LQ* tracking control design.

Section 4 estimates the time-frequency model developed in section 3. We run simulations that compare the optimal fiscal policy trajectories under the wavelet decomposition model with fiscal policy under the aggregate model with no decomposition. These simulations show that fiscal policy will be more active within the wavelet decomposition framework than in the model without decomposition. The simulations also show that the aggregate model would consistently produce consumption levels that are overvalued relative to consumption levels in the wavelet-based model, and would produce investment levels that are undervalued relative to the wavelet-based model throughout the planning horizon. This demonstrates the suboptimal fiscal policy and the errors in the projected consumption and investment trajectories that would likely result from using an aggregate model with no decomposition.

The government debt resulting from using the optimal wavelet policy is similar to the debt levels under the aggregate model, but the comparative levels depend on the relative priorities of the targeted state and control variables. When the tracking errors for government spending are assigned a high relative weight, the debt stock in the wavelet model at the end of the planning period will be slightly lower than debt that results from the aggregated model. When consumption and investment are more heavily weighted, the final debt stock is slightly higher under the wavelet model. Thus, another advantage of employing the wavelet-based system is the increased ability to project the impact of fiscal policy on the government stock of debt.

The simulations also explore *FHEC* (Frequency Harmonizing Emphasis Control) strategies. This approach allows the fiscal policymakers to place different weights on the tracking errors for government spending, consumption, and investment, at different frequency ranges. This is not possible under an aggregate model without time-frequency decomposition. For example, the optimal *FHEC* policy allows the government to place more emphasis on the time horizon where spending is targeted, and/or to place more emphasis on the consumption and investment intervals that will be most affected. This allows for an entirely new operational procedure for packaging the optimal fiscal policy, and for determining the likely effects at each frequency range. We simulate the model under three different *FHEC* policymaker priorities: (1) long-term business cycle and productivity targeting, (2) political cycle targeting, and (3) short-term stabilization targeting. The results show that fiscal policy can be structured to effectively reduce consumption and investment gaps for the cycles over the frequency ranges that are most emphasized under the different objectives.

1.2 Comparison of the Results to Robust Modeling

Our analysis shows that the optimal policy under the wavelet model is more active in the sense that the trajectory has much greater variation than the policy under the aggregated model. The finding that optimal policy is overly passive, or rigid, under a standard deterministic or stochastic control model is consistent with some of the findings in the robust control literature. Robust modeling analyzes the optimal control when the model form, parameters, or error structure are unknown. Hudgins and Na (2014) find that fiscal policy is more aggressive under a variety of robust error modeling structures. Dennis et al. (2009) find that optimal robust monetary policy for central banks within a New Keynesian model is more activist in order to curb the additional persistence of shock-induced inflation, and Bernhard (2002) and Onatski and Stock (2002) find that optimal robust monetary policy is generally more aggressive. Diebold (2005) shows that robust monetary policy can still be overly complacent due to problems of global uncertainty, rather than local uncertainty. The more variable wavelet-derived policy shows that these robust-derived policies may have a stabilizing effect when the system is sub-optimally modelled due to the failure to employ a time-frequency decomposition through wavelet filtering.

The Brainard Principle holds when increased uncertainty or robust errors reduce the level of policy activism (Brainard, 1967). Barlevy (2011) upholds the Brainard principle and finds that robust control strategies are often more passive in response to incoming news policies without uncertainty. Zakovic et al. (2007) also upholds the Brainard principle and finds that monetary policy rules under a minimax design approach do not lead to extreme activism in a model of the euro area. The wavelet decompositions employed here increase the policy activism, but reduce the uncertainty in modeling procedure by providing a more complete model for the policymaker. The wavelet model thus reduces the need to rely on robust considerations in the control formulation. The wavelet-based model has been designed, however, so that it can be used in conjunction with robust control methods, as in Hudgins and Na (2014).

2 *MODWT* Wavelet Analysis

Here we provide a brief overview of the mathematical background of time-frequency analysis and the wavelet methodology. As mentioned previously, time-domain analysis cannot provide a proper basis for analysis when frequencies are changing; in other words, time-domain methods generally cannot reveal valuable and helpful information which are hidden in different frequencies. The most well-known frequency domain method is the Fourier Transform (*FT*). This method transforms time series data from the time domain to the frequency domain, but the problem with the *FT* approach is that the data is transformed into just the frequency domain, so there is no ability to simultaneously analyze relationships in both the time and frequency domains. Indeed, with the *FT* method, the time series under consideration should be locally and globally stationary; but unfortunately, this also imposes a limitation since a significant number of economic and financial time series are locally and globally non-stationary. This is because of trade-offs between the frequency resolution and time resolution (Weedon, 2003) and the *FT* cannot capture this properly. Although the *Windowed FT* is often used to address these shortcomings with the *FT*, it does so only partially, as it suffers from a further problem which lies in the Heisenberg uncertainty principle.[1]

2.1 Wavelet Analysis

Wavelet analysis has its roots in multi-scale decomposition, the so-called multiscale analysis or multiresolutional analysis, which was developed by Meyer (1986), Mallat (1989a,b), Strang (1989), and Daubechies (1986). Technically speaking, multiscale analysis is an approximation operation through a dense vector space (Hilbert space) with empty intersects from coarsest to less detailed information.

[1] For more information on orthogonal transforms, see Wang (2012) pp. 461-481.

Following Mallat's explanation for the pyramid algorithm and multiresolutional analysis, the value of a variable x at time instant k, x_k, can be written as follows:

$$x_k \approx S_{J,k} + d_{J,k} + d_{J-1,k} + \ldots + d_{1,k} \tag{1}$$

where $d_{j,k}$ are detail components (wavelet "crystals"), $j = 1,\ldots, J$; $S_{J,k}$ is a trend component (the wavelet "smooth"); and J stands for the number of scales (frequency bands). Equations (1) – (3) summarize the *DWT* (Discrete Wavelet Transform) process. A variable x_k is filtered by a (low-pass) filter, l, and a wavelet (high-pass) filter at each step. In other words, we filter out information at a different set of frequencies in each step until we reach an approximated variable which contains only the trend. In this regard, in the first step x_k is decomposed into $d_{1,k}$ (the high frequency part) and $S_{1,k}$ (the low frequency part); consequently, the decomposed signal at scale 1 ($J = 1$) can be written as follows:

$$x_k \approx S_{1,k} + d_{1,k} \tag{2}$$

Then, this same process is performed on $S_{1,k}$ while the signal will be subsampled by 2, so that:

$$S_{1,k} \approx S_{2,k} + d_{2,k} \tag{3}$$

This recursive procedure is continued until we reach scale J. So finally, we have a set of detailed variables (high frequency components or "crystals") and a smoothed trend component (or the "smooth") as in equation (1). This approach then enables us to simultaneously investigate the data in both the time and frequency domain.

It should be noted that there are many different wavelet filter functions that are used in discrete wavelet analysis, such as the Symlet, Coiflet, Haar, Discrete Meyer, Biorthogonal, Daubechies and so on, which can be approximated for use in the filtering process as pairs of low pass and high pass filters. In this paper, we employ a Daubechies 4-tap (D4) wavelet as the wavelet function, which is an asymmetric wavelet. What is more, we employ the *MODWT* (Maximum Overlap Discrete Wavelet Transform) as our method of time-frequency decomposition. Since the *DWT* suffers from two shortcomings, namely (1) "dyadic data requirements" and (2) the fact that the *DWT* is "non-shift invariant" (see Crowley (2007)), the *MODWT* is used as an alternative method which addresses the aforementioned drawbacks and provides some other advantages.[2] As a robustness check, the decompositions were also repeated for national output (Y) with a different wavelet filter function, notably a Symlet 8-tap wavelet. The results were virtually identical to the ones that we present here.

[2] For detailed information see Crowley (2007)

2.2 MODWT Wavelet Decomposition Results

The empirical analysis uses U.S. Bureau of Economic Analysis (BEA) Real GDP component data in billions of chained 2005 dollars for the period 1947 quarter 1 to 2012 quarter 3. The *MODWT* is applied to this data using a two-step procedure in order to obtain the crystals and the smooth trend at frequencies $j = 1,\ldots, 5$. One of the problems in using the *MODWT* filter approach is that the crystals do not always aggregate up to the original series due to the loss of orthogonality as the higher order (lower frequency) wavelets are passed, observation by observation, through the series. Thus, two wavelet decompositions were undertaken, one by level to obtain a smooth, and then another in terms of differences of the absolute values of each series, to obtain crystals. Then, to ensure consistency in terms of the decomposition, a residual was added to the smooth to create a *modified smooth* (S), so that the sum of the summed crystals and the *modified smooth* was equal to the actual observation. The reason for this two-stage decomposition is that the trend in the real variables is difficult to identify using level data, as any cyclical activity is overwhelmed by the trend. So, to more effectively isolate cyclical activity in the second stage of the decomposition, we use differences and then use summed values of the differences which then converts the crystals back to level cycles. The interpretation of the crystals in terms of the frequency ranges that they contain are given in Table 1 as follows, and the crystals and the actual data are shown in Figures 1 – 4:

Table 1

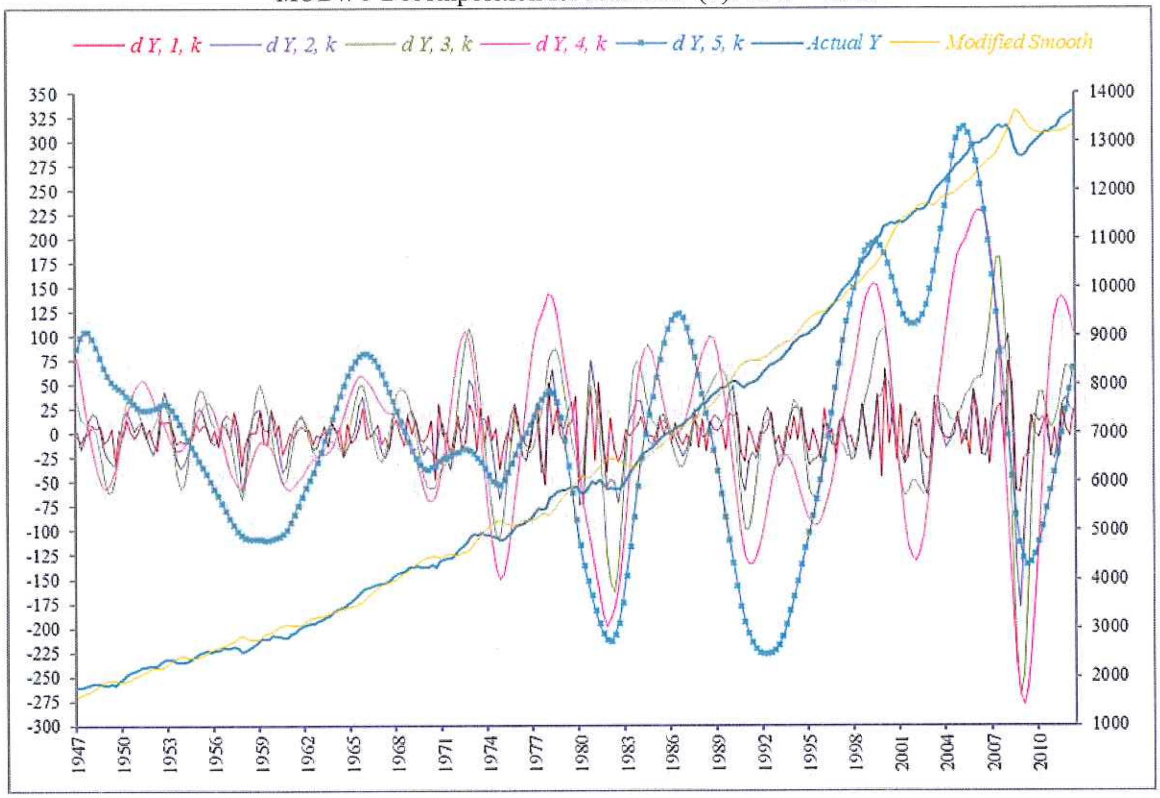
The time intervals associated with each of the frequencies

j	*Time interval*
1	6 months to 1 year
2	1 – 2 years
3	2 – 4 years
4	4 – 8 years
5	8 – 16 years

In Figure 1, each crystal is displayed as series $d_{Y,1,k}$ to $d_{Y,5,k}$, which refers to the reconstructed fluctuations in the level data for Y at frequencies up to a 16 year cycle. The actual series (which is measured relative to the right hand side axis), refers to the actual level *real GDP* series (Y), and the *modified smooth* residual trend (S) referred to above. As can be seen, the modified smooth and the actual data series are quite closely aligned except when there are sharp turning points in the data, such as in the beginning of the "great recession", when there is some divergence, implying that the *MODWT* decomposition does not model the data too well over these specific data ranges.

Figure 1

MODWT Decomposition for *real GDP* (*Y*): 1947 – 2012



The crystals exhibit a wide degree of variation in the type of fluctuations that are embodied in the original series. The higher frequency variations appear to be almost noise, but the lowest frequency variations captured by $d_{Y,5,k}$ capture the business cycle and appear to be highly irregular cycles, as one might expect. Interestingly there is a more volatile higher frequency cycle of 2 – 4 years (corresponding to $d_{Y,3,k}$) apparent in the data, corroborating the findings of Crowley and Hughes Hallett (2014). The *MODWT* filter results for Government Purchases, Consumption, and Private Investment are illustrated in figures 2, 3, and 4, respectively.

In figure 2, the government expenditure decomposition is shown. There is clearly a smaller degree of fluctuation in any cycles detected in government expenditures at higher frequencies, with the modified smooth containing much of the variation in government expenditures, with longer cycles contained in the $d_{G,4,k}$ and $d_{G,5,k}$ crystals. The government stimulus is correctly identified mostly in the $d_{G,4,k}$ crystal (4-8 year cycles), with some of the stimulus bleeding over into the $d_{G,5,k}$ crystal. There is a sharp slowdown in government expenditure detected though in recent data, which stems from the withdrawal of the stimulus.

Figure 2
MODWT Decomposition for *Government Spending* (*G*): 1947 – 2012

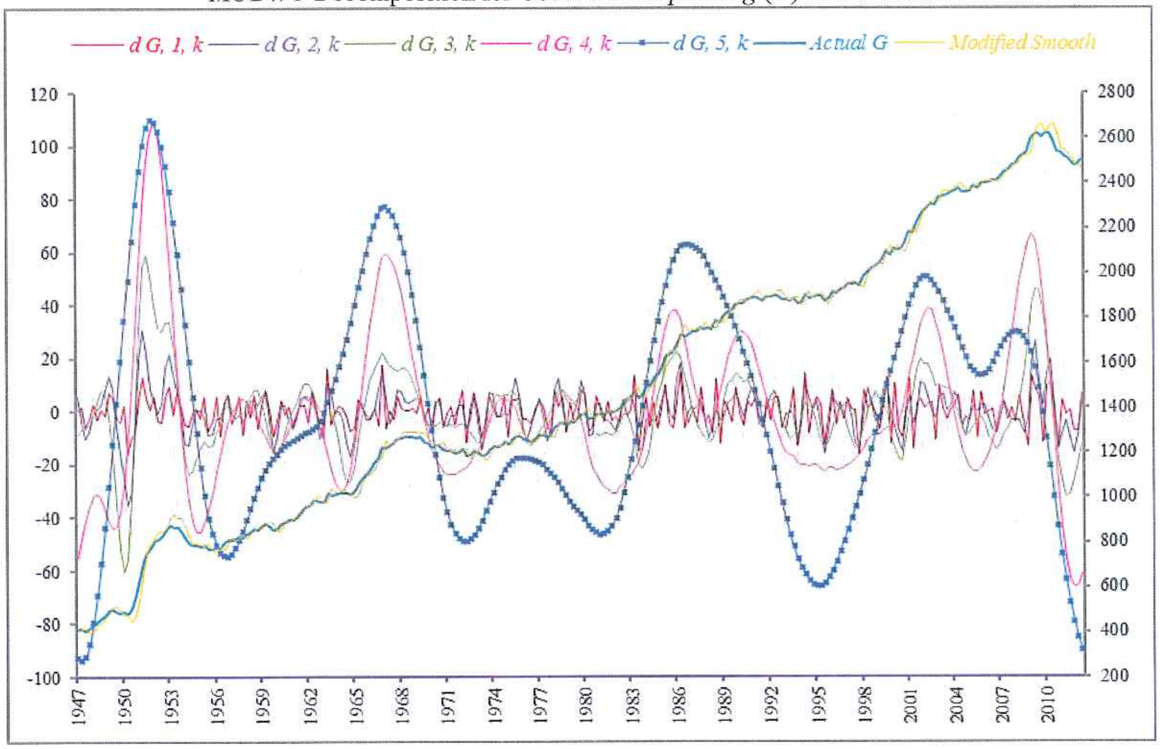
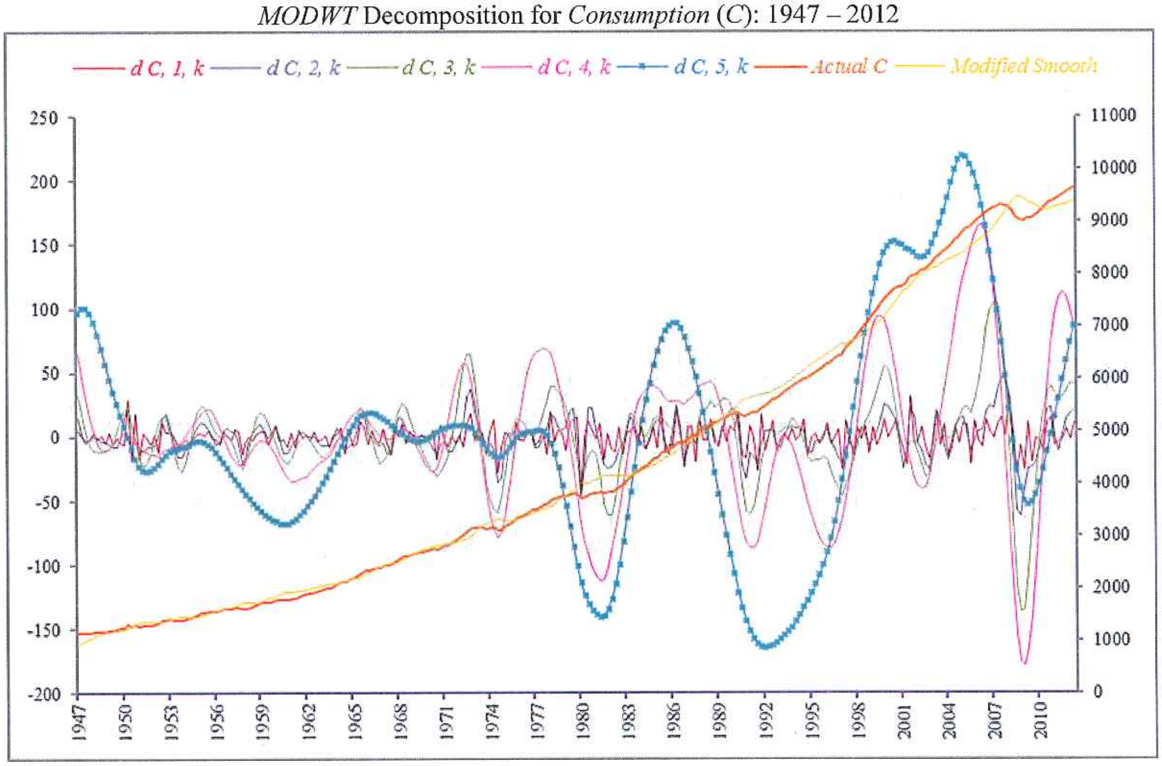


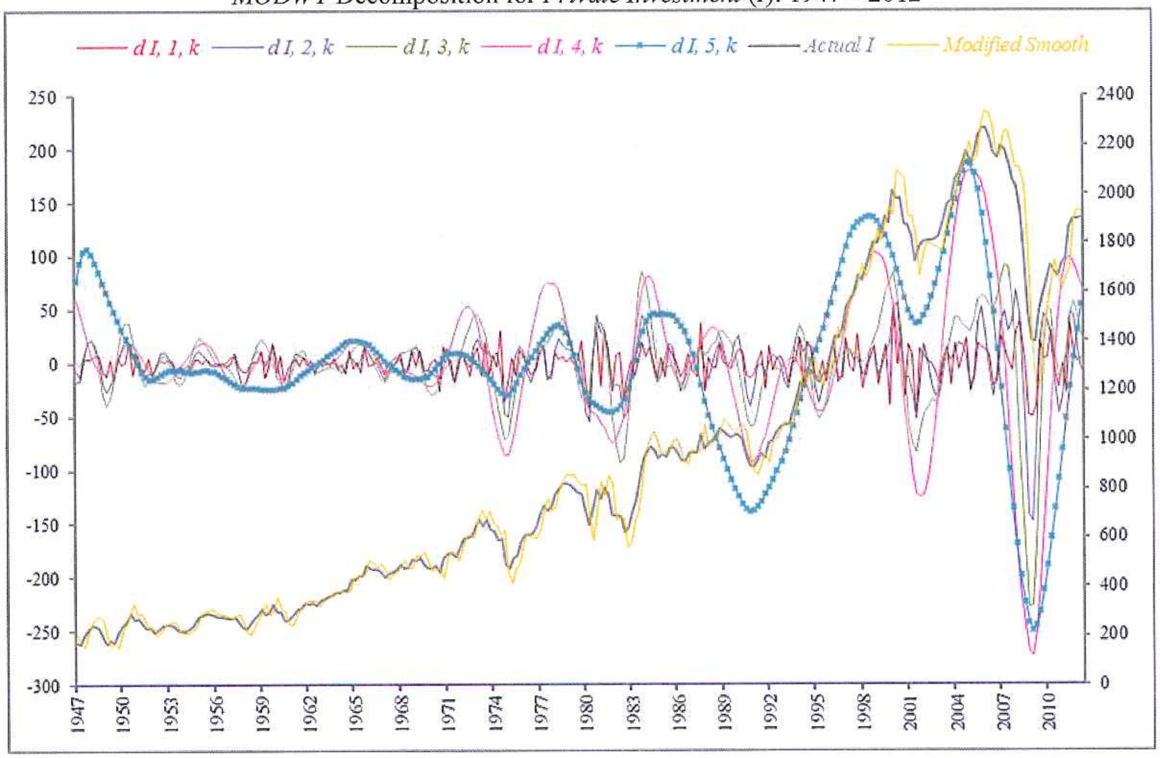
Figure 3
MODWT Decomposition for *Consumption* (*C*): 1947 – 2012



In figure 3, the consumption expenditure decomposition is shown. As might be expected, consumption expenditure cycles and the modified smooth are very similar to those observed for real GDP (Y), but with some differences. For example, the $d_{C,5,k}$ crystal appears to be much less volatile in the 1960s and 1970s than in the case of real GDP. The cyclical activity embodied in the detail crystals once again shows a large downturn during the great recession, and then a rebound.

Figure 4

MODWT Decomposition for *Private Investment* (I): 1947 – 2012



In figure 4, the wavelet decomposition for investment is shown. The actual data and the modified smooth are much more volatile than for the other components of output, while the fall in investment during the great recession is large at all frequencies. Also, it is clear that the level of investment has still not yet recovered to its pre-recession levels.

3 Macroeconomic Model Derivation

This section uses the *MODWT* crystals to develop a macroeconomic accelerator model, where the foundation is based on the closed-economy partial accelerator models of Kendrick (1981) and Kendrick and Shoukry (2013), and Hudgins and Na (2014). Let the variables be defined as follows:

C_k = total personal consumption expenditures in period k (2005 dollars)

C_k^* = desired consumption, or target consumption, in period k

I_k = gross private domestic investment in period k (2005 dollars)

I_k^* = desired investment, or target investment, in period k

Y_k = gross national product in period k (2005 dollars)

G_k = government purchases of goods and services in period k (2005 dollars)

G_k^* = desired government purchases in period k

NX_k = net exports in period k (2005 dollars)

$d_{C,j,k}$ = the value of the consumption expenditure crystal for frequency j in quarter k, where $j = 1, \ldots, 5$

$d_{I,j,k}$ = the value of the private domestic investment crystal for frequency j in quarter k, where $j = 1, \ldots, 5$

$d_{G,j,k}$ = the value of the government purchases crystal for frequency j in quarter k, where $j = 1, \ldots, 5$

$S_{C5,k}$ = the value of the consumption modified smooth in quarter k, where $J = 5$

$S^*_{C5,k}$ = target value for the modified smooth trend consumption, in period k

$S_{I5,k}$ = the value of the investment modified smooth in quarter k, where $J = 5$

$S^*_{I5,k}$ = target value for the modified smooth trend investment, in period k

$S_{G5,k}$ = the value of the government purchases modified smooth in quarter k, where $J = 5$

$S^*_{G5,k}$ = target value for the modified smooth trend in government spending, in period k

$C_{j,k}$ = the prevailing consumption expenditure at frequency j in quarter k, which includes the sum of the consumption crystal and the consumption modified smooth, where $j = 1, \ldots, 5$

$C^*_{j,k}$ = the target consumption expenditure at frequency range j in quarter k

$I_{j,k}$ = the prevailing private domestic investment at frequency j in quarter k, which includes the sum of the investment crystal and the investment modified smooth, where $j = 1, \ldots, 5$

$I^*_{j,k}$ = the target investment expenditure at frequency range j in quarter k

$G_{j,k}$ = the prevailing government purchases at frequency range j in quarter k, which includes the sum of the government purchases crystal and the government purchases modified smooth, where $j = 1, \ldots, 5$

$G^*_{j,k}$ = the target government expenditure at frequency range j in quarter k

$G^d_{j,k}$ = the current cycle trend government purchases at frequency range j in quarter k.

T_k = net government taxes and income in quarter k, which equals total government tax and income minus total government transfer payments.

DEF_k	=	total government budget deficit in quarter k, which equals government purchases of goods and services minus net government taxes
$DEBT_k$	=	total government debt in quarter k
i_k	=	quarterly interest rate on government debt in quarter k
τ_k	=	rate of net tax (tax minus transfers) collection in quarter k

The actual prevailing level of consumption at frequency range j in period k follows an accelerator framework, where it is determined by the previous period's consumption level at frequency range j, plus a fraction of the difference between the current targeted level and the last period level, as shown in equation (4). The target consumption level ($C^*_{j,k}$) for each frequency j in the wavelet frequency range during period k, is expressed in equation (5) as a linear function of the expected rate of national output production, where $\beta_{j,k}$ is the *MPC* (marginal propensity to consume). The net *MPC*, $b_{j,k}$, is formulated based on the net disposable income, where τ_k is the rate of net (taxes minus transfers) government tax and revenue collection.

$$C_{j,k} = C_{j,k-1} + \gamma_{j,k}(C^*_{j,k} - C_{j,k-1}) \qquad j = 1, \ldots, 5 \qquad (4)$$

$$C^*_{j,k} = a_{j,k} + b_{j,k} Y^e_{j,k} \qquad b_{j,k} = (1-\tau_k)\beta_{j,k} \qquad j = 1, \ldots, 5 \qquad (5)$$

Consumption at each frequency consists of the specific contribution to the consumption spending at that frequency, given by the consumption crystal $d_{C,j,k}$, plus the modified smooth $S_{C5,k}$, which reflects the base-level trend consumption. This prevailing level of consumption $C_{j,k}$ at frequency range j is given by equation (6), where the contributions from the other four frequency ranges have been removed. Equations (7) and (8) specify this same time-frequency contribution for the prevailing levels of private domestic investment $I_{j,k}$, and government spending $G_{j,k}$, respectively.

$$C_{j,k} = d_{C,j,k} + S_{C5,k} \qquad j = 1, \ldots, 5 \qquad (6)$$

$$I_{j,k} = d_{I,j,k} + S_{I5,k} \qquad j = 1, \ldots, 5 \qquad (7)$$

$$G_{j,k} = d_{G,j,k} + S_{G5,k} \qquad j = 1, \ldots, 5 \qquad (8)$$

The expected rate of output production is given by equation (9). The expected contribution of government purchases of goods and services at frequency range j in period k is given by equation (10).

$$Y^e_{j,k} = C_{j,k} + I_{j,k} + G^e_{j,k} + NX^e_{j,k} \qquad j = 1, \ldots, 5 \qquad (9)$$

$$G^e_{j,k} = \phi_{j,k}\left[G_{j,k-1} - \pi_k(DEBT_{k-1} - DEBT_0)\right] + (1-\phi_{j,k})\hat{G}^d_{j,k-1};\ 0 < \phi < 1; j = 1, \ldots, 5 \quad (10)$$

$$NX^{e}_{j,k} = n_{0,j,k} \qquad j = 1, \ldots, 5 \qquad (11)$$

Equations (9) and (10) model rational expectations behavior, where the effective contribution of government spending toward national output production is both limited and crowded out by any stock of national debt that exceeds its initial value. Any new fiscal policy regime will pulse the current cycle, and the analysis below explores cases where government spending is determined through optimal control.

The variable $G^{d}_{j,k}$ defines the current time-frequency trend of government purchases. Equation (10) shows that the expected value of government purchases in any period k is determined based on a weighted average of the actual spending in the previous period, and the trend value along the wavelet frequency range in the previous period. Since government purchases only affect the economy through the expected national output term in equations (5) and (9), all government spending changes have a limited impact. The higher is the value of $\phi_{j,k}$, the greater will be the effectiveness of fiscal policy at any given frequency range. The expected value of net exports in equation (11) is expected to be constant for the closed economy model.

Equations (12) and (13) specify the investment functions. The targeted level of investment in equation (12) follows Kendrick's (1981) specification, where target investment is a linear function of the difference between the current period consumption and the consumption level in the previous period. The current level of investment in equation (13) follows its own accelerator function, similar to that for consumption.

$$I^{*}_{j,k} = e_{j,k} + f_{j,k}(C_{j,k} - C_{j,k-1}) \qquad j = 1, \ldots, 5 \qquad (12)$$

$$I_{j,k} = I_{j,k-1} + \theta_{j,k}(I^{*}_{j,k} - I_{j,k-1}) \qquad j = 1, \ldots, 5 \qquad (13)$$

Substitute equations (10) and (11) into equation (6), and then substitute equation (6) into equation (5). Next, substitute equation (5) into (4), and substitute equation (12) into (13). After rearranging and including the disturbance term variables $\omega_{1,j,k}$ and $\omega_{2,j,k}$, this yields equations (14) and (15).

$$C_{j,k} = \delta_{0,j} + \delta_{1,j}C_{j,k-1} + \delta_{2,j}I_{j,k-1} + \delta_{3,j}G_{j,k-1} \qquad (14)$$

$$+ \delta_{4,j}\hat{G}^{d}_{j,k-1} + \delta_{5,j}DEBT_{k-1} + \delta_{6,j}\omega_{1,j,k-1} \qquad j = 1, \ldots, 5$$

$$I_{j,k} = \lambda_{0,j} + \lambda_{1,j}C_{j,k-1} + \lambda_{2,j}I_{j,k-1} + \lambda_{3,j}G_{j,k-1} \qquad (15)$$

$$+ \lambda_{4,j}\hat{G}^{d}_{j,k-1} + \lambda_{5,j}DEBT_{k-1} + \lambda_{6,j}\omega_{2,j,k-1} \qquad j = 1, \ldots, 5$$

where

$$z_{j,k} = \frac{1}{1 - \gamma_{j,k}b_{j,k}(1 + \theta_{j,k}f_{j,k})} \qquad (16)$$

$$\delta_{0,j} = z_{j,k}[\gamma_{j,k}(a_{j,k} + b_{j,k}\theta_{j,k}e_{j,k}) + \gamma_{j,k}b_{j,k}(n_{0,j,k} + \phi\,\pi_k DEBT_0)]$$

$$\delta_{1,j,k} = \frac{z_{j,k}[1-\gamma_{j,k} - \gamma_{j,k}b_{j,k}(1-\gamma_{j,k} + \theta_{j,k}f_{j,k} - \gamma_{j,k}b_{j,k}\theta_{j,k}f_{j,k})]}{(1-\gamma_{j,k}b_{j,k})}$$

$$\delta_{2,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}(1-\theta_{j,k}); \qquad \delta_{3,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}\phi_{j,k}$$

$$\delta_{4,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}(1-\phi_{j,k}); \qquad \delta_{5,j,k} = -z_{j,k}\gamma_{j,k}b_{j,k}\phi_{j,k}\pi_{j,k}$$

$$\lambda_{0,j,k} = z_{j,k}\theta_{j,k}[e_{j,k} - e_{j,k}\gamma_{j,k}b_{j,k} + f_{j,k}\gamma_{j,k}a_{j,k} + f_{j,k}\gamma_{j,k}b_{j,k}(n_{0,j,k} + \phi_{j,k}\pi_k DEBT_0)]$$

$$\lambda_1 = z_{j,k}\gamma_{j,k}\theta_{j,k}f_{j,k}(b_{j,k}-1); \qquad \lambda_{2,j,k} = z_{j,k}(1-\gamma_{j,k}b_{j,k})(1-\theta_{j,k})$$

$$\lambda_{3,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}\theta_{j,k}f_{j,k}\phi_k \qquad \lambda_{4,j,k} = z_{j,k}\gamma_{j,k}b_{j,k}\theta_{j,k}f_{j,k}(1-\phi_k)$$

$$\lambda_{5,j,k} = -z_{j,k}\gamma_{j,k}b_{j,k}\theta_{j,k}f_{j,k}\phi_{j,k}\pi_k$$

The trend process for government purchases, $\hat{G}^{d}_{j,k}$, at each frequency range is defined by inserting the actual current level of government purchases into the following state equation, where $\rho_{1,j}$ is the growth coefficient.

$$\hat{G}^{d}_{j,k} = \rho_{1,j}G_{j,k-1} + \rho_{2,j}\omega_{3,j,k-1} \qquad j = 1, \ldots, 5 \qquad (17)$$

The aggregate levels of consumption, investment, and government purchases in the model are recovered from the time-frequency decomposition as follows. Applying equation (1) shows that the modified smooth expresses the trend residual for the original data after all five of the crystals from each frequency range have been removed, so that

$$S_{C5,k} = C_k - d_{C,1,k} - d_{C,2,k} - d_{C,3,k} - d_{C,4,k} - d_{C,5,k} \qquad (18)$$

$$S_{I5,k} = I_k - d_{I,1,k} - d_{I,2,k} - d_{I,3,k} - d_{I,4,k} - d_{I,5,k} \qquad (19)$$

$$S_{G5,k} = G_k - d_{G,1,k} - d_{G,2,k} - d_{G,3,k} - d_{G,4,k} - d_{G,5,k} \qquad (20)$$

The model is closed by equations (21) through (29). Recall that the prevailing consumption $C_{j,k}$ at each frequency range (j = 1, …, 5) consists of both the crystal $d_{C,j,k}$ and the modified smooth, $S_{C5,k}$, where the contributions to consumption from the other frequency ranges have been removed. Equation (21) shows that aggregate consumption can be found by summing the consumption levels at each frequency range, and then subtracting the modified smooth out four times. The aggregate consumption level in equation (21) is an algebraically identical to the identity relationship given by equation (18), where actual aggregate consumption C_k during each period k is the sum of the five consumption crystals $d_{C,j,k}$ at each frequency plus the modified smooth, $S_{C5,k}$. This construction of the aggregate variables follows the same procedure for private investment (I_k) and government purchases (G_k), as shown in equation (21).

$$C_k = \sum_{j=1}^{5} C_{j,k} - 4S_{C5,k} \quad I_k = \sum_{j=1}^{5} I_{j,k} - 4S_{I5,k} \quad G_k = \sum_{j=1}^{5} G_{j,k} - 4S_{G5,k} \tag{21}$$

The modified smooth trend processes for consumption, investment, and government purchases are modeled as first-order difference equations in (22) – (24).

$$S_{C5,k} = s_{C,1} S_{C5,k-1} + s_{C,2} C_{k-1} + s_{C,3}\, \omega_{4,k-1} \tag{22}$$

$$S_{I5,k} = s_{I,1} S_{I5,k-1} + s_{I,2} I_{k-1} + s_{I,3}\, \omega_{5,k-1} \tag{23}$$

$$S_{G5,k} = s_{G,1} S_{G5,k-1} + s_{G,2} G_{k-1} + s_{G,3}\, \omega_{6,k-1} \tag{24}$$

In equations (22) – (24), the coefficients on the lagged modified smooth trend variable, and the coefficients on the lagged aggregate consumption, investment, and government spending, produce a weighted average growth contribution toward the current trend values of each series.

$$Y_k = C_k + I_k + G_k + NX_k \tag{25}$$

$$NX_k = n_0 \qquad \text{for all } k = 1, \ldots, K \tag{26}$$

The national income identity is given by equation (25). For the domestic model, equation (26) assumes that the value of net exports NX_k is constant. Net taxes in quarter k are defined as the total government tax and income minus total government transfer payments. Net taxes are assumed to be generated as a constant percentage τ of national output, as shown in equation (27). Government tax income and transfer payments could each be modeled as separate fiscal policy variables. However, as in Kendrick and Shoukry (2013), this analysis will treat them as passively determined variables for simplicity. This is also consistent with findings of Kliem and Kriwoluzky (2014), which show that that there is little evidence in the U.S. for the typical simple fiscal policy rules derived in *DSGE* (Dynamic Stochastic General Equilibrium) models where tax rates respond to output. Thus, the only actively determined fiscal policy variables are government spending over each frequency range, $G_{j,k}, j = 1, \ldots, 5$.

$$T_k = \tau Y_k \tag{27}$$

$$DEF_k = G_k - T_k \tag{28}$$

$$DEBT_k = 0.25\, DEF_k + (1 + i_k) DEBT_{k-1} \tag{29}$$

The resulting government budget deficit (or surplus, if it is negative) in quarter k is given by DEF_k in equation (28). Equation (29) defines the national debt, $DEBT_k$ as the sum of the current budget deficit (converted from annualized rates to quarterly levels) and the previous period debt stock, which is growing at the quarterly interest rate, i_k.

This model in equations (14) through (29) has several advantages. It can be specified with either constant coefficients, as is the case in this paper, or with time varying coefficients. It is also derived from the macroeconomic accelerator framework which has been widely used, such as in Chow (1967), Kendrick (1981), Kendrick and Shoukry (2013), and Hudgins and Na (2014). It also includes a rational expectation component where the level of government debt affects the impact of fiscal policy. Equations (14) and (15) also do not explicitly require the exact background model specification of the constituent equations that led to their final reduced form, since other theoretical underpinnings would also lead to these same two equations. This model is not meant to be directly applied as a complete econometric forecasting model. Its purpose is to simulate the optimal tracking control policy in the time-frequency domain, and thereby show how the technique could be employed within a larger model, such as the 135-equation model in Taylor (1993). Since the deterministic, stochastic, and robust optimal feedback control can all be simulated within this *MODWT* wavelet-based accelerator framework, our model offers considerable insight.

3.1. Optimal Control

The *LQ* tracking problem can be stated as follows. The objective is for the fiscal policymaker to choose the level of government purchases at each of the five frequencies so that it will minimize the quadratic performance index given in (30) subject to the linear state equations given by (14) – (29).

$$
\begin{aligned}
\min_{G_{j,k}} J = & \frac{1}{2}\left[q_{1,f}(C_{K+1} - C^*_{K+1})^2 + q_{2,f}(I_{K+1} - I^*_{K+1})^2\right] \\
& + \frac{1}{2}\left[q_{S,C5,f}(S_{C5,K+1} - S^*_{C5,K+1})^2 + q_{S,I5,f}(S_{I5,K+1} - S^*_{I5,K+1})^2\right] \\
& + \frac{1}{2}\left[\sum_{j=1}^{5} q_{3,j,f}(C_{j,K+1} - C^*_{j,K+1})^2 + \sum_{j=1}^{5} q_{4,j,f}(I_{j,K+1} - I^*_{j,K+1})^2\right] \\
& + \frac{1}{2}\sum_{k=1}^{K}\left[q_{1,k}(C_k - C^*_k)^2 + q_{2,k}(I_k - I^*_k)^2\right] \\
& + \frac{1}{2}\sum_{k=1}^{K}\left[q_{S,C5,k}(S_{C5,k} - S^*_{C5,k})^2 + q_{S,I5,k}(S_{I5,k} - S^*_{I5,k})^2\right] \\
& + \frac{1}{2}\sum_{k=1}^{K}\left[\sum_{j=1}^{5} q_{3,j,k}(C_{j,k} - C^*_{j,k})^2 + \sum_{j=1}^{5} q_{4,j,k}(I_{j,k} - I^*_{j,k})^2\right. \\
& + q_{5,k}(DEF_k - DEF^*_k)^2 + q_{6,k}(DEBT_k - DEBT^*_k)^2 \\
& + \sum_{j=1}^{5} q_{7,j,k}\left[(G_{j,k} - G_{j,k-1}) - (G_{j,k-1} - G_{j,k-1})^*\right]^2 \\
& \left. + q_{8,k}(G_k - G^*_k)^2 + q_{S,G5,k}(S_{G5,k} - S^*_{G5,k})^2 + \sum_{j=1}^{5} r_{j,k}(G_{j,k} - G^*_{j,k})^2\right]
\end{aligned}
\qquad (30)
$$

If the coefficients on the error terms in equations (14), (15), (17), (22), (23), and (24) are set equal to 1, then the model can be simulated as a stochastic *LQG* design, as in Chow (1975), Kendrick (1981), and Kendrick and Shoukry (2013), or as a robust design as in Basar and Bernhard (1991) and Hudgins and Na (2014), or as a mixed H^{∞} /stochastic *LQG* design (Hudgins and Na, 2014). If the disturbance coefficients are zero, then the model becomes deterministic.

This *LQ* tracking problem has five control variables ($G_{j,k}$, $j = 1, \ldots, 5$) and sixty-five state variables. The benefits and drawbacks of the symmetric quadratic performance index for economic and engineering applications need not be discussed here, since they are well known, and have been discussed in Kendrick (1981). The first two terms in (30) penalize the tracking errors for aggregate consumption and investment, respectively, in the final period at the end of the planning horizon. The third and fourth terms specify penalties for the final period tracking errors of the modified smooth trends for consumption and investment. The fifth and sixth terms in (30) penalize the final period tracking errors over each frequency range for consumption and investment. Policymakers will assign higher weights to frequency ranges where that time cycle interval is emphasized. For example, the weighting parameters $q_{3,2,f}$ and $q_{4,2,f}$ on the final respective tracking errors for consumption and investment at frequency 2 will be assigned large values if policymakers are primarily concerned with 1 to 2 year cycles.

The seventh and eighth terms assign penalties for the aggregate consumption and investment tracking errors in each period, and the ninth and tenth terms provide a penalty for the tracking errors for the modified smooth trends for consumption and investment in each period. The eleventh and twelfth set of terms provide penalties for the consumption and investment tracking errors over each individual frequency range for each period. The thirteenth and fourteenth terms penalize the tracking errors associated with the current period budget deficit, and the current period national debt, respectively.

The fifteenth term penalizes the policymaker for large changes in government spending between periods. Hudgins and Na (2014) employ this term as a pragmatic consideration, because government policymakers will prefer more stable spending patterns in the ongoing budget appropriation process, and will not desire large fluctuations from the previous budget. This term reflects the fact that most new budgets are largely designed by adjusting the prevailing budget on a line-by-line basis, which also supports a constant frequency range structure where the resistance to change across each frequency range can be decoupled and penalized separately. Hudgins and Na (2014) demonstrate the substantial additional cost savings in control effort that result when including this term under robust modeling designs, versus the case where government is not penalized for changing its spending above or below the stated quarterly target growth, which our analysis specifies as a rate of $g_{G,j,k}$ = 0.5% (which is about 2% annualized growth) between periods for each of the 5 frequency ranges, $j = 1, \ldots, 5$.

The sixteenth term assigns a penalty for deviations of aggregate government purchases from its target value, and the seventeenth term assigns a penalty for tracking error of the modified smooth trend level of government purchases. The last term represents the control variables, and it provides a penalty for the government purchases tracking error at each of the 5 frequencies.

We transform the *LQ*-tracking problem into a *LQ*-regulator problem by restructuring the state-space equations. Transforming the tracking problem into a regulator problem requires increasing the number of state variables in order to incorporate the targets into the state-space. The problem can be rewritten as an *LQ*-regulator problem with 65 state variables, 65 state equations, and 5 control variables. Although this transformation creates a higher dimensional state-space, it greatly simplifies the subsequent solution procedures for deterministic, stochastic, and H^∞-optimal control problems. This conversion method is similar to that used in Hudgins and Na (2014). Although this is a large scale system, the state-space construction procedures and the accompanying MATLAB program that we have developed have proven to be efficient and feasible to employ. This wavelet-based system framework can easily be adopted within the context of larger base models, such as the model developed by Taylor (1993), although the inclusion of the different frequency ranges would substantially increase the size of the larger econometric models.

There are two methods for handling the constant terms when writing the state-space system in a standard form. Kendrick and Amman (2006) use a vector of ones for the system, and then multiply this by a coefficient matrix that consists of the constants in each of the state equations. This leads to an extra additive matrix term in the system state equation. The present transformation alternatively uses the Hudgins and Na (2014) approach that avoids the use of the additional additive term in the linear matrix state equation. The tradeoff is that this necessitates an additional state variable. The constants can be effectively incorporated into the set of state variables as an additional state variable as follows. Rather than using a vector of ones, equation (31) defines a variable that is a sequences of recurring ones, for all $k = 1, \ldots, K$.

$$c_{k+1} = 1 c_k \quad ; \qquad c_1 = 1 \tag{31}$$

Based on equation (28) and its initial value of 1, $c_k = 1$ for all $k = 1, \ldots, K$. This new state variable has the value of 1 in each period, and thus serves as a placeholder in each state equation, where the coefficient of this c_k variable in each individual equation is the constant. So, the constant terms in equations (14) and (15) become $\delta_{0,j} c_k$ and $\lambda_{0,j} c_k$, respectively, in the state space equations. This analysis only utilizes time invariant coefficients, but the method used in equation (31) also works in the time variant case. The variable c_k will grow or shrink over time if the coefficient in equation (31) is greater than 1, or less than 1, respectively. The constants in equations (14) and (15) would thus grow or shrink over the time horizon in those time variant cases.

The model allows for optimal aggregate consumption, investment, government purchases, and the optimal modified smooth values of each of these to grow at quarterly target rates of $g_{C,k}$, $g_{I,k}$, $g_{G,k}$, $g_{S,C5,k}$, $g_{S,I5,k}$, $g_{S,G,k}$, respectively, that are specified by the fiscal policymaker, which results in an annual growth rate of $[(1 + g_{i,k})^4 - 1]$ per year. The quarterly consumption, investment, and government purchases tracking equations for each frequency range, and the aggregate consumption, investment, and government purchases, along with the modified smooth for each of these can thus be written, respectively, as

$$C^*_{j,k+1}=(1+g_{C,j,k})C^*_{j,k}\,;\quad I^*_{j,k+1}=(1+g_{I,j,k})I^*_{j,k}\,;\quad G^*_{j,k+1}=(1+g_{G,j,k})G^*_{j,k} \tag{32}$$

$$C^*_{k+1}=(1+g_{C,k})C^*_k\,;\quad I^*_{k+1}=(1+g_{I,k})I^*_k;\quad G^*_{k+1}=(1+g_{G,k})G^*_k$$

$$S^*_{C5,k+1}=(1+g_{S,C5,k})S^*_{C5,k}\,;\qquad S^*_{I5,k+1}=(1+g_{S,I5,k})S^*_{I5,k}\,;$$

$$S^*_{G5,k+1}=(1+g_{S,G5,k})S^*_{G5,k}$$

The 65-dimensional state vector is defined as follows:

$$x_k = \left[x_{1,k}\,;\, x_{2,k}\,;\,\ldots\,;\, x_{65,k}\right]^T \tag{33}$$

where

$$\begin{aligned} x_k = [\, & C_{1,k};\, C_{2,k};C_{3,k};C_{4,k};C_{5,k};S_{C5,k} \mid I_{1,k};\, I_{2,k};I_{3,k};I_{4,k};I_{5,k};S_{I5,k} \mid c_k \mid \\ & C^*_{1,k};C^*_{2,k};C^*_{3,k};C^*_{4,k};C^*_{5,k} \mid I^*_{1,k};I^*_{2,k};I^*_{3,k};I^*_{4,k};I^*_{5,k} \mid G^*_{1,k};\, G^*_{2,k};G^*_{3,k};G^*_{4,k};G^*_{5,k} \mid \\ & \hat{G}^d_{1,k};\hat{G}^d_{2,k};\hat{G}^d_{3,k};\hat{G}^d_{4,k};\hat{G}^d_{5,k};S_{G5,k} \mid C_k;I_k;G_k \mid C^*_k;I^*_k;G^*_k \mid NX_k \mid Y_k \mid \\ & T_k;DEF_k;DEBT_k \mid G_{1,k-1};G_{2,k-1};G_{3,k-1};G_{4,k-1};G_{5,k-1} \mid \\ & \mid G_{1,k-1}-G_{1,k-2};G_{2,k-1}-G_{2,k-2};G_{3,k-1}-G_{3,k-2};G_{4,k-1}-G_{4,k-2};G_{5,k-1}-G_{5,k-2} \mid \\ & (G_{1,k-1}-G_{1,k-2})^*;(G_{2,k-1}-G_{2,k-2})^*;(G_{3,k-1}-G_{3,k-2})^*;(G_{4,k-1}-G_{4,k-2})^*; \\ & (G_{5,k-1}-G_{5,k-2})^* \mid DEF^*_k;\, DEBT^*_k \mid S^*_{C5,k};S^*_{I5,k};S^*_{G5,k}\,]^T \end{aligned}$$

Define the control vector so that the elements are difference between the actual and targeted level of government purchases at each frequency:

$$u_k = \left[u_{1,k}\,;\,u_{2,k}\,;\,u_{3,k}\,;\,u_{4,k}\,;\,u_{5,k}\right]^T \qquad u_{j,k}=G_{j,k}-G^*_{j,k} \tag{34}$$

The disturbance vector for stochastic and robust design cases is defined by

$$\begin{aligned} \omega_k = \Big[& \omega_{1,1,k};\,\omega_{1,2,k};\,\omega_{1,3,k};\,\omega_{1,4,k};\,\omega_{1,5,k} \mid \omega_{2,1,k};\,\omega_{2,2,k};\,\omega_{2,3,k}; \\ & \omega_{2,4,k};\omega_{2,5,k} \mid \omega_{3,1,k};\,\omega_{3,2,k};\,\omega_{3,3,k};\,\omega_{3,4,k};\,\omega_{3,5,k};\,\omega_{4,k};\,\omega_{5,k};\,\omega_{6,k}\Big]^T \end{aligned} \tag{35}$$

The disturbance vector is 0 for the deterministic case.

Since the control variables, $u_{j,k}$, $j = 1, \ldots, 5$, include the negative of the targeted levels of government purchases at each frequency, $G^*_{j,k}$, these target variables are added to state equations 1 – 5 for the individual frequencies of consumption, and to the five frequency equations 7 – 11 for investment. The net of effect of adding and subtracting the same variable is 0, but this allows the problem to be written in standard *LQ*-regulator format. Once the optimal control has been simulated to produce the values for $u_{j,k}$ over each frequency range, the target level of government purchases, $G^*_{j,k}$, will have to be added to $u_{j,k}$ in order to recover the values for government purchases, $G_{j,k}$ over each frequency range. However, these values are also automatically recovered with one lag in state equations 46 – 50 by adding the target values to the state values of government purchases.

The matrix state-space equation system can be written as follows, where the 65 state equations are given in Appendix 1.

$$x_{k+1} = A_k x_k + B_k u_k + D_k \omega_k \tag{36}$$

$dim\, x = (65, 1)$	$dim\, u = (5, 1)$	$dim\, \omega = (18, 1)$
$dim\, A = (65, 65)$	$dim\, B = (65, 5)$	$dim\, D = (65, 18)$

3.2. Transformed Deterministic Regulator Design

Consider the deterministic *LQ*-regulator problem where the disturbance vector is zero, or $\omega_k = 0$, or alternatively, where the disturbance coefficient vector is $D_k = 0$. After rewriting expression (30) based on the state space system in (36), the objective is to minimize the performance index

$$\min_u J(u) = x_{K+1}^T Q_f \, x_{K+1} + \sum_{k=1}^{K} \left[x_k^T Q_k x_k + u_k^T R_k u_k \right] \tag{37}$$

subject to

$$x_{k+1} = A_k x_k + B_k u_k \quad ; \quad x(1) = x_1 \tag{38}$$

where the size of the penalty weighting matrices are

$dim\, Q_f = (65, 65) \quad dim\, Q_k = (65, 65) \quad dim\, R_k = (5, 5)$

The solution to the *LQ* regulator problem is found by computing the recursive equations (36) and (37) offline in retrograde time.

$$F_k = \left(B_k^T P_{k+1} B_k + R_k \right)^{-1} B_k^T P_{k+1} A_k \tag{39}$$

$$P_k = Q_k + A_k^T P_{k+1}\left(A_k - B_k F_k\right); \qquad P_{k+1} = Q_f \tag{40}$$

These recursive equations are much simpler to compute than the longer recursive equations employed by Chow (1975), Kendrick (1981), Amman (1996), and others that arise when solving the *LQ*-tracking problem. Using the values computed in (39) and (40), the unique optimal feedback control policy is computed in forward time by

$$u_k^* = -F_k x_k \tag{41}$$

The optimal closed-loop state trajectory is given by

$$x_{k+1} = \left(A_k - B_k F_k\right) x_k \; ; \qquad x(1) = x_1 \tag{42}$$

The control equations in (39) and (40) are the same for the stochastic *LQG* form of the model with perfect state information. The state variable trajectory, however, would be calculated by equation (36), rather than (42). The control vector in equation (41) would then be computed by using equations (39) and (36).

4 Estimation and Simulation

The crystals and smoothed trend values from the *MODWT* decomposition were used to run *OLS* regressions in order to obtain the estimated coefficients for consumption, investment, and government purchases in equations (14), (15), and (17), respectively. The regression results with *t-statistics* are given in tables 1 – 5.

The model procedure assumes that the economy would have some reaction to any announced, consistent government policy regime. Since no such control policy has yet been historically implemented, there was no past distinction between government spending under the optimal control policy, and the spending trajectory along the current cycle trajectory that reflects existing expectations. The rational reactions involving an adjustment for government debt under the lack of any announced consistent policy are zero, since there was no such policy against which to react. Thus, the lag of the current government spending trend variable and the lag of government debt variables are not included in equations (14) and (15). Instead, the values for these coefficients are assigned and evaluated under different scenarios in control system policy simulations.

Tables 2 – 4 show that the estimated equations for all of the consumption, investment, and government purchases equations over each frequency range have a good fit. All of the coefficients have the expected sign, and almost all of the coefficients are statistically significant. The consumption equation coefficients in table 2 show that both investment and government purchases have a crowding-in effect on consumption. The investment coefficients in table 3 show that consumption has a crowding-in effect, but government purchases has a crowing-out effect on investment. The government purchases coefficients in table 4 shows that the average quarterly growth rate is close to .005 per quarter (about 2% per year) at all frequencies.

Table 2

Estimated coefficients for $C_{j,k}$ at each frequency

j	$Constant_j$	Coefficient $C_{j,k-1}$	Coefficient $I_{j,k-1}$	Coefficient $G_{j,k-1}$	R^2
1	6.6655	0.9890	0.0336	0.0292	
t-statistic	*0.5522*	*106.6919*	*1.5088*	*1.1829*	0.9998
2	1.1387	0.9765	0.0677	0.0492	
t-statistic	*0.0839*	*92.8565*	*2.6425*	*1.7681*	0.9997
3	-0.3681	0.9737	0.0761	0.0534	
t-statistic	*-0.0263*	*90.2204*	*2.8132*	*1.8739*	0.9997
4	-5.2582	0.9761	0.0625	0.0578	
t-statistic	*-0.3573*	*89.4979*	*2.2220*	*1.9609*	0.9997
5	-12.5060	0.9706	0.0739	0.0720	
t-statistic	*-0.8240*	*93.9012*	*17.9697*	*2.4360*	0.9997

Table 3

Estimated coefficients for $I_{j,k}$ at each frequency

j	$Constant_j$	Coefficient $C_{j,k-1}$	Coefficient $I_{j,k-1}$	Coefficient $G_{j,k-1}$	R^2
1	18.5121	0.0335	0.9028	-0.0480	
t-statistic	*1.1006*	*2.5946*	*29.1273*	*-1.3930*	0.9922
2	23.9509	0.0456	0.8699	-0.0673	
t-statistic	*1.3290*	*3.2659*	*25.5620*	*-1.8225*	0.9911
3	23.8170	0.0560	0.8351	-0.0767	
t-statistic	*1.2585*	*3.8329*	*22.7990*	*-1.9890*	0.9905
4	17.8292	0.0644	0.7973	-0.0745	
t-statistic	*0.8728*	*4.2509*	*20.4059*	*-1.8201*	0.9895
5	17.5519	0.0449	0.8584	-0.0542	
t-statistic	*0.8329*	*3.1313*	*24.8021*	*-1.3207*	0.9883

Table 4

Estimated coefficients for $G_{j,k}$ at each frequency

j	Coefficient $G_{j,k-1}$	R^2
1	1.0049	
t-statistic	*1193.6343*	0.9998
2	1.0049	
t-statistic	*1045.6836*	0.9998
3	1.0048	
t-statistic	*979.9708*	0.9997
4	1.0046	
t-statistic	*960.2556*	0.9997
5	1.0045	
t-statistic	*921.3503*	0.9997

Table 5

Estimated coefficients for the *modified smooth* trend residuals for *Consumption*, *Investment*, and *Government Purchases* at each frequency range

	$s_{C,1}$	$s_{C,2}$	R^2
$S_{C5,k}$	0.8927	0.1133	
t-statistic	*48.8170*	*6.2208*	0.9999

	$s_{I,1}$	$s_{I,2}$	R^2
$S_{I5,k}$	0.8194	0.1861	
t-statistic	*37.5588*	*8.5257*	0.9967

	$s_{G,1}$	$s_{G,2}$	R^2
S_{G5}	0.8609	0.1441	
t-statistic	*46.0681*	*7.7119*	0.9997

Table 5 gives the paths for the modified smooth trends after extracting the crystals from all 5 frequency ranges for consumption, investment, and government purchases, as specified in equations (22), (23), and (24), respectively. The summation of the two coefficients in each of the equations forms a weighted average trend growth rate. In consumption trend series equation, the coefficient on the lagged value of the series is $s_{C,1}$ = 0.89, which is much larger than coefficient on the lagged value of aggregate consumption, given by $s_{C,2}$ = 0.11. This pattern holds for the investment and government purchases modified smooth trend series, where the coefficients on the lagged value of both series is over 0.8, while the coefficients on the lagged aggregate investment and aggregate government purchases are less than 0.2. All three equations obtain a good fit, with statistically significant coefficients.

4.1 Simulation Analysis

The simulations define the initial values for the state variables in period 1 to correspond to the U.S. (annualized) data in 2012, quarter 3, measured in billions of \$2005. Net exports are set at a constant value of $n_0 = -420$. The initial values for net taxes, the government deficit, and stock of government debt are given respectively by $T_0 = 2{,}166$, $DEF_0 = 1{,}200$, and $DEBT_0 = 16{,}339$. The quarterly interest rate on the debt is set at $i_0 = .0025$, which is 1% per year. Following Kendrick and Shoukry (2013), the tax rate as a percentage of national income is fixed at $\tau_0 = 0.16$. In equation (10), the weight for the current level of government purchases in the expectation formation equation is set at $\phi_{j,k} = 0.90$ and the parameter weight for the adjustment for the national debt differential in the expectation is set at $\pi_{j,k} = 0.0005$ for all frequency ranges in all periods.

The simulations fix the state variables at their initial values at period $k = 1$. Recall from equations (36) – (42), that the government chooses the optimal level of government spending at each frequency range, $j = 1, \ldots, 5$, starting in period $k = 1$. So, the level of government spending in period $k = 1$ has its first effect on the state variables in period $k = 2$. In the shorter 2-year planning simulations, the policymaker selects the final values for government spending in quarter $K = 8$, which determines the values of the state variables at the end of the planning horizon in quarter $K + 1 = 9$. In the longer 4-year simulations, optimal government spending in quarter $K = 16$ determines the level of consumption, investment, and the other state variables in period $K + 1 = 17$.

Since the U.S. was in a recessionary phase over the period 2008 – 2012, the simulations set the initial target for each frequency of government purchases in period 0, and have targeted aggregate government purchases in period 0 to be 1% below the level of actual aggregate government purchases. The quarterly target growth rate for government purchases and its modified smooth are set at $g_{G,j,k} = g_{S,G5,k} = 0.005$, which is about 2% per year. The consumption and investment values at each frequency and the aggregate levels are initially set at their 2012 quarter 3 levels. The initial targets for consumption and investment are set to be 1% above the aggregate consumption and investment levels, respectively. These same initial targets are also respectively assigned to the decomposed series for consumption and investment at each frequency, $j = 1, \ldots, 5$. The quarterly target rates for aggregate and decomposed consumption and investment at all frequency ranges $j = 1, \ldots, 5$, are all fixed, respectively at $g_{C,k} = g_{I,k} = g_{C,j,k} = g_{I,j,k} = g_{S,C5,k} = g_{S,I5,k} = 0.0075$, which is about 3% per year. These growth rates for consumption and investment are used in Kendrick (1981) and Hudgins and Na (2014), whereas Kendrick and Shoukry (2013) use a higher target quarterly growth rate for national output of 0.015, as it progresses from a deep recession in 2008 through a recovery phase.

The scenario presents a tradeoff for the policymaker. The targeted levels and growth rates for consumption and investment are above their current values due to the recession. Since the initial values are higher than they were during the economic trough, the initial crystals for consumption and investment are positive at each frequency range, leading to initial values for the decomposed series that are above there modified smooth values for all frequency ranges, with the exception of investment at frequency range 2 (1 to 2 years). Although the government desires and expansionary policy to achieve its consumption and investment targets, the desired government target spending level is lower than current spending, and the target growth rate for government spending is lower than that for consumption and investment. Additionally, the government is penalized for large changes in spending between periods. The simulated trajectories reflect the optimal balance over each period of the planning horizon.

4.2 Identical Penalty Weights on the Tracking Errors for Each Frequency Range

We simulate the model for two categories of objectives. The first set of simulations analyzes the wavelet model's behavior when the penalty parameters on the tracking errors over each frequency range and the modified smooth are the same. The control tracking error penalty parameter weights for government purchases are identically set at $r_{j,k} = 1$ for each of the frequency ranges, $j = 1, \ldots, 5$, for all periods $k = 1, \ldots, K$. The penalty on the tracking error for the government spending modified smooth is also set at $q_{S,G5,k} = 1$ for each period.

The penalty weights for the state tracking errors on consumption and investment and their modified smooth trends are set at $q_{3,j,k} = q_{4,j,k} = q_{S,C5,k} = q_{S,I5,k} = 0.2$ for each of the frequency ranges, $j = 1, \ldots, 5$, in all periods $k = 1, \ldots, K$. The final tracking error parameters for consumption and investment at each frequency range and for their respective modified smooth trends are set at $q_{3,j,f} = q_{4,j,f} = q_{S,C5,f} = q_{S,I5,f} = 2$. The government deficit and debt tracking errors are set at $q_{5,k} = q_{6,k} = 0.20$, respectively, for all periods, and the penalty parameters for changes in government spending are set at $q_{7,j,k} = 0.2$ for all periods. The penalty parameters on aggregate consumption and investment tracking errors are set to 0 in all periods. Although the MATLAB program that we have developed allows the user to input any number of periods for the planning horizon, the first group of simulations set the number of periods at $K = 8$ quarters, so that the horizon runs from the initial period $k = 1$ to the final period $K + 1 = 9$.

Note that only the relative values of the parameter weights affect the optimal control policies, whereas the absolute numbers of the parameters are irrelevant. The relative weights are consistent with the range values utilized in previous literature. Kendrick (1981) uses a final state tracking error weight that is 100 times greater than the weights for the state and control tracking errors in all other periods. Kendrick and Shoukry (2013) uses a relative penalty weight for the state tracking errors that is 10 times that of the control tracking errors, and Hudgins and Na (2014) consider a weighting scheme with a final aggregate tracking error penalty weight that is 10 times greater than the state and control tracking error weights in all other periods.

After simulating the wavelet-based model, we explore an aggregate version of the model where there is no frequency decomposition. The aggregate model re-estimates equations (14) and (15) for aggregate consumption and investment where there is no frequency decomposition. Then, the initial values and parameters at each frequency are set to be identical, so that the levels of consumption, investment, and government purchases at each frequency are the same as their respective aggregate values. This allows for a direct comparison of wavelet-based model under frequency decomposition, and the aggregate model with no decomposition. As in the wavelet model, the aggregate model assigns a final penalty weight for the tracking errors in consumption and investment that is twice as large as the weight on government purchases, and is 10 times the weight assigned for all tracking errors for every other period, and for each of the other variables. The trajectories for government purchases, consumption, and investment are shown in Figures 5 – 7, respectively.

Figure 5

Government Purchases Optimal Forecast Trajectories

Identical penalty weights on the tracking errors for each frequency

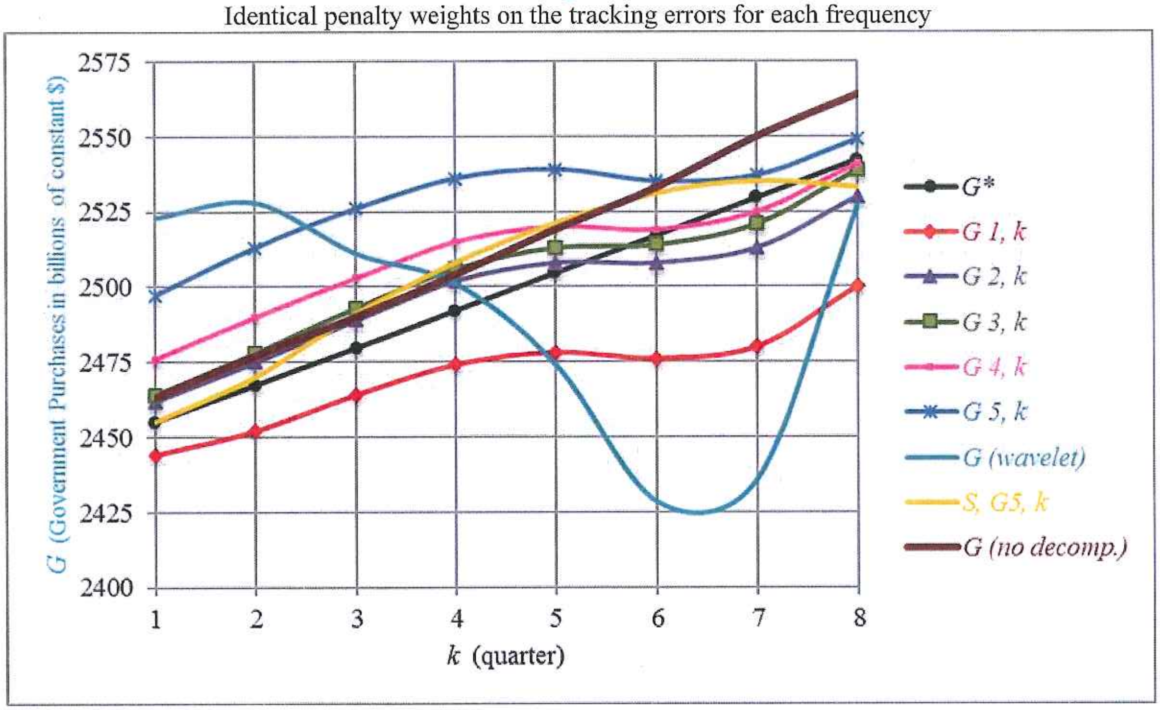


Figure 5 graphs the target values for government purchases (G^*), and the government spending level at each of the five frequency ranges and the modified smooth trend spending level. The *G (wavelet)* curve shows the aggregate level of government purchases that results under the optimal control scheme with wavelet decomposition, and the *G (no decomp.)* shows the total government purchases that will result from using optimal control in the aggregated model with no wavelet decomposition. Figure 5 shows that government spending is optimally structured so that it is the most aggressive at the longest frequency range $j = 5$, which corresponds to projects that impact the cycle at 8 – 16 years. This intensity is followed in descending frequency order by frequency ranges 4, 3, 2, and 1. This emphasis is consistent with the findings of Leeper et al. (2010), whose results suggest that effective expansionary government investment should be focused on obtaining relatively high productivity in the long-run, since otherwise, relatively low productivity on government investment will call for a relatively more contractionary fiscal stance. Note, however, that this pattern where spending decreases at successively higher frequencies (shorter time intervals) depends on the initial conditions of all of the variables, and the penalty parameters chosen; other scenarios would lead to different behaviors at the various frequency ranges.

Government spending under the aggregate model with no decomposition consistently grows, and is consistently above the target, whereas the wavelet-based aggregate government spending is more active with greater variation. Figure 5 shows that aggregate government spending is larger during the first three periods when the wavelet decomposition is used, is about the same in period 4, and then falls below the spending under the aggregate model throughout the remainder of the horizon. Spending at frequency 5 is consistently above the modified smooth, while spending at frequency 1 is consistently below the smooth. Spending at the other frequencies starts above smooth, then falls below, and then extends above the smooth in the last period. This shows that the optimal policy can be improved by using the wavelet-based model rather than the aggregate model with no decomposition. Since the aggregate model with no decomposition suffers from a lack of the relevant frequency information in its design, it does not fully capture the system dynamics. In this current scenario, the optimal policy under the wavelet-based model is therefore more flexible than the policy under the aggregate model with no decomposition.

Robust control designs are used to model systems so that the control policies will be stabilizing in the worst cases, where the model or error structures are unknown. The optimal fiscal and/or monetary policies can be either more or less aggressive as the level of uncertainty increases. Figure 5 shows that the more active fiscal policy under the wavelet-based scheme is consistent with the more aggressive control policies that result in some robust designs, including Bernhard (2002), Onatski and Stock (2002), Diebold (2005), Dennis et al. (2009) Hudgins and Na (2014), which find that optimal robust policy is generally more aggressive. Thus, the robust approaches to aggregate models under the worst-case design would move the optimal policy thrust in the correct direction, and would likely improve the performance of the aggregate model versus cases where the aggregate model is placed within a stochastic framework. Nonetheless, the best approach would be to employ the wavelet-based model, which directly captures all of the relevant frequency information within its design.

Some robust designs, such as Zakovic et al. (2007) and Barlevy (2011), find that the optimal policy under the worst-case robust designs is more passive. This would support the Brainard Principle that suggests that increased uncertainty should reduce the level of policy activism (Brainard, 1967). The more active policy under the wavelet-based model results shown in Figure 5 suggest any excessively passive robust policy derived from an aggregate model with no decomposition could be more destabilizing, since it would lead to limited changes in the path of government spending. This would prescribe the opposite of the behavioral alteration that is called for in figure 5 when re-modeling the aggregate model with a wavelet-based approach. The wavelet-based model increases the information contained in the model and thus provides a more complete model for the policymaker. Regardless of the change in intensity that would have resulted from any robust design, the wavelet-based approach reduces the need to rely on robust considerations in the control formulation. Additionally, robust fiscal policies can still be generated for the wavelet model by simulating the worst-case disturbance errors using the minimax methods developed in Hudgins and Na (2014).

It is also insightful to investigate the projected stock of government debt at the end of the planning horizon. In current simulations shown in figure 5, the final debt level is about \$6.4 billion, or about 0.03%, larger in the aggregate model versus the wavelet model. However, when the penalty weights on consumption and investment tracking errors in the final period are increased to be 10 times the control weights, all else constant, the resulting final debt stock in the wavelet model is about \$284 billion (or about 1.6%) larger than the value in the aggregate model with no decomposition. Thus, the debt levels are similar under both models, but the final debt stock can be either larger or smaller in the wavelet model versus the model with no decomposition, depending upon the relative importance that the policymaker assigns to the different tracking errors.

Figure 6

Consumption Optimal Forecast Trajectories

Identical penalty weights on the tracking errors for each frequency range

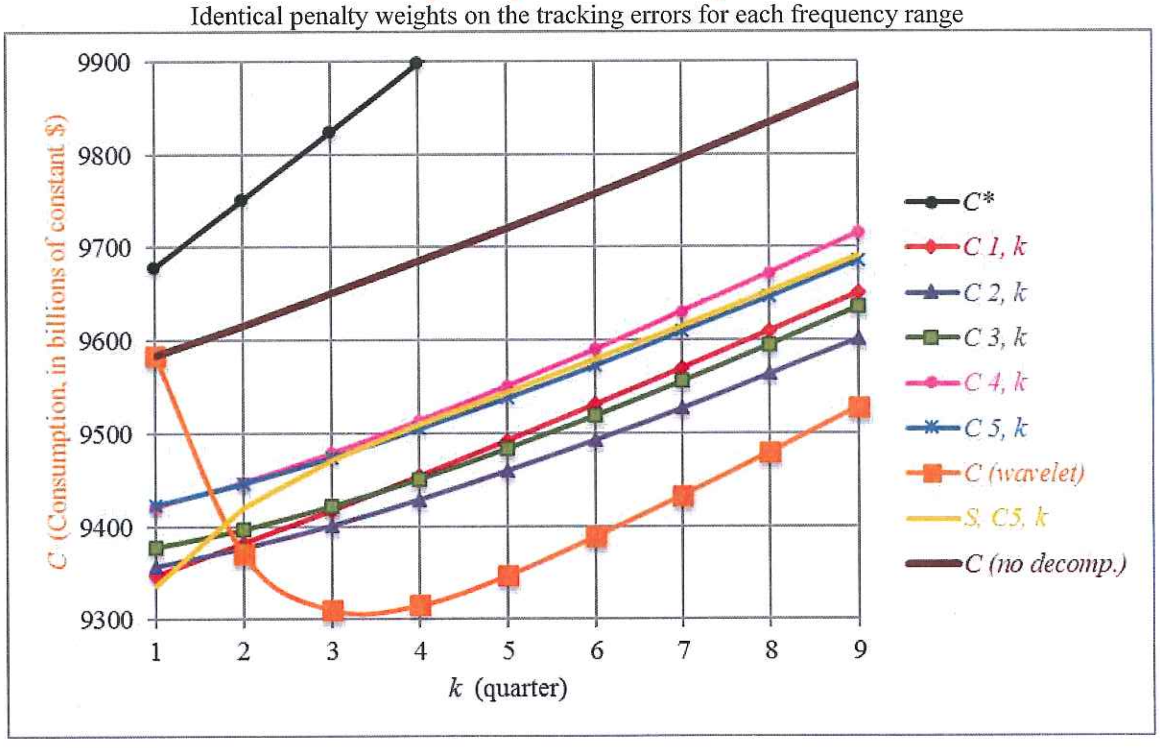


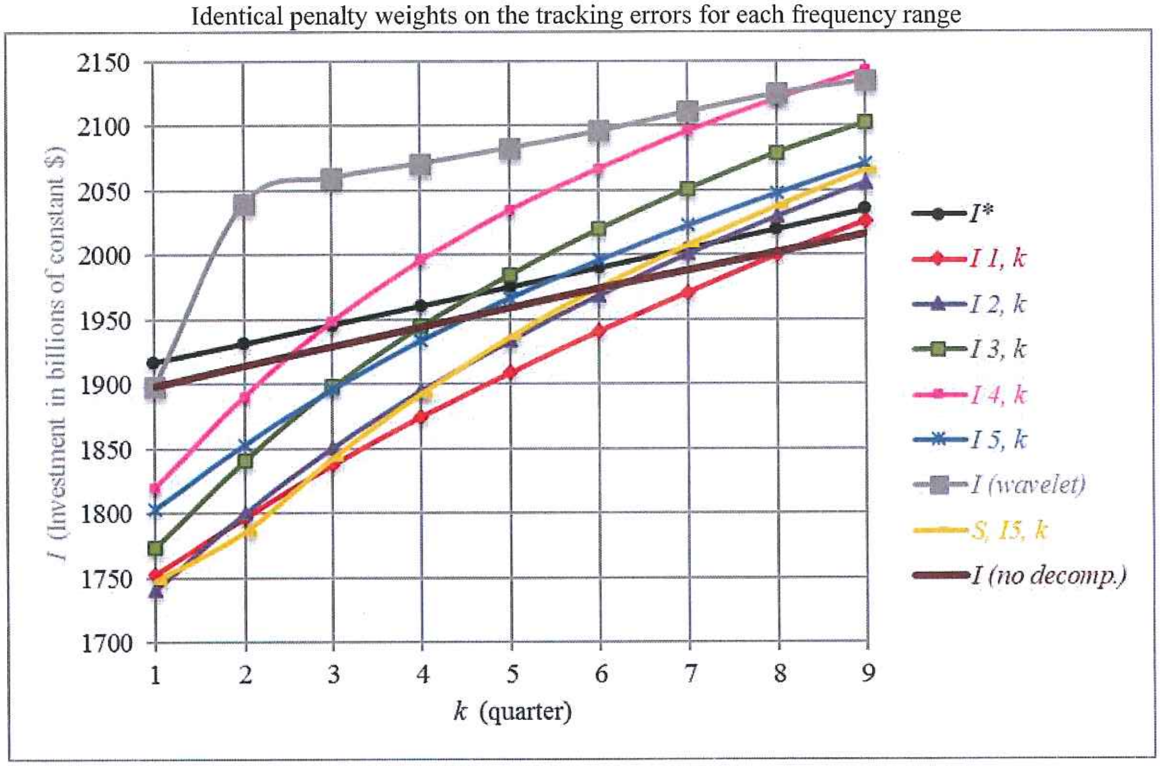
Figure 6 shows the consumption target trajectories and the paths of consumption at each frequency ragne and the modified smooth trend. It also compares the aggregate consumption that would result from the optimal control policy that was employed under wavelet-based with the predicted values for consumption under the aggregate model with no decomposition. Since the model is misspecified in the absence of the wavelet decomposition, the consumption values predicted without frequency decomposition would not have actually resulted. Government spending has a different effect upon consumption at each of the five different frequencies, and the aggregate model fails to account for this. The aggregated model would have incorrectly predicted that higher levels consumption could have been obtained through the utilization of more constantly growing path of government spending. Even with the higher government spending levels during the early periods under the wavelet-based model, aggregate consumption will lag behind the individual frequency ranges during the first few periods before beginning to close the gap at the end of the planning horizon. The fall in aggregate consumption is due to the negative crystals in frequency ranges 1 – 3, which causes consumption over those frequency ranges to fall below the smoothed consumption trend.

Unlike aggregate consumption, figure 7 shows that aggregate investment is larger under the optimal control with wavelet decomposition than its predicted value using optimal control derived from the aggregate model with no wavelet decomposition. The aggregate model would have predicted that aggregate investment would have consistently remained below the target, whereas the wavelet-based model produces investment levels that are consistently above the target, since the large positive crystals at frequencies 3 – 5 are dominating the negative crystals that start to occur after period 4 at frequency ranges 1 and 2.

Figure 7

Investment Optimal Forecast Trajectories

Identical penalty weights on the tracking errors for each frequency range



4.3 FHEC: Long-run Business Cycle Targeting

Although the first set of simulations shows some improvement from utilizing the decomposed form of the model, *FHEC* (Frequency Harmonizing Emphasis Control) extols the primary implementation benefits of the wavelet-decomposed optimal control model. Under *FHEC*, the relative penalty parameter weights in the performance index on the sets of control and state tracking errors are different for each frequency range. Thus, the *FHEC* scheme cannot be compared to an aggregated optimal control policy, which effectively forces the policymaker to use the same relative weight for all frequencies. The second set of simulations in figures 8 – 10 show the trajectories under *FHEC* where the policymaker focuses primarily on influencing the economic cycle and trend growth.

The simulations assume that the fiscal policymaker primarily emphasizes long-run government investment spending at frequency range $j = 5$, which corresponds to an 8 to 16-year time horizon. It views the next priority as being equally distributed between frequency range $j = 4$ (4 to 8 years) and the modified smooth trend, followed by frequency range $j = 3$ (2 to 4 years), with the last priority equally divided between frequency ranges $j = 1$ (6 months to a year) and $j = 2$ (1 to 2 years). The penalty parameter weights for the tracking errors in aggregate consumption and investment, the weights at each frequency range, and the weights for the smooth trends in the final period $K+1$ are set as follows:

$$q_{1,f} = q_{2,f} = 2;\ q_{3,5,f} = q_{4,5,f} = 16;\ q_{3,4,f} = q_{4,4,f} = 4;\ q_{3,3,f} = q_{4,3,f} = 2;$$
$$q_{3,1,f} = q_{3,2,f} = q_{4,1,f} = q_{4,2,f} = 1;\ q_{S,C5,f} = q_{S,I5,f} = 4.$$

These tracking weights for all other periods $k = 1, \ldots, K$ are assigned weights that are ten times less than their respective final penalty weights, so that:

$$q_{1,k} = q_{2,k} = 0.2;\ q_{3,5,k} = q_{4,5,k} = 1.6;\ q_{3,4,k} = q_{4,4,k} = 0.4;\ q_{3,3,k} = q_{4,3,k} = 0.2;$$
$$q_{3,1,k} = q_{3,2,k} = q_{4,1,k} = q_{4,2,k} = 0.1;\ q_{S,C5,k} = q_{S,I5,k} = 0.4.$$

The control tracking error penalty weights for each of the frequency ranges, $j = 1, \ldots, 5$, in all periods $k = 1, \ldots, K$ are as follows: $r_{1,k} = r_{2,k} = r_{3,k} = 1;\ r_{4,k} = r_{5,k} = 2$. The control penalty weights are twice as high for frequency ranges 4 and 5, but the relative weight on consumption and investment errors are still 8 times the control weight for frequency 5, and twice as much for frequency range 4. The penalty weights on all of the other tracking errors in the performance index set at 0.2 for all periods. The simulations are calculated over a 4-year (16-quarter) time horizon.

In figure 8, government spending is consistently the largest at frequency range $j = 5$, followed in descending order by frequency ranges $j = 4$, 3, 2, and 1. Thus, the frequency pattern is still in descending order, as it was in figure 5. However, the smooth is above government spending at frequency ranges 1, 2, and 3 throughout the planning horizon, due to the negative crystals at those frequencies. Government spending at frequency 5 is still considerably above the smooth, and spending at frequency range 4 is also above the smooth trend for all periods.

Aggregate government purchases follow a *sforzando* trajectory, where they are initially large, then decline in periods 2 – 4, and then increase until period 12. Then, aggregate government spending decreases in the next two periods, before finally increasing in the last two quarters. Except for frequency range 5, aggregate government spending is initially larger at than all frequencies. It falls below the smooth in period 4, and below the spending at frequency 4 during period 5, before closely tracking both of these trajectories in the middle periods. After a decrease in period 14, aggregate spending surpassing the smooth and frequency range 4 spending at the end of the horizon. The trajectories in figure 8 provide further pragmatic information for fiscal policymakers than does any aggregate model, since they show the optimal allocation of resources that should be dedicated toward the projects based upon the impact associated with their time respective time cycle horizons. For the long-term business cycle targeting emphasis, the most aggressive spending is occurs on projects that are aimed at stabilizing the longer-term cyclic components of consumption and investment, and the modified smooth trend that is largely driven by productivity.

Figure 8

Government Purchases Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Long-Run Business Cycle Targeting

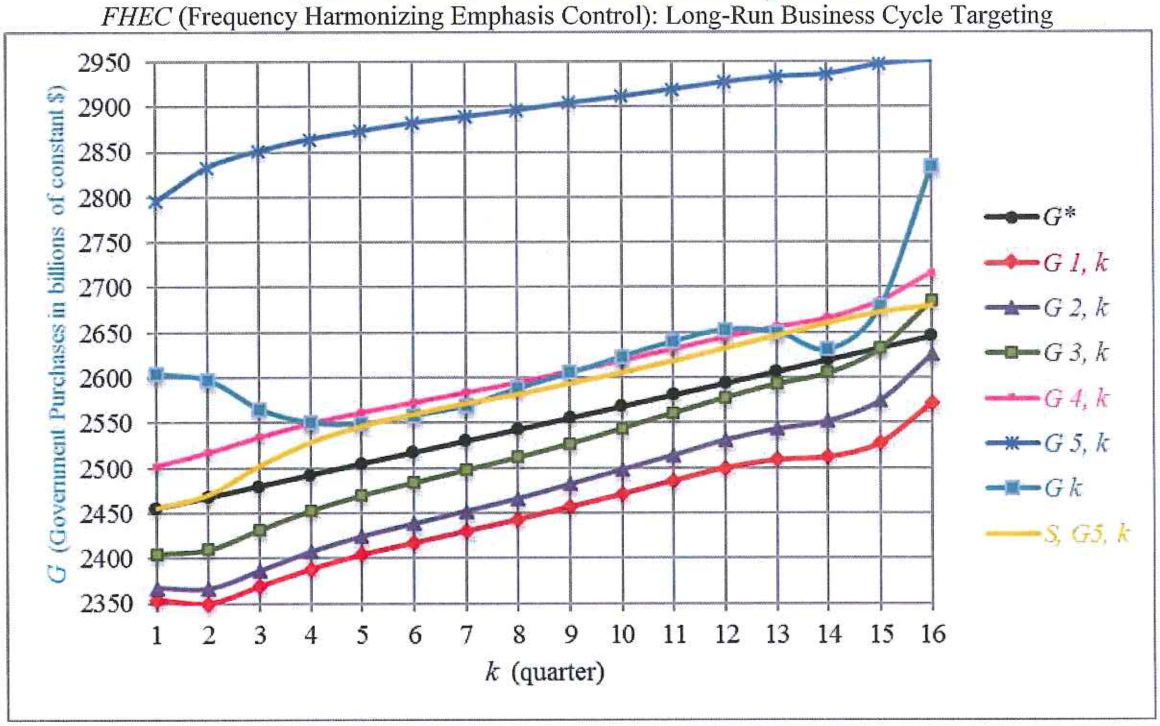


Figure 9 shows the various consumption trajectories under *FHEC* with a long-term emphasis. Consumption is the largest at frequency range 5, followed by frequency 4, and then the smooth trend. The larger consumption pattern at these lower frequencies (long cycles) reflects the heavy emphasis of the business cycle and trend under the long-term targeting scenario. Consumption at frequencies 1 and 3 follow similar growth trajectories, while consumption at frequency 2 is the lowest. As in figure 6, aggregate consumption initially falls in the first two periods, and then begins to recover during the following periods. This trajectory is largely consistent with the predictions of

Kriwoluzky (2012), where consumption initially falls during the preannouncement period, and then recovers after the policy has been implemented. The behavior of the consumption and investment state variables cannot be predicted without computing them in the control system framework, and the trajectories depend upon the initial conditions and the penalty parameters.

Figure 9

Consumption Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Long-Run Business Cycle Targeting

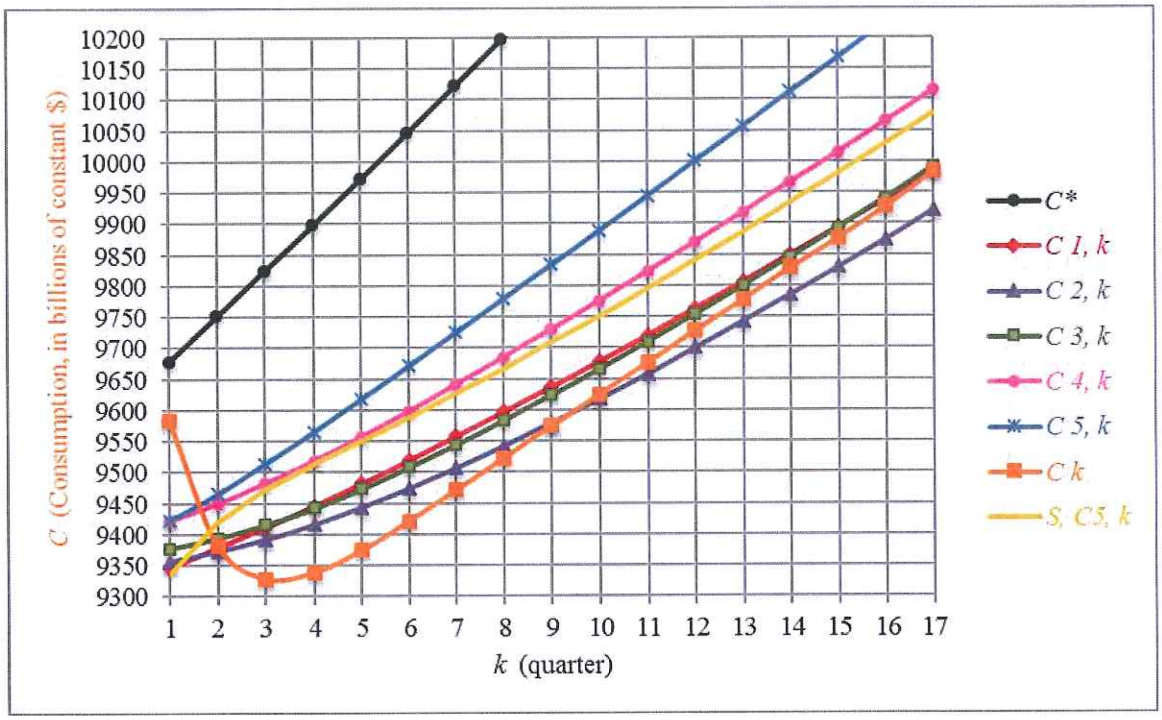
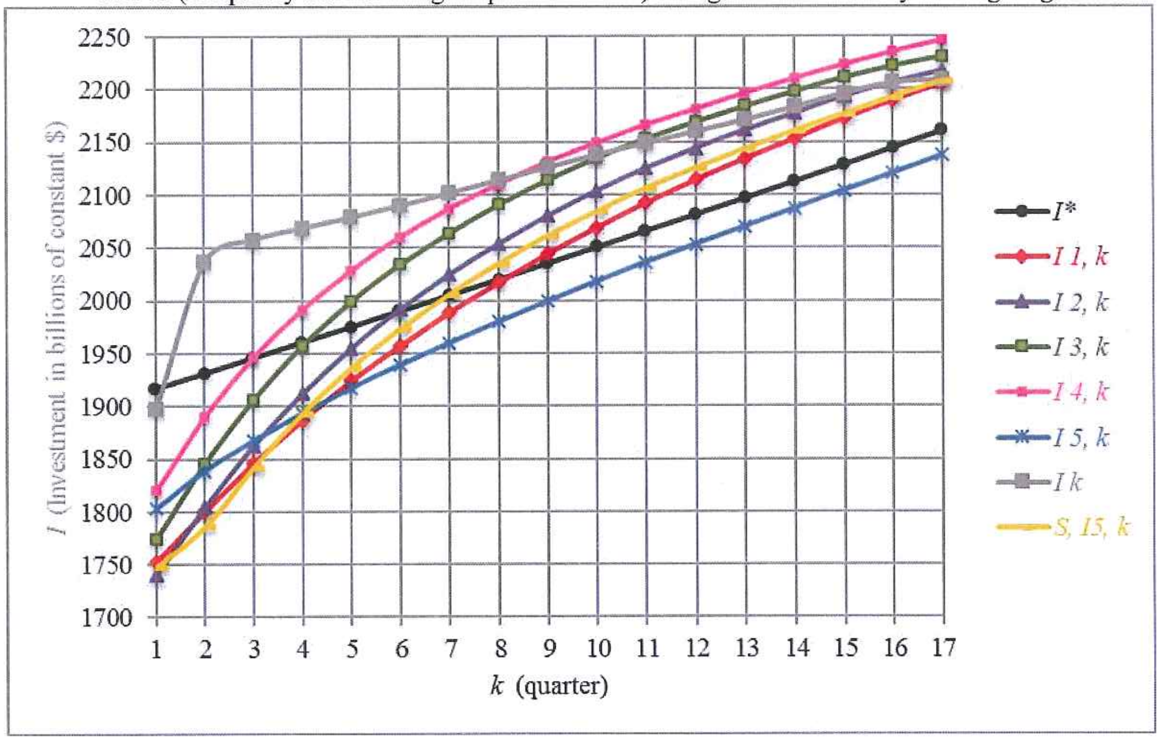


Figure 10 illustrates the investment trajectories under the long-term *FHEC* design. Investment is consistently the largest at frequency range $j = 4$, followed by frequency range 3. Some of the long-term investment at frequency range 5 has been crowded out. Investment levels at frequency ranges 1 and 5 are below the modified smooth trend, while investment frequency ranges 2, 3, and 4 have large positive crystals and consistently lie well above the smooth trend. Thus, aggregate investment is consistently above the smooth, and its growth begins to lag and starts to close the gap with the smooth and the target at the end of the planning horizon.

Figure 10

Investment Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Long-Run Business Cycle Targeting



4.4 FHEC: Political Cycle Targeting

The third set of simulations in figures 11 – 13 show the trajectories under *FHEC* where the fiscal policy is formulated based on the political cycle. Here, the emphasis is primarily on the frequency range j = 4 (4 to 8 years), followed by frequency 3 (2 to 4 years), and then the modified smooth trend. All other frequency ranges are assumed to be a lesser priority, and each of these carry equal emphasis. The penalty parameter weights for the tracking errors in aggregate consumption and investment, the weights at each frequency range, and the weights for the smooth trends in the final period $K + 1$ are set as follows:

$$q_{1,f} = q_{2,f} = 2;\ q_{3,4,f} = q_{4,4,f} = 16;\ q_{3,3,f} = q_{4,3,f} = 4;$$

$$q_{3,1,f} = q_{3,2,f} = q_{3,5,f} = q_{4,1,f} = q_{4,2,f} = q_{4,5,f} = 1;\ q_{S,C5,f} = q_{S,I5,f} = 2.$$

These tracking weights for all other periods $k = 1, \ldots, K$ are assigned weights that are ten times less than their respective final penalty weights, so that:

$$q_{1,k} = q_{2,k} = 0.2;\ q_{3,4,k} = q_{4,4,k} = 1.6;\ q_{3,3,k} = q_{4,3,k} = 0.4;$$

$$q_{3,1,k} = q_{3,2,k} = q_{3,5,k} = q_{4,1,k} = q_{4,2,k} = q_{4,5,k} = 0.1;\ q_{S,C5,k} = q_{S,I5,k} = 0.2.$$

The control tracking error penalty weights for each of the frequency ranges, $j = 1, \ldots, 5$, in all periods $k = 1, \ldots, K$ are as follows: $r_{1,k} = r_{2,k} = r_{5,k} = 1$; $r_{3,k} = r_{4,k} = 2$. The penalty weights on all of the other tracking errors in the performance index set at 0.2 for all periods. As in the previous simulations, both the control and state variable tracking errors are more heavily weighted at the focal frequency ranges.

Figure 11 illustrates that government spending is consistently the largest at frequency range $j = 4$, followed in descending order by frequency ranges $j = 3, 5, 2$, and 1. The smooth increases above spending at frequency range 3 after period 4, and the smooth rises closely tracks aggregate government spending during periods 4 through 8. After period 8, aggregate spending exceeds the smooth, except for period 14. Government spending at frequency 5 closely tracks the target levels. Spending at frequency range 2 is consistently below the target, and spending a frequency 1 is always the lowest. Thus, the fiscal spectrum in figure 11 shows the dominance of thrust placed upon activity at the longer 4 to 8 year political cycle and the shorter 2 to 4 year cycle. Aggregate government spending is initially much lower than in figure 8. After a slight initial decline and a modest upward trend, it rises, but ends the horizon at a lower level than in figure 8.

Figure 11

Government Purchases Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Political Cycle Targeting

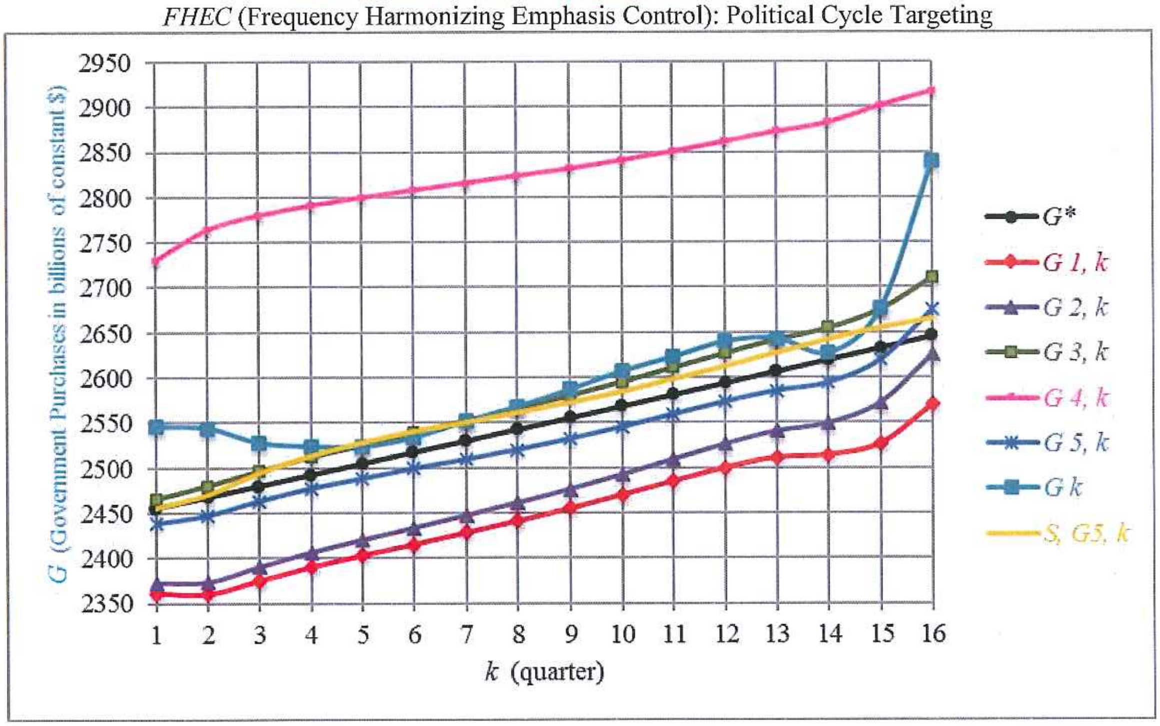
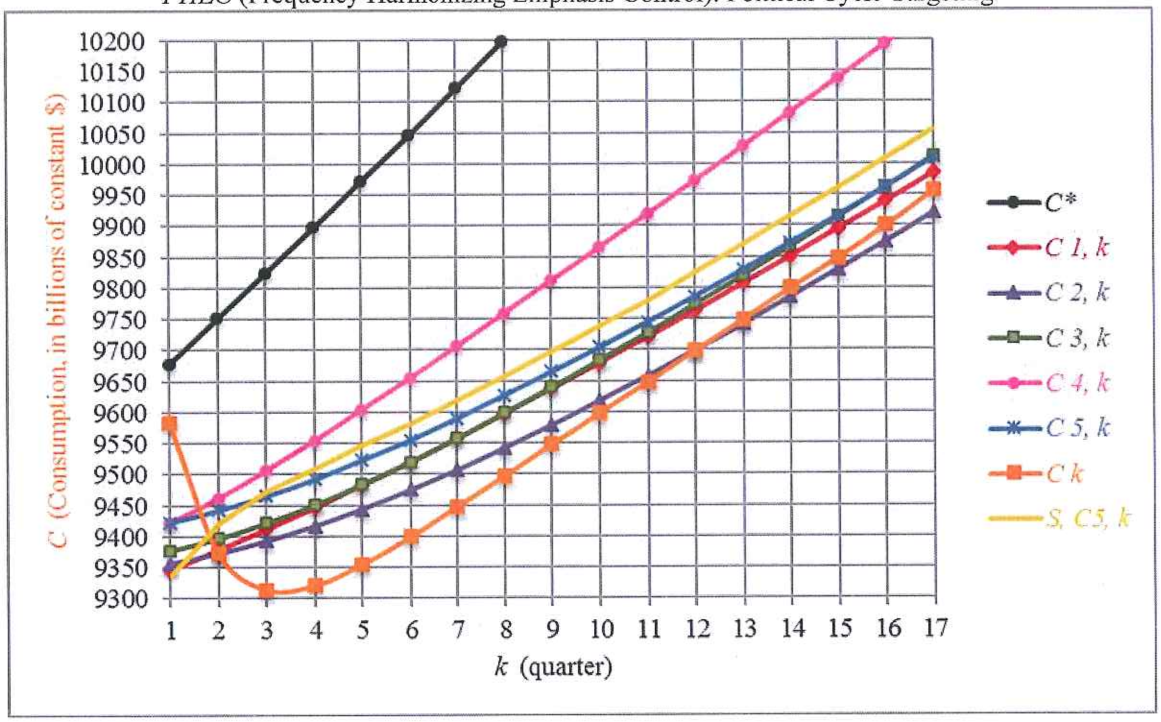


Figure 12 shows that consumption is the largest at the at frequency range $j = 4$ (4 to 8 years), and that trajectory is the only frequency where consumption always exceeds the smooth trend. Consumption at the longest cycle, frequency range 5, falls below the trend after period 3. Consumption at frequency ranges 1 and 3 follow similar paths, and consumption is still the smallest at frequency range $j = 2$. Again, aggregate consumption begins to increase after period 3, and thereafter consistently narrows the gap under its modified smooth trend.

Figure 12

Consumption Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Political Cycle Targeting

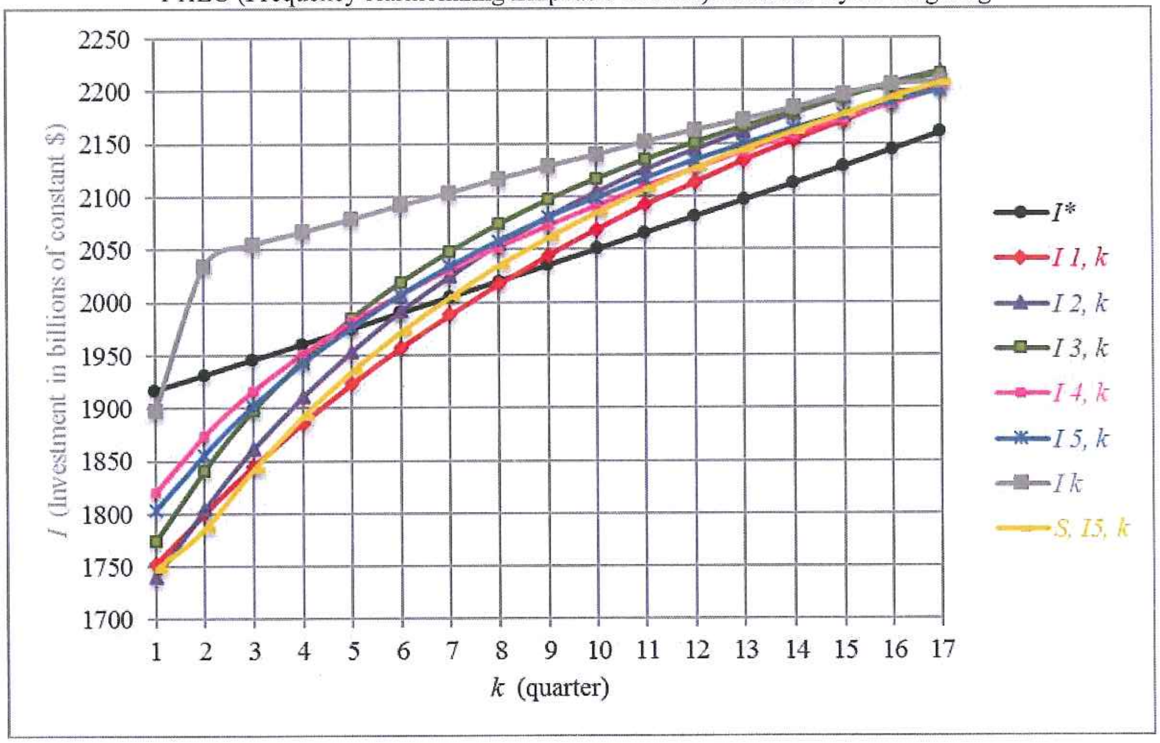


In figure 13, investment is initially the largest at frequency range 4, and consistently lies above the smooth at all frequency ranges, except for frequency 1. After period 5, investment at frequency ranges 3 and 5 surpass investment at frequency 4. The preponderance of positive crystals in the consolidated spectrum forms an aggregate investment trajectory that is consistently above the smooth. At the end of the horizon, the growth in aggregate investment and its levels at each frequency range converge, and slow to approach the target growth rate.

Figure 13

Investment Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Political Cycle Targeting



4.5 FHEC: Short-Term Stabilization Targeting

Figures 14 – 16 simulates a scenario where fiscal policymakers pursue a short-term stabilization policy in an attempt to mediate the center of the cyclic frequency spectrum. The penalty parameter weights for the tracking errors in aggregate consumption and investment, the weights at each frequency range, and the weights for the smooth trends in the final period $K + 1$ are set as follows:

$$q_{1,f} = q_{2,f} = 2;\ q_{3,3,f} = q_{4,3,f} = 16;\ q_{3,2,f} = q_{3,4,f} = q_{4,2,f} = q_{4,4,f} = 4;$$
$$q_{3,1,f} = q_{3,5,f} = q_{4,1,f} = q_{4,5,f} = 1;\ q_{S,C5,f} = q_{S,I5,f} = 2.$$

These tracking weights for all other periods $k = 1, \ldots, K$ are again assigned weights that are ten times less than their respective final penalty weights, so that:

$$q_{1,k} = q_{2,k} = 0.2;\ q_{3,3,k} = q_{4,3,k} = 1.6;\ q_{3,2,k} = q_{3,4,k} = q_{4,2,k} = q_{4,4,k} = 0.4;$$
$$q_{3,1,k} = q_{3,5,k} = q_{4,1,k} = q_{4,5,k} = 0.1;\ q_{S,C5,k} = q_{S,I5,k} = 0.2.$$

As in the previous simulations, both the control and state variable tracking errors are more heavily weighted at the focal frequency ranges, which are now $j = 2$, 3, and 4. The largest weight is placed on frequency range 3 (2 to 4 years). The control tracking error penalty weights for each of the frequency ranges in all periods $k = 1, \ldots, K$ are as follows: $r_{1,k} = r_{5,k} = 1$; $r_{2,k} = r_{3,k} = r_{4,k} = 2$. The penalty weights on all of the other tracking errors in the performance index maintained at 0.2 for all periods.

Figure 14

Government Purchases Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Short-Term Stabilization Targeting

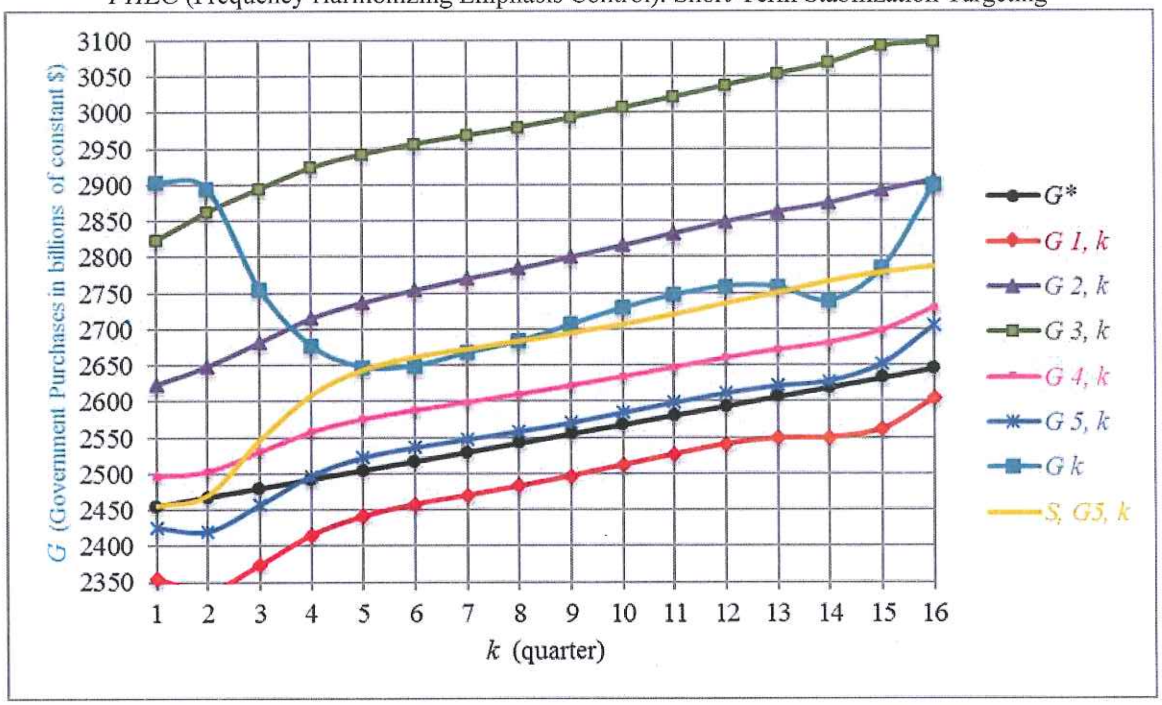


Figure 14 shows that the government spending trajectories consistently are the largest at frequency range 3 (2 to 4 years), followed in descending order by frequency ranges 2, 4, 5, and 1. The modified smooth trend is consistently above the target. Government spending at frequency ranges 2 and 3 is consistently above the smooth trend, and government spending at the other frequencies is below the smooth, except for frequency 4 in the first two periods. Thus, fiscal policy primarily engages the cyclic spending cycles between 1 and 4 years. Aggregate government spending is the most aggressive under this scenario. Although aggregate spending falls in the middle periods, it is consistently larger in each period than it was in each corresponding period in both figure 8 and figure 11.

Figure 15 illustrates the consumption behavior. Consumption at frequency range 1 has the slowest growth rate across the horizon, and falls below the trend after the first period. Consumption at frequency range 3 maintains the fastest growth, followed by consumption at frequency range 4. Both surpass the smooth after period 5, with increasing growth throughout the horizon. The paths of consumption for frequencies 2 and 5 follow almost identical trajectories, and are consistently below the smooth trend. Aggregate consumption has a final value that is larger than it was in figure 9 or figure 12.

Figure 15

Consumption Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Short-Term Stabilization Targeting

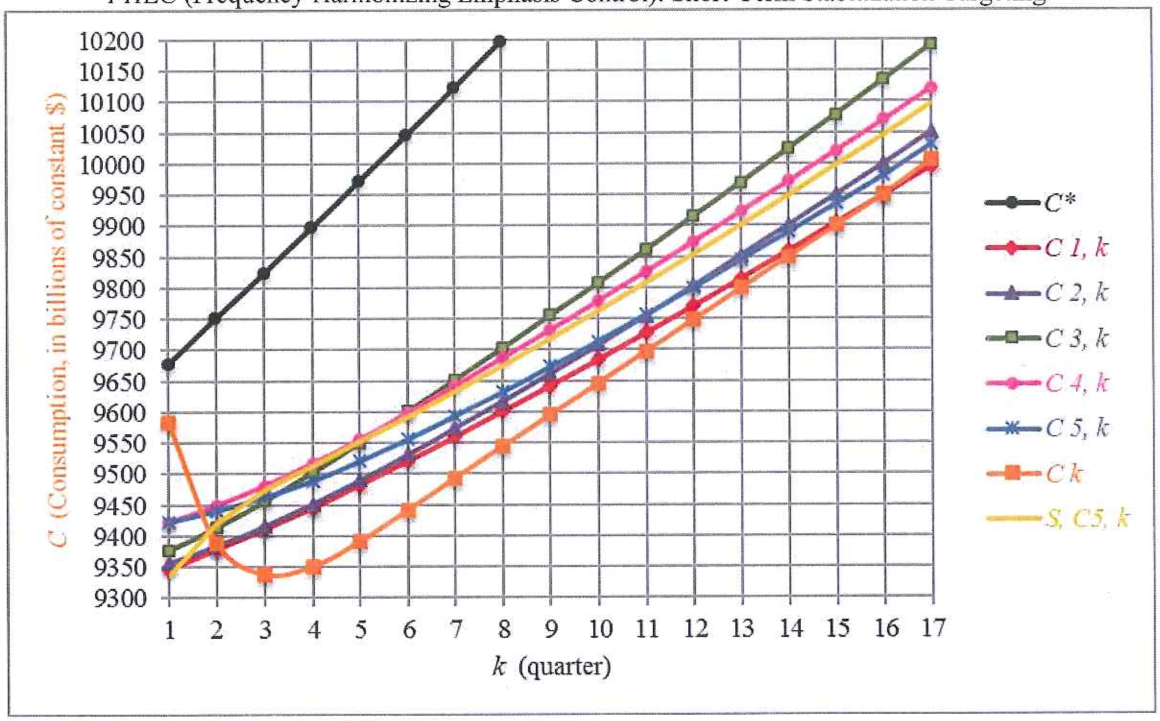
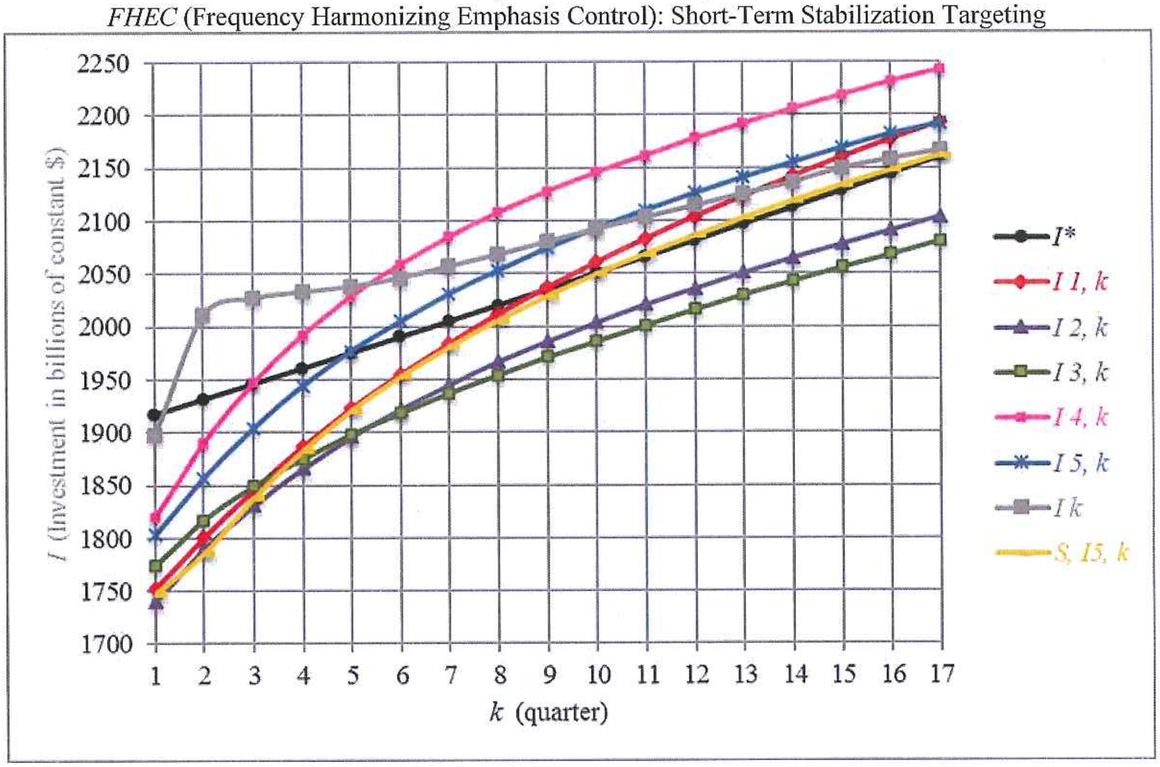


Figure 16

Investment Optimal Forecast Trajectories

FHEC (Frequency Harmonizing Emphasis Control): Short-Term Stabilization Targeting



In figure 16, the investment trajectory at frequency range 4 is consistently the largest, followed be frequency 5. Investment at frequency range 1 and the smooth trend both exceed the target level at the end of the horizon, while investment at frequency ranges 2 and 3 obtain values that are below the target in the last period. The growth in aggregate investment levels off after the second period, and both the final level of aggregate investment and the smooth reaches the target at the end of the horizon.

5 Conclusion

The wavelet-based design explored above provides a pragmatic approach to gain additional insight into macroeconomic control models through time-frequency decomposition. *FHEC* applied within the wavelet-based model provides two primary benefits to policymakers. It allows for more precise predictions regarding the likely impact of various policies, thus improving the predictions of the system. Furthermore, it aids in the actual composition of government spending, since it gives insight into the types of spending that can be optimally targeted, and the frequency ranges in consumption and investment that are likely to be most heavily impacted. The contribution of the methods above are not limited to United States data, since they can be employed using the data from any country.

The model can be extended to analyze the wavelet-based model performance under the *LQG* stochastic error structure, and the robust design with a worst-case disturbance structure. Due to length considerations, the authors are exploring these extensions in future papers. The paper can also be extended to include an exploration of monetary policy through the interest rate, exchange rate, and the price level.

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Appendix A1

The 65 state equations are listed as follows:

$$x_{1,k+1} = \delta_{1,1}x_{1,k} + \delta_{2,1}x_{7,k} + \delta_{0,1}x_{13,k} + \delta_{3,1}x_{24,k} + \delta_{4,1}x_{29,k} + \delta_{5,1}x_{45,k} + \delta_{3,1}u_{1,k} + \delta_{6,1}\omega_{1,1,k}$$

$$x_{2,k+1} = \delta_{1,2}x_{2,k} + \delta_{2,2}x_{8,k} + \delta_{0,2}x_{13,k} + \delta_{3,2}x_{25,k} + \delta_{4,2}x_{30,k} + \delta_{5,2}x_{45,k} + \delta_{3,2}u_{2,k} + \delta_{6,2}\omega_{1,2,k}$$

$$x_{3,k+1} = \delta_{1,3}x_{3,k} + \delta_{2,3}x_{9,k} + \delta_{0,3}x_{13,k} + \delta_{3,3}x_{26,k} + \delta_{4,3}x_{31,k} + \delta_{5,3}x_{45,k} + \delta_{3,3}u_{3,k} + \delta_{6,3}\omega_{1,3,k}$$

$$x_{4,k+1} = \delta_{1,4}x_{4,k} + \delta_{2,4}x_{10,k} + \delta_{0,4}x_{13,k} + \delta_{3,4}x_{27,k} + \delta_{4,4}x_{32,k} + \delta_{5,4}x_{45,k} + \delta_{3,4}u_{4,k} + \delta_{6,4}\omega_{1,4,k}$$

$$x_{5,k+1} = \delta_{1,5}x_{5,k} + \delta_{2,5}x_{11,k} + \delta_{0,5}x_{13,k} + \delta_{3,5}x_{28,k} + \delta_{4,5}x_{33,k} + \delta_{5,5}x_{45,k} + \delta_{3,5}u_{5,k} + \delta_{6,5}\omega_{1,5,k}$$

$$x_{6,k+1} = s_{C,1}x_{6,k} + s_{C,0}x_{13,k} + s_{C,2}x_{35,k} + s_{C,3}\omega_{4,k}$$

$$x_{7,k+1} = \lambda_{1,1}x_{1,k} + \lambda_{2,1}x_{7,k} + \lambda_{0,1}x_{13,k} + \lambda_{3,1}x_{24,k} + \lambda_{4,1}x_{29,k} + \lambda_{5,1}x_{45,k} + \lambda_{3,1}u_{1,k} + \lambda_{6,1}\omega_{2,1,k}$$

$$x_{8,k+1} = \lambda_{1,2}x_{2,k} + \lambda_{2,2}x_{8,k} + \lambda_{0,2}x_{13,k} + \lambda_{3,2}x_{25,k} + \lambda_{4,2}x_{30,k} + \lambda_{5,2}x_{45,k} + \lambda_{3,2}u_{2,k} + \lambda_{6,2}\omega_{2,2,k}$$

$$x_{9,k+1} = \lambda_{1,3}x_{3,k} + \lambda_{2,3}x_{9,k} + \lambda_{0,3}x_{13,k} + \lambda_{3,3}x_{26,k} + \lambda_{4,3}x_{31,k} + \lambda_{5,3}x_{45,k} + \lambda_{3,3}u_{3,k} + \lambda_{6,3}\omega_{2,3,k}$$

$$x_{10,k+1} = \lambda_{1,4}x_{4,k} + \lambda_{2,4}x_{10,k} + \lambda_{0,4}x_{13,k} + \lambda_{3,4}x_{27,k} + \lambda_{4,4}x_{32,k} + \lambda_{5,4}x_{45,k} + \lambda_{3,4}u_{4,k} + \lambda_{6,4}\omega_{2,4,k}$$

$$x_{11,k+1} = \lambda_{1,5}x_{5,k} + \lambda_{2,5}x_{11,k} + \lambda_{0,5}x_{13,k} + \lambda_{3,5}x_{28,k} + \lambda_{4,5}x_{33,k} + \lambda_{5,5}x_{45,k} + \lambda_{3,5}u_{5,k} + \lambda_{6,5}\omega_{2,5,k}$$

$$x_{12,k+1} = s_{I,1}x_{12,k} + s_{I,0}x_{13,k} + s_{I,2}x_{36,k} + s_{I,3}\omega_{5,k}$$

$$x_{13,k+1} = x_{13,k}$$

$$x_{14,k+1} = (1+g_{C,1,k})x_{14,k}$$

$$x_{15,k+1} = (1+g_{C,2,k})x_{15,k}$$

$$x_{16,k+1} = (1+g_{C,3,k})x_{16,k}$$

$$x_{17,k+1} = (1+g_{C,4,k})x_{17,k}$$

$$x_{18,k+1} = (1+g_{C,5,k})x_{18,k}$$

$$x_{19,k+1} = (1+g_{I,1,k})\,x_{19,k}$$
$$x_{20,k+1} = (1+g_{I,2,k})\,x_{20,k}$$
$$x_{21,k+1} = (1+g_{I,3,k})\,x_{21,k}$$
$$x_{22,k+1} = (1+g_{I,4,k})\,x_{22,k}$$
$$x_{23,k+1} = (1+g_{I,5,k})\,x_{23,k}$$
$$x_{24,k+1} = (1+g_{G,1,k})\,x_{24,k}$$
$$x_{25,k+1} = (1+g_{G,2,k})\,x_{25,k}$$
$$x_{26,k+1} = (1+g_{G,3,k})\,x_{26,k}$$
$$x_{27,k+1} = (1+g_{G,4,k})\,x_{27,k}$$
$$x_{28,k+1} = (1+g_{G,5,k})\,x_{28,k}$$
$$x_{29,k+1} = \rho_{0,1}x_{13,k} + \rho_{1,1}x_{24,k} + \rho_{1,1}u_{1,k} + \rho_{2,1}\omega_{3,1,k-1}$$
$$x_{30,k+1} = \rho_{0,2}x_{13,k} + \rho_{1,2}x_{25,k} + \rho_{1,2}u_{2,k} + \rho_{2,2}\omega_{3,2,k-1}$$
$$x_{31,k+1} = \rho_{0,3}x_{13,k} + \rho_{1,3}x_{26,k} + \rho_{1,3}u_{3,k} + \rho_{2,3}\omega_{3,3,k-1}$$
$$x_{32,k+1} = \rho_{0,4}x_{13,k} + \rho_{1,4}x_{27,k} + \rho_{1,4}u_{2,k} + \rho_{2,4}\omega_{3,4,k-1}$$
$$x_{33,k+1} = \rho_{0,5}x_{13,k} + \rho_{1,5}x_{28,k} + \rho_{1,5}u_{5,k} + \rho_{2,5}\omega_{3,5,k-1}$$
$$x_{34,k+1} = s_{G,0}x_{13,k} + s_{G,1}x_{34,k} + s_{G,2}x_{37,k} + s_{G,3}\omega_{6,k}$$
$$\begin{aligned} x_{35,k+1} = {} & \delta_{1,1}x_{1,k} + \delta_{1,2}x_{2,k} + \delta_{1,3}x_{3,k} + \delta_{1,4}x_{4,k} + \delta_{1,5}x_{5,k} - 4x_{6,k} + \delta_{2,1}x_{7,k} \\ & + \delta_{2,2}x_{8,k} + \delta_{2,3}x_{9,k} + \delta_{2,4}x_{10,k} + \delta_{2,5}x_{11,k} + \sum_{j=1}^{5}\delta_{0,j}x_{13,k} + \delta_{3,1}x_{24,k} \\ & + \delta_{3,2}x_{25,k} + \delta_{3,3}x_{26,k} + \delta_{3,4}x_{27,k} + \delta_{3,5}x_{28,k} + \delta_{4,1}x_{29,k} + \delta_{4,2}x_{30,k} \\ & + \delta_{4,3}x_{31,k} + \delta_{4,4}x_{32,k} + \delta_{4,5}x_{33,k} + \sum_{j=1}^{5}\delta_{5,j}x_{45,k} + \sum_{j=1}^{5}\delta_{3,j}u_{j,k} \\ & + \sum_{j=1}^{5}\delta_{6,j}\omega_{1,j,k} \end{aligned}$$

$$\begin{aligned} x_{36,k+1} = {} & \lambda_{1,1}x_{1,k} + \lambda_{1,2}x_{2,k} + \lambda_{1,3}x_{3,k} + \lambda_{1,4}x_{4,k} + \lambda_{1,5}x_{5,k} + \lambda_{2,1}x_{7,k} \\ & + \lambda_{2,2}x_{8,k} + \lambda_{2,3}x_{9,k} + \lambda_{2,4}x_{10,k} + \lambda_{2,5}x_{11,k} - 4x_{12,k} + \sum_{j=1}^{5}\lambda_{0,j}x_{13,k} \\ & + \lambda_{3,1}x_{24,k} + \lambda_{3,2}x_{25,k} + \lambda_{3,3}x_{26,k} + \lambda_{3,4}x_{27,k} + \lambda_{3,5}x_{28,k} + \lambda_{4,1}x_{29,k} \\ & + \lambda_{4,2}x_{30,k} + \lambda_{4,3}x_{31,k} + \lambda_{4,4}x_{32,k} + \lambda_{4,5}x_{33,k} + \sum_{j=1}^{5}\lambda_{5,j}x_{45,k} \end{aligned}$$

$$+ \sum_{j=1}^{5} \lambda_{3,j} u_{j,k} + \sum_{j=1}^{5} \lambda_{6,j} \omega_{2,j,k}$$

$$x_{37,k+1} = x_{24,k} + x_{25,k} + x_{26,k} + x_{27,k} + x_{28,k} - 4x_{34,k} + u_{1,k} + u_{2,k} + u_{3,k} + u_{4,k} + u_{5,k}$$

$$x_{38,k+1} = (1 + g_{C,k}) x_{38,k}$$

$$x_{39,k+1} = (1 + g_{I,k}) x_{39,k}$$

$$x_{40,k+1} = (1 + g_{G,k}) x_{40,k}$$

$$x_{41,k+1} = n_0 x_{13,k}$$

$$x_{42,k+1} = x_{35,k} + x_{36,k} + x_{37,k} + x_{41,k}$$

$$x_{43,k+1} = \tau x_{42,k}$$

$$x_{44,k+1} = x_{24,k} + x_{25,k} + x_{26,k} + x_{27,k} + x_{28,k} - 4x_{34,k} - x_{43,k}$$
$$+ u_{1,k} + u_{2,k} + u_{3,k} + u_{4,k} + u_{5,k}$$

$$x_{45,k+1} = 0.25 x_{44,k} + (1 + i) x_{45,k}$$

$$x_{46,k+1} = x_{24,k} + u_{1,k}$$

$$x_{47,k+1} = x_{25,k} + u_{2,k}$$

$$x_{48,k+1} = x_{26,k} + u_{3,k}$$

$$x_{49,k+1} = x_{27,k} + u_{4,k}$$

$$x_{50,k+1} = x_{28,k} + u_{5,k}$$

$$x_{51,k+1} = x_{24,k} - x_{46,k} + u_{1,k}$$

$$x_{52,k+1} = x_{25,k} - x_{47,k} + u_{2,k}$$

$$x_{53,k+1} = x_{26,k} - x_{48,k} + u_{3,k}$$

$$x_{54,k+1} = x_{27,k} - x_{49,k} + u_{4,k}$$

$$x_{55,k+1} = x_{28,k} - x_{50,k} + u_{5,k}$$

$$x_{56,k+1} = g_{G,1,k} x_{24,k}$$

$$x_{57,k+1} = g_{G,2,k} x_{25,k}$$

$$x_{58,k+1} = g_{G,3,k} x_{26,k}$$

$$x_{59,k+1} = g_{G,4,k} x_{27,k}$$

$$x_{60,k+1} = g_{G,5,k} x_{28,k}$$

$$x_{61,k+1} = (1 + g_{DEF,k}) x_{61,k}$$

$$x_{62,k+1} = (1 + g_{DEBT,k}) x_{62,k}$$

$$x_{63,k+1} = (1+g_{S,C5,k})\,x_{63,k}$$
$$x_{64,k+1} = (1+g_{S,I5,k})\,x_{64,k}$$
$$x_{65,k+1} = (1+g_{S,G5,k})\,x_{65,k}$$

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