Mikael Bask

Adaptive learning in an expectational difference equation with several lags: selecting among learnable REE

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Suomen Pankki
Bank of Finland
P.O.Box 160
FI-00101 HELSINKI
Finland
☎ + 358 10 8311

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* Postal address: Monetary Policy and Research Department, Bank of Finland, P.O. Box 160, FIN-00101 Helsinki, Finland. E-mail address: mikael.bask@bof.fi

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Mikael Bask
Monetary Policy and Research Department

Abstract

It is demonstrated in this paper that adaptive learning in least squares sense may be incapable to reduce, in a satisfactory way, the number of attainable equilibria in a rational expectations model. The model investigated, as an illustration, is the monetary approach to exchange rate determination that is augmented with technical trading in the currency market in the form of moving averages since it is the most commonly used technique according to questionnaire surveys. Because of technical trading in foreign exchange, the current exchange rate is dependent on j_{max} lags of the exchange rate, and the model has, therefore $j_{max} + 1$ nonbubble rational expectations equilibria (REE), where most of them are adaptively learnable. However, by assuming that a solution to the model should have a solution to a nested model as its limit, it is possible to single out a unique equilibrium among the adaptively learnable equilibria that is economically meaningful.

Key words: asset pricing, heterogenous agents, least squares learnability, rational expectations equilibria and technical trading

JEL classification numbers: C62, F31, G12

Adaptiivinen oppiminen ja yksikäsitteisen rationaalisten odotusten tasapainon valinta odotuksilla täydennetyssä usean viipeen dynaamisessa yhden yhtälön mallissa

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Mikael Bask
Rahapolitiikka- ja tutkimusosasto

Tiivistelmä

Tässä työssä osoitetaan, että pienimmän neliösumman menetelmään perustuva adaptiivinen oppimismekanismi ei tyydyttävällä tavalla rajoita mahdollisten tasapainojen joukkoa rationaalisten odotusten mallissa. Tämän havainnollistamiseksi työssä käytetään monetaarista valuuttakurssin määräytymistä kuvaavaa mallia, jota on täydennetty teknisellä kaupankäynnillä. Tekninen kaupankäynti – trendin metsästys – perustuu yksinkertaiseen valuuttakurssin liukuvan keskiarvon ennustemalliin, jonka mukaan kauppaa käyvät ennustavat valuuttakurssin tulevia arvoja sen historiallisten arvojen painotettuina keskiarvoina. Teknisen kaupankäynnin vuoksi valuuttakurssin talouden perustekijöistä riippuvia rationaalisten odotusten tasapainoja on yksi enemmän kuin valuuttakurssin viipeitä sen ratkaisuyhtälössä. Useimmat näistä tasapainoista ovat lisäksi adaptiivisesti opittavia. Näiden mahdollisten tasapainojen joukosta voidaan kuitenkin valita yksikäsitteinen, taloudellisesti mielekäs tasapaino vaatimalla, että alkuperäisen laajemman mallin erikoistapauksen ratkaisut saadaan rajankäynnillä laajemman mallin ratkaisuista.

Avainsanat: varallisuuden hinnoittelu, heterogeeniset taloudenpitäjät, pienimmän neliösumman ennusteet, rationaalisten odotusten tasapaino, tekninen kaupankäynti

JEL-luokittelu: C62, F31, G12

Contents

1 Introduction

The first aim of this paper is to investigate whether the solutions to the following equation are adaptively learnable in least squares sense

$$s[t] = x_1 f[t] - x_2 \sum_{j=1}^{j_{\max}} \exp(-jv)\, s[t-j] + x_3 E[s[t+1]]. \tag{1.1}$$

The expectational difference equation in (1.1) is derived by augmenting a basic asset pricing model for exchange rate determination with technical trading in the currency market. That is, the difference equation in (1.1) is a foreign exchange model in which the current nominal exchange rate, $s[t]$, not only depends on current fundamentals, $f[t]$, and the (mathematically) expected value of the next time period's exchange rate, $E[s[t+1]]$, but also on $j_{\max}$ lags of the exchange rate, $s[t-j]$, $j \geq 1$. The reason for the latter dependence is, of course, the use of technical trading in foreign exchange. Specifically, we will investigate the adaptive learnability of the forward-solutions to (1.1) in which the expected fundamentals in future time periods are part of the solutions, and not the simpler minimum state variable solution (MSV) suggested by McCallum (1983).

Adaptive learning has been proposed in the literature as a selection criterion when there is a multiplicity of *rational expectations equilibria* (REE) in a model, because it is a well-known fact that models in economics and finance, in which agents have rational expectations regarding a variable in the model, may exhibit this phenomenon. In fact, it is shown below that the model in (1.1) has as many as $j_{\max} + 1$ REE, even if we disregard from rational bubble solutions. However, by focusing on the REE that are a possible result of an adaptive learning process for the agents, it may be possible to reduce the number of attainable equilibria. Specifically, it can be assumed that the agents' expectations are formed by a correctly specified model, ie, a model that corresponds to the REE, but without having perfect knowledge about the parameters in the model. However, using past and current values of the variables in the model, the parameters are learned over time since the beliefs are revised as new information is gained. See Evans and Honkapohja (2001) for a nice introduction to this literature.

Unfortunately, it is not always the case that adaptive learning as a selection criterion is able to reduce the number of attainable REE in a satisfactory way. It is demonstrated below that the foreign exchange model in (1.1) is an example of such a model since it has too many adaptively learnable REE in least squares sense. Therefore, it is necessary to find another selection criterion that is more successful in reducing the number of attainable REE. The second aim of this paper is to argue that *continuity* should hold for a REE to be economically meaningful, meaning that if the model in focus nests another model, then a solution to the general model should have a solution to the nested model as its limit. Therefore, the proposed selection criterion is in the spirit of McCallum's (1983) procedure to single out the REE that *'do not possess peculiar or aberational properties'* (p. 141).

To be more specific, we will argue that, after deriving the forward-solutions to the expectational difference equation in (1.1), that the parameter for the time-$t-1$ exchange rate should have the limit 0 when there is no technical trading in foreign exchange to have an economically meaningful equilibrium. The intuition behind this selection criterion is straightforward. Since it is the presence of technical trading that is causing the current exchange rate to depend on past rates, the parameters for these exchange rates must vanish when technical trading is absent in currency trade. Moreover, since all parameters for past exchange rates depend on the parameter for the time-$t-1$ exchange rate, as is shown below, all attention should be focused on the behavior of the latter parameter. It turns out that the proposed selection criterion is able to single out a unique equilibrium that is economically meaningful.

At a first sight, it may seem that the proposed *continuity criterion* has a limited applicability. However, having in mind the large and growing literature on heterogeneous agents in economics and finance, we believe the contrary to be true. Thus, in many cases, it is possible to shrink a heterogenous agents model to one or several homogenous agents models, eg, one model for each type of agent, and use the proposed selection criterion to find the REE that are economically meaningful. The foreign exchange model that is explored in this paper is a nice example of this method.[1]

The rest of the paper is organized as follows: The expectational difference equation in (1.1) is derived and solved in Section 2, whereas the adaptive learnability of the forward-solutions to this equation is in focus in Section 3. The proposed selection criterion is put forward in Section 4, and Section 5 concludes the paper. Finally, the Appendix contains some proofs.

2 Deriving and solving the difference equation

The aim of this section is to derive and solve the expectational difference equation in (1.1). To be more specific, we will derive the forward-solutions to this equation, and not the simpler MSV solution suggested by McCallum (1983), which is the solution to a linear difference equation that depends linearly on a set of variables such that there does not exist a solution that depends linearly on a smaller set of variables.[2]

[1] In Bask (2006), the announcement and implementation of temporary as well as permanent monetary policy are analyzed using the model in (1.1). Among other things, it is shown that the exchange rate is much more sensitive to changes in money supply than when technical trading is absent in currency trade, which is an interesting result since it sheds light on the so-called exchange rate disconnect puzzle in international finance.

[2] The MSV solution is not appropriate to analyze the announcement effects on exchange rate movements since expected future fundamentals are not part of the solution (see Bask (2006)).

2.1 Deriving the difference equation

This section is divided into two parts. In the first part, the expectations formations at the currency market are presented and discussed, whereas, in the second part, the basic asset pricing model for exchange rate determination is derived that also is the monetary approach to foreign exchange.

2.1.1 Expectations formations

Herein, we will describe the expectations formations used in the monetary exchange rate model that is outlined below. Specifically, the market expectations and the expectations formed by chartism and fundamental analysis about the next time period's exchange rate are formulated and discussed.

Chartism

Chartism, or technical analysis, utilizes past exchange rates to detect patterns that are extrapolated into the future. Focusing on past exchange rates is not considered as a shortcoming for agents using any of these technical trading techniques since a primary assumption behind technical analysis is that all relevant information about future exchange rate movements is contained in past movements. Thus, chartism is purely behavioristic in nature and does not examine the underlying reasons of currency traders.

That chartism is used extensively in currency trade is confirmed in several questionnaire surveys. Examples include Cheung and Chinn (2001), who conducted a survey at the U.S. market; Lui and Mole (1998), who conducted a survey at the Hong Kong market; Menkhoff (1997), who conducted a survey at the German market; Oberlechner (2001), who conducted a survey at the markets in Frankfurt, London, Vienna and Zurich; and Taylor and Allen (1992), who conducted a survey at the London market. An extensive exploration of the psychology in currency trade may also be found in Oberlechner (2004), which is based on surveys conducted at the European and the North American markets.

The most commonly used technical trading technique in foreign exchange is moving averages (see Lui and Mole (1998) and Taylor and Allen (1992)). According to this trading technique, buying and selling signals are generated by two moving averages; a short-period moving average and a long-period moving average. Specifically, a buy (sell) signal is generated when the short-period moving average rises above (falls below) the long-period moving average. In its simplest form, the short-period moving average is the current exchange rate and the long-period moving average is an exponentially weighted moving average of current and past exchange rates.

Thus, it is expected that the exchange rate will increase (decrease) when the current exchange rate is higher (lower) than an exponentially weighted moving average of current and past exchange rates

$$s_c^e[t+1] - s[t] = \gamma (s[t] - MA[t]), \tag{2.1}$$

where $s_c^e[t+1]$ is the expectations formed by chartism about the next time period's exchange rate, and where the superscript e denotes expectations. The exchange rate is defined as the domestic price of the foreign currency. Moreover, the long-period moving average, $MA[t]$, is formulated as[3]

$$MA[t] = (1 - \exp(-v)) \sum_{j=0}^{\infty} \exp(-jv)\, s[t-j], \tag{2.2}$$

where the weights given to current and past exchange rates sum up to 1

$$(1 - \exp(-v)) \sum_{j=0}^{\infty} \exp(-jv) = 1. \tag{2.3}$$

Note that when $v \to 0$ or $v \to \infty$, the long-period moving average in (2.2) does not depend at all on past exchange rates. Specifically, for small v, all weights in the long-period moving average get small, including the weight given to the current exchange rate, while for large v, only the weights for past exchange rates get small, but the weight given to the current exchange rate approaches 1.

Fundamental analysis

When fundamental analysis is used in currency trade, it is assumed that the agents have rational expectations regarding the next time period's exchange rate

$$s_f^e[t+1] = E[s[t+1]], \tag{2.4}$$

where $s_f^e[t+1]$ is the expectations formed by fundamental analysis about the next time period's exchange rate, and where $E[s[t+1]]$ is equal to the mathematical expectation of $s[t+1]$ based on the information set available at time t, which includes the knowledge of the complete model as well as the realized values of all variables in the model up to and including time t. Thus, because currency trade based on chartism is affecting the exchange rate, currency trade based on fundamental analysis will take this into account when forming exchange rate expectations.

Market expectations

According to the questionnaire surveys mentioned above, the relative importance of technical versus fundamental analysis in the currency market depends on the time horizon in currency trade. For shorter time horizons, more weight is placed on technical analysis, or chartism, while more weight is placed on fundamental analysis for longer horizons. This observation is formulated as

$$s^e[t+1] = \omega(\tau)\, s_f^e[t+1] + (1 - \omega(\tau))\, s_c^e[t+1], \tag{2.5}$$

[3] In the analysis below, in Sections 2.2 and 3, we will focus on the case when the long-period moving average in (2.2) is a moving average of $j_{\max}$ past exchange rates. An obvious justification for this is the lack of an infinite amount of data.

where $s^e[t+1]$ is the market expectations about the next time period's exchange rate. Moreover, $\omega(\tau)$ is a weight function that depends on the 'artificial' time horizon, τ, in currency trade

$$\omega(\tau) = 1 - \exp(-\tau), \tag{2.6}$$

which is exogenously given in the model.

2.1.2 The monetary approach to foreign exchange

We will now derive the monetary approach to exchange rate determination that will be augmented with the trading behavior in the currency market that was described in the previous part.

The baseline model

The baseline model is the monetary approach to exchange rate determination that consists of two parity conditions, uncovered interest rate parity (UIP) and purchasing power parity (PPP), as well as equilibrium conditions at the domestic and foreign money markets.

The first parity condition is UIP, which states that the expected change of the exchange rate is equal to the difference between the domestic and foreign interest rates

$$s^e[t+1] - s[t] = i[t] - i^*[t], \tag{2.7}$$

where $i[t]$ and $i^*[t]$ are the domestic and foreign nominal interest rates, respectively. The parity condition in (2.7) is based on the assumption that domestic and foreign assets are perfect substitutes, which only holds if there is perfect capital mobility. Since the latter also is assumed, only the slightest difference in expected yields would draw the entire capital into the asset that offers the highest expected yield. Thus, the parity condition in (2.7) is an equilibrium condition at the international asset market.

The second parity condition is PPP, which states that the exchange rate is equal to the difference between the domestic and foreign price levels

$$s[t] = p[t] - p^*[t], \tag{2.8}$$

where $p[t]$ and $p^*[t]$ are the domestic and foreign nominal price levels, respectively. The parity condition in (2.8) means that the domestic and foreign price levels, expressed in a common currency, are equal to each other. Thus, according to PPP, a relative increase (decrease) in the domestic price level not only means that the domestic price of the foreign currency increases (decreases), it also means that the increase (decrease) in the exchange rate is of such a magnitude that the price levels, expressed in a common currency, are still equal to each other.

Equilibrium at the domestic and foreign money markets hold when real money supply is equal to real money demand

$$m[t] - p[t] = \alpha y[t] - \beta i[t], \tag{2.9}$$

and

$$m^* [t] - p^* [t] = \alpha y^* [t] - \beta i^* [t] , \tag{2.10}$$

where $m[t]$ and $m^*[t]$ are the domestic and foreign nominal money supplies, and $y[t]$ and $y^*[t]$ are the domestic and foreign real incomes, respectively. Thus, real money demand increases (decreases) when real income increases (decreases) or the interest rate decreases (increases). Note that we assume that the real income elasticities, α, in (2.9)–(2.10) are equal to each other. The same assumption is made for the interest rate semi-elasticities, β, in the same equations.

If we substitute the conditions for money market equilibrium in (2.9)–(2.10) into the condition for PPP in (2.8), we have an equation describing the monetary approach to exchange rate determination

$$s[t] = m[t] - m^*[t] - \alpha (y[t] - y^*[t]) + \beta (i[t] - i^*[t]) . \tag{2.11}$$

According to (2.11), the exchange rate depreciates (appreciates) if the relative money supply increases (decreases). To be more specific, an increases (decrease) in the domestic money supply, relative to the foreign money supply, causes a one-to-one depreciation (appreciation) of the exchange rate. Moreover, the exchange rate depreciates (appreciates) when the relative income decreases (increases) as well as when the relative interest rate increases (decreases). The magnitudes of the two latter effects depend on the real income elasticity and the interest rate semi-elasticity, respectively.

Incorporating chartism

Now, substitute the condition for UIP in (2.7) into the monetary approach to exchange rate determination in (2.11), and solve for the current exchange rate

$$s[t] = \frac{m[t] - m^*[t] - \alpha (y[t] - y^*[t])}{1+\beta} + \frac{\beta s^e [t+1]}{1+\beta}. \tag{2.12}$$

The main difference between the monetary approach in (2.11), and the same approach in (2.12), is that the market expectations about the next time period's exchange rate has replaced the relative interest rate. According to (2.12), the exchange rate depreciates (appreciates) in the current time period when the exchange rate is expected to depreciate (appreciate) in the next time period. Thus, (2.12) is characterized by a kind of self-fulfilling expectations, meaning that if the market believe that the currency will be weaker (stronger) in the next time period, it will be weaker (stronger) already in the current time period.

Technical trading is introduced into the baseline model in (2.12) via the expected exchange rate. This is accomplished in two steps. Firstly, the expectations formed by chartism and fundamental analysis in (2.1) and (2.4), respectively, are substituted into the market expectations in (2.5), where also the long-period moving average in (2.2) and the weight function in (2.6) are used in the derivations. Thus, an equation describing how the expected exchange rate is determined by technical trading and fundamental analysis is derived. Secondly, this equation is substituted into (2.12), the resulting equation is solved for the current exchange rate, and we have, finally, derived the monetary approach to exchange rate determination that has been augmented with technical trading in foreign exchange. See Proposition 2.1 below and the proof of it for details.

Proposition 2.1 *The expectational difference equation for the foreign exchange model is*

$$s[t] = x_1 f[t] - x_2 \sum_{j=1}^{\infty} \exp(-jv)\, s[t-j] + x_3 E[s[t+1]], \tag{2.13}$$

where the fundamentals are

$$f[t] \equiv m[t] - m^*[t] - \alpha(y[t] - y^*[t]), \tag{2.14}$$

and where

$$\begin{cases} x_1 \equiv \frac{1}{1+\beta(1-\exp(-\tau)-\gamma\exp(-\tau-v))} \\ x_2 \equiv \frac{\beta\gamma\exp(-\tau)(1-\exp(-v))}{1+\beta(1-\exp(-\tau)-\gamma\exp(-\tau-v))} \\ x_3 \equiv \frac{\beta(1-\exp(-\tau))}{1+\beta(1-\exp(-\tau)-\gamma\exp(-\tau-v))} \end{cases}. \tag{2.15}$$

Obviously, since both chartism and fundamental analysis are used in currency trade, the current exchange rate is affected by past exchange rates (see the second term at the right-hand side of (2.13)) as well as expectational matters (see the third term at the right-hand side of (2.13)).

2.2 Solving the difference equation

The aim is here to determine the solutions to (1.1) with time-t dating of exchange rate expectations in which the expected fundamentals in future time periods are part of the general solution. Observe that the focus is on the expectational difference equation in (1.1), where $j_{\max}$ may be large, and not on the difference equation in (2.13) in Proposition 1, where $j_{\max} \to \infty$.

Thus, we will determine the forward-solutions to the expectational difference equation in (1.1). Therefore, a suggested general solution is

$$s[t] = \sum_{j=1}^{j_{\max}} \beta_j s[t-j] + \sum_{k=0}^{k_{\max}} \beta_{j_{\max}+1+k} E[f[t+k]], \tag{2.16}$$

where $\{\beta_j\}_{j=1}^{j_{\max}+1+k_{\max}}$ are parameters to be determined, and where $k_{\max}$ is large. Assuming that the general solution in (2.16) is correct, determine the rationally formed forecast of the next time period's exchange rate

$$E[s[t+1]] = \sum_{j=1}^{j_{\max}} \beta_j s[t-j+1] + \sum_{k=0}^{k_{\max}} \beta_{j_{\max}+1+k} E[f[t+k+1]], \tag{2.17}$$

substitute this forecast into the difference equation in (1.1), and solve the resulting equation for $s[t]$

$$\begin{aligned} s[t] &= \frac{1}{1-\beta_1 x_3} \cdot \sum_{j=1}^{j_{\max}-1} \left(\beta_{j+1} x_3 - x_2 \exp(-jv)\right) s[t-j] - \\ & \frac{x_2 \exp(-j_{\max} v)}{1-\beta_1 x_3} \cdot s[t-j_{\max}] + \frac{x_1}{1-\beta_1 x_3} \cdot f[t] + \\ & \frac{x_3}{1-\beta_1 x_3} \cdot \sum_{k=0}^{k_{\max}} \beta_{j_{\max}+1+k} E[f[t+k+1]]. \end{aligned} \tag{2.18}$$

Then, the solutions to the following equation system determine the parameters in (2.16)

$$\begin{cases} \beta_{j_0} = \frac{\beta_{j_0+1} x_3 - x_2 \exp(-j_0 v)}{1-\beta_1 x_3} \\ \beta_{j_{\max}} = -\frac{x_2 \exp(-j_{\max} v)}{1-\beta_1 x_3} \\ \beta_{j_{\max}+1} = \frac{x_1}{1-\beta_1 x_3} \\ \beta_{j_1} = \frac{\beta_{j_1-1} x_3}{1-\beta_1 x_3} \end{cases}, \tag{2.19}$$

where $j_0 \in \{1, ..., j_{\max}-1\}$ and $j_1 \in \{j_{\max}+2, ..., j_{\max}+1+k_{\max}\}$. Note that all parameters for past exchange rates depend on β_1.

If the equation system in (2.19) is partly solved via recursion, a general solution to (1.1) is

$$s[t] = \sum_{j=1}^{j_{\max}} \beta_j s[t-j] + \frac{x_1}{1-\beta_1 x_3} \cdot \sum_{k=0}^{k_{\max}} x_3^k E[f[t+k]], \tag{2.20}$$

or, when $k_{\max} \to \infty$,

$$s[t] = \sum_{j=1}^{j_{\max}} \beta_j s[t-j] + \frac{x_1}{1-\beta_1 x_3} \cdot \sum_{k=0}^{\infty} x_3^k E[f[t+k]]. \tag{2.21}$$

Of course, we can also solve for $\{\beta_j\}_{j=1}^{j_{\max}}$ in (2.20). However, we skip these derivations, except the derivation of β_1, because it is not necessary to make explicit use of all parameters for past exchange rates in the analysis. Obviously, (2.20) is not easy to analyze since, according to Proposition 2.2 below, there are $j_{\max}+1$ roots to the equation that determines β_1, meaning that there are as many as $j_{\max}+1$ solutions to (1.1), even if we disregard from rational bubble solutions.

Proposition 2.2 *β_1 satisfy the following equation:*

$$\beta_1 = -x_2 \sum_{j=1}^{j_{\max}} \frac{x_3^{j-1} \exp(-jv)}{(1-\beta_1 x_3)^j}, \tag{2.22}$$

which has $j_{\max}+1$ roots, but

$$\beta_1 \neq \frac{1}{x_3}. \tag{2.23}$$

Let us now pose the question whether the solutions in (2.20) are adaptively learnable in least squares sense.

3 Adaptive learning as a criterion among REE

The assumption in (2.4) is that when fundamental analysis is used in currency trade, the agents have rational expectations in the sense that the expected exchange rate is equal to the mathematical expectation of the exchange rate conditioned on all information available to the currency trader. Thus, since this information not only includes past and current values of the variables in the model, but also a complete knowledge about the structure of the model, rational expectations is a rather strong assumption. This assumption has, therefore, in the more recent literature been complemented by an analysis of the possible convergence to the REE.

It will be assumed below that the expectations formed by fundamental analysis are formed by a correctly specified model, ie, a model that corresponds to the REE, but without having perfect knowledge about the parameters in the model. However, using past and current values of the variables in the model, the parameters are learned over time since the beliefs are revised as new information is gained. Thus, one may think of the agents that use fundamental analysis that they act as econometricians who adaptively learn the parameters in the model. Specifically, it will be investigated whether the model is characterized by least squares learnability. However, since expectational stability, ie, E-stability, implies least squares learnability (see, eg, Evans and Honkapohja (2001)), the focus in the analysis will be on E-stability. This is because the latter concept is easier to handle mathematically.

Now, if we allow for non-rational expectations in (1.1) and (2.16)–(2.18), note that the suggested general solution in (2.16) is the perceived law of motion (PLM) of the exchange rate, where $\{\beta_j\}_{j=1}^{j_{\max}+1+k_{\max}}$ are the parameters that are estimated by the agents, and that (2.18) is the actual law of motion (ALM) of the exchange rate. Moreover, note that there is a mapping, $\mathbf{M} : \mathbb{R}^{j_{\max}+1+k_{\max}} \longrightarrow \mathbb{R}^{j_{\max}+1+k_{\max}}$, from the parameters in the PLM to the parameters in the ALM

$$\mathbf{M}\begin{pmatrix} \beta_{j_0} \\ \beta_{j_{\max}} \\ \beta_{j_{\max}+1} \\ \beta_{j_1} \end{pmatrix} = \begin{pmatrix} \frac{\beta_{j_0+1}x_3 - x_2\exp(-j_0 v)}{1-\beta_1 x_3} \\ -\frac{x_2\exp(-j_{\max}v)}{1-\beta_1 x_3} \\ \frac{x_1}{1-\beta_1 x_3} \\ \frac{\beta_{j_1-1}x_3}{1-\beta_1 x_3} \end{pmatrix}, \tag{3.1}$$

where $j_0 \in \{1, ..., j_{\max} - 1\}$ and $j_1 \in \{j_{\max} + 2, ..., j_{\max} + 1 + k_{\max}\}$. Then, consider the differential equation

$$\frac{d}{d\tau_a}\begin{pmatrix} \beta_{j_0} \\ \beta_{j_{\max}} \\ \beta_{j_{\max}+1} \\ \beta_{j_1} \end{pmatrix} = \mathbf{M}\begin{pmatrix} \beta_{j_0} \\ \beta_{j_{\max}} \\ \beta_{j_{\max}+1} \\ \beta_{j_1} \end{pmatrix} - \begin{pmatrix} \beta_{j_0} \\ \beta_{j_{\max}} \\ \beta_{j_{\max}+1} \\ \beta_{j_1} \end{pmatrix} \tag{3.2}$$
$$= \begin{pmatrix} \frac{\beta_{j_0+1}x_3 - x_2\exp(-j_0 v)}{1-\beta_1 x_3} \\ -\frac{x_2\exp(-j_{\max}v)}{1-\beta_1 x_3} \\ \frac{x_1}{1-\beta_1 x_3} \\ \frac{\beta_{j_1-1}x_3}{1-\beta_1 x_3} \end{pmatrix} - \begin{pmatrix} \beta_{j_0} \\ \beta_{j_{\max}} \\ \beta_{j_{\max}+1} \\ \beta_{j_1} \end{pmatrix},$$

where $j_0 \in \{1, ..., j_{\max} - 1\}$ and $j_1 \in \{j_{\max} + 2, ..., j_{\max} + 1 + k_{\max}\}$, and where τ_a is "artificial" time.

Clearly, all parameters for current and expected future fundamentals are locally asymptotically stable under (3.2),

$$\operatorname{Re}\left[\frac{d\left(\frac{d\beta_j}{d\tau_a}\right)}{d\beta_j}\right] = -1 < 0, \tag{3.3}$$

where $j \in \{j_{\max} + 1, ..., j_{\max} + 1 + k_{\max}\}$. However, it may not be true that all parameters for past exchange rates are locally asymptotically stable under (3.2), which means that the general solution in (2.20) may not be characterized by least squares learnability. Therefore, the question is: which solutions in (2.20) are adaptively learnable? To solve this problem, we start with the cases $j_{\max} = 1$ and $j_{\max} = 2$, and, then, continue with the general case in which $j_{\max}$ may be large.

Case: $j_{\max} = 1$

In this case, when there is only one lagged exchange rate in the general solution in (2.20), the differential equation in (3.2) reduces to

$$\frac{d}{d\tau_a}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_j \end{pmatrix} = \begin{pmatrix} -\frac{x_2\exp(-v)}{1-\beta_1 x_3} \\ \frac{x_1}{1-\beta_1 x_3} \\ \frac{\beta_{j-1}x_3}{1-\beta_1 x_3} \end{pmatrix} - \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_j \end{pmatrix}, \tag{3.4}$$

where $j \in \{3, ..., 2 + k_{\max}\}$, and where it has already been concluded that all parameters for current and expected future fundamentals are locally asymptotically stable under the relevant differential equation. Thus, since it is also true that

$$\operatorname{Re}\left[\frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1}\right] = -\operatorname{Re}\left[\frac{x_2 x_3\exp(-v)}{(1-\beta_1 x_3)^2}\right] - 1 < 0, \tag{3.5}$$

both forward-solutions in (2.20) to the expectational difference equation in (1.1) are characterized by least squares learnability.

Case: $j_{\max} = 2$

Let us now turn to the case when there are two lagged exchange rates in the general solution in (2.20). Firstly, since it is true that

$$\mathrm{Re}\left[\frac{d\left(\frac{d\beta_2}{d\tau_a}\right)}{d\beta_2}\right] = -1 < 0, \tag{3.6}$$

the parameter β_2 is locally asymptotically stable under the differential equation in (3.2). Secondly, since it has already been concluded that all parameters for current and expected future fundamentals are locally asymptotically stable under the same differential equation, it is the behavior of the parameter β_1 that determines which solutions in (2.20) that are characterized by least squares learnability. According to (3.2), if the right-hand side of the following equation is negative for a specific β_1, the specific solution in (2.20) that is associated with this β_1 is adaptively learnable

$$\begin{aligned} \frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1} &= \frac{\beta_2 x_3^2 - x_2 x_3 \exp(-v)}{(1-\beta_1 x_3)^2} - 1 \\ &= -\frac{x_2 x_3^2 \exp(-2v)}{(1-\beta_1 x_3)^3} - \frac{x_2 x_3 \exp(-v)}{(1-\beta_1 x_3)^2} - 1, \end{aligned} \tag{3.7}$$

where the second equation in (2.19) has been used in the second step in (3.7). Moreover, the equation that determines the solutions of β_1, ie, (2.22) in Proposition 2.2, reduces in the case of two lagged exchange rates to

$$\beta_1 = -x_2 \sum_{j=1}^{2} \frac{x_3^{j-1} \exp(-jv)}{(1-\beta_1 x_3)^j} = -\frac{x_2 \exp(-v)}{1-\beta_1 x_3} - \frac{x_2 x_3 \exp(-2v)}{(1-\beta_1 x_3)^2}, \tag{3.8}$$

or, after multiplying each side of (3.8) by $\frac{x_3}{1-\beta_1 x_3}$ and rearranging

$$-\frac{x_2 x_3^2 \exp(-2v)}{(1-\beta_1 x_3)^3} = \frac{\beta_1 x_3}{1-\beta_1 x_3} + \frac{x_2 x_3 \exp(-v)}{(1-\beta_1 x_3)^2}, \tag{3.9}$$

which is substituted into the real part of (3.7)

$$\mathrm{Re}\left[\frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1}\right] = \mathrm{Re}\left[\frac{\beta_1 x_3}{1-\beta_1 x_3}\right] - 1. \tag{3.10}$$

Since it is clear that the right-hand side of (3.10), in general, is not negative, we have to evaluate the adaptive learnability of each solution in (2.20).

If we solve numerically for each β_1, where all structural parameters in the model have unit values, the value of the right-hand side of (3.10) as a function of the time horizon in currency trade can be found in Figure 1.[4,5]

According to Figure 1, there are three stable solutions of β_1, meaning that all three solutions in (2.20) are characterized by least squares learnability.

[4] MATLAB routines for this purpose are available on request from the author.
[5] We will return below, in Section 4, to 'the meaningful solution' in Figures 1–9.

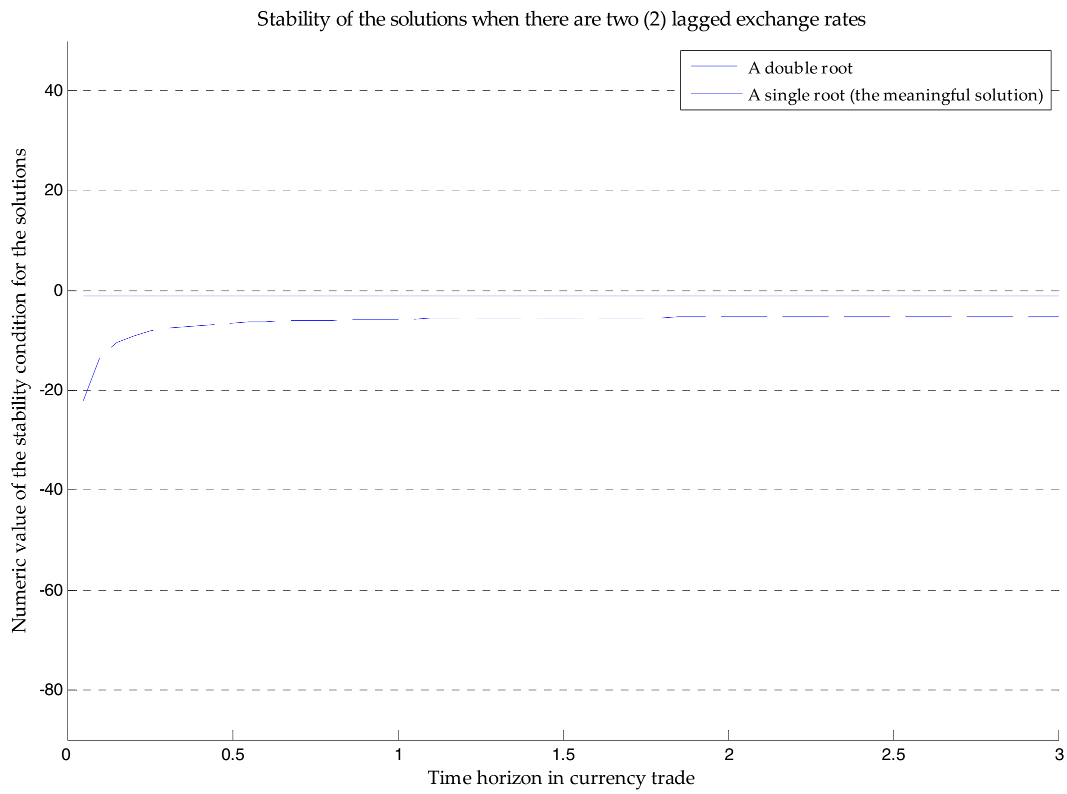

Figure 1: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are two (2) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are three (3) stable solutions.

General case

Again, since it has already been concluded that all parameters for current and expected future fundamentals are locally asymptotically stable under the differential equation in (3.2), we can turn our focus to the parameters for lagged exchange rates. That the parameters $\{\beta_j\}_{j=2}^{j_{\max}}$ are locally asymptotically stable under (3.2) is clear since

$$\operatorname{Re}\left[\frac{d\left(\frac{d\beta_j}{d\tau_a}\right)}{d\beta_j}\right] = -1 < 0, \tag{3.11}$$

where $j \in \{2, ..., j_{\max}\}$. Then, if we turn to the parameter β_1, a specific β_1 implies least squares learnability if the left-hand side of (3.12) is negative, which is stated and proved in Proposition 3.1 below.

Proposition 3.1 *A specific solution in (2.20) is characterized by least squares learnability if the following inequality holds for the specific β_1 that is associated with this solution:*

$$\operatorname{Re}\left[\frac{\beta_1 x_3}{1-\beta_1 x_3}\right] - 1 < 0. \tag{3.12}$$

Now, if we solve numerically for each β_1, when there are three lagged exchange rates, and where all structural parameters in the model have unit values, the value of the left-hand side of (3.12) as a function of the time horizon in currency trade can be found in Figure 2.

According to Figure 2, there are two stable solutions and two unstable solutions of β_1. However, when the time horizon in currency trade is short enough, there are four stable solutions of this parameter. All this means that either two or all four solutions in (2.20) are characterized by least squares learnability.

If we continue with four, five, six and seven lagged exchange rates, respectively, and solve numerically for each β_1, the value of the left-hand side of (3.12) as a function of the time horizon in currency trade can be found in Figures 3–6, respectively. Again, all structural parameters in the model have unit values.

According to Figure 3, there are three stable solutions and two unstable solutions of β_1, whereas, according to Figure 4, there are four stable solutions and two unstable solutions of the same parameter. Moreover, according to Figure 5, there are five stable solutions and two unstable solutions of β_1, whereas, according to Figure 6, there are six stable solutions and two unstable solutions of the same parameter. Thus, in each case, all solutions in (2.20), except two of them, are characterized by least squares learnability.

Finally, if we continue with eight, nine and ten lagged exchange rates, respectively, and solve numerically for each β_1, the value of the left-hand side of (3.12) as a function of the time horizon in currency trade can be found in

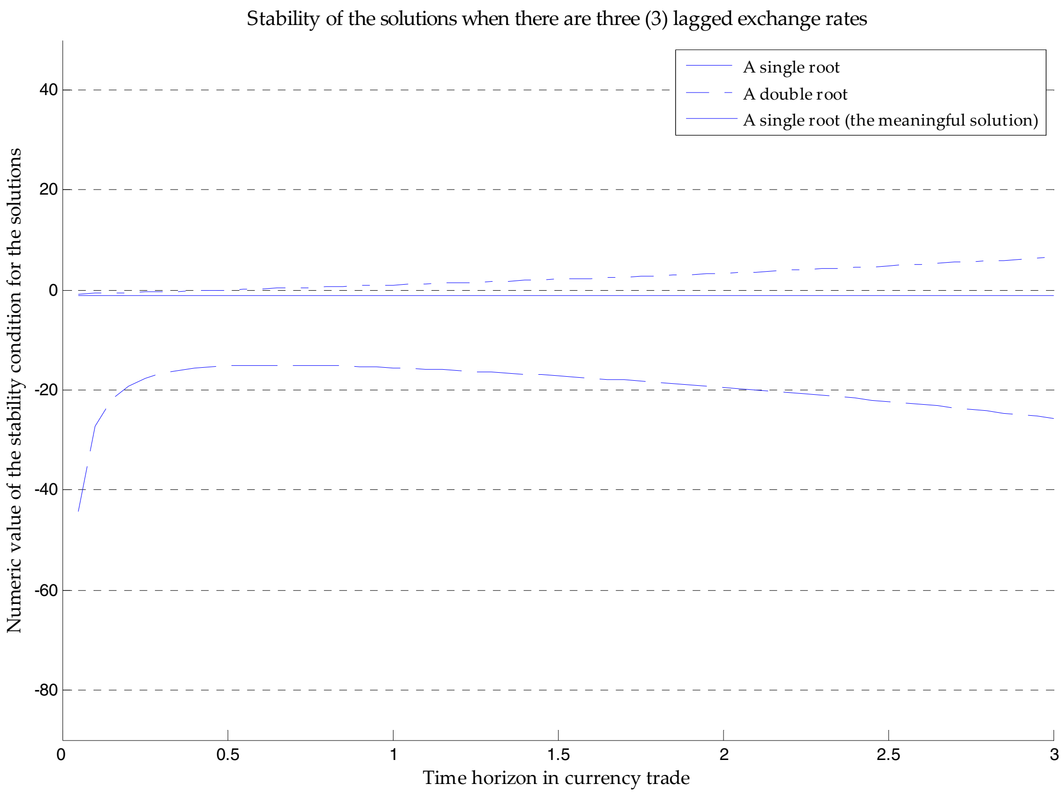

Figure 2: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are three (3) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are two (2) stable solutions and two (2) unstable solutions (when the time horizon in currency trade is not too short).

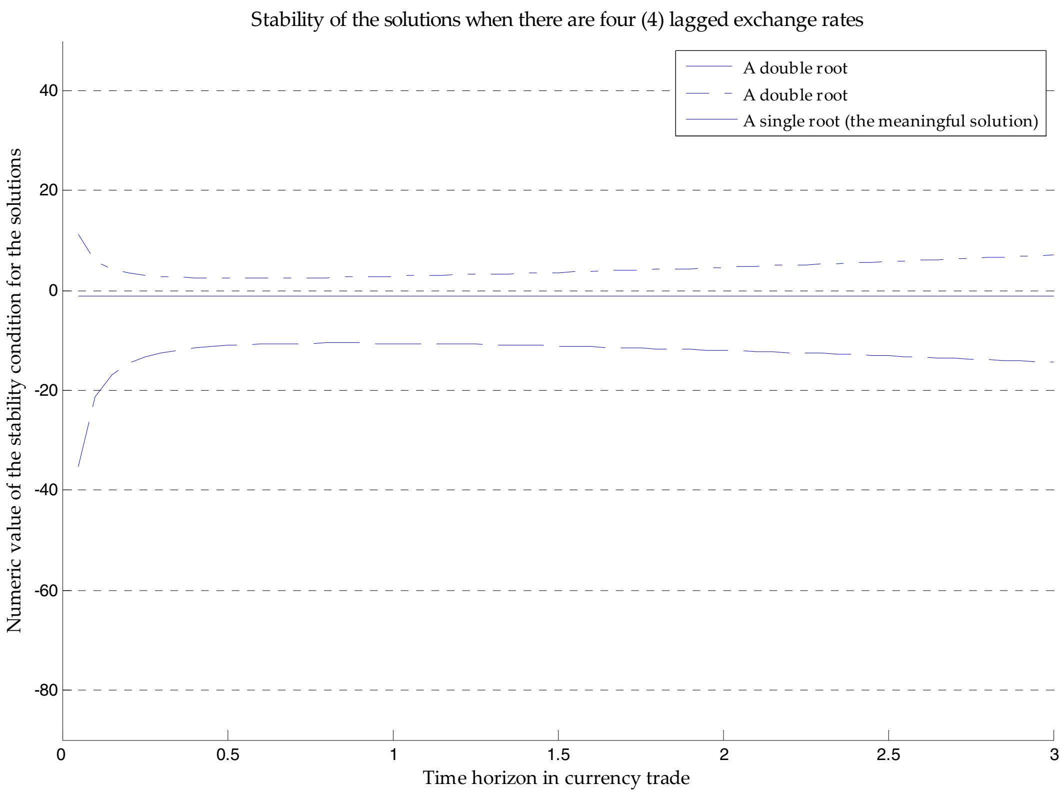

Figure 3: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are four (4) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are three (3) stable solutions and two (2) unstable solutions.

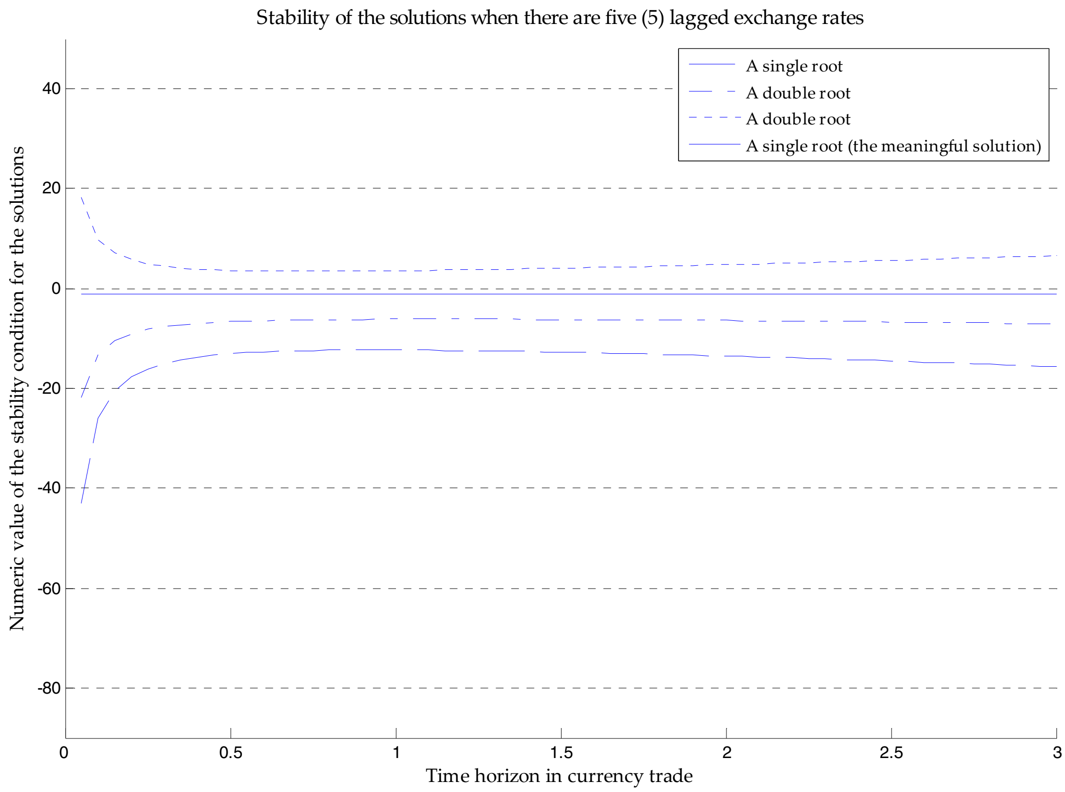

Figure 4: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are five (5) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are four (4) stable solutions and two (2) unstable solutions.

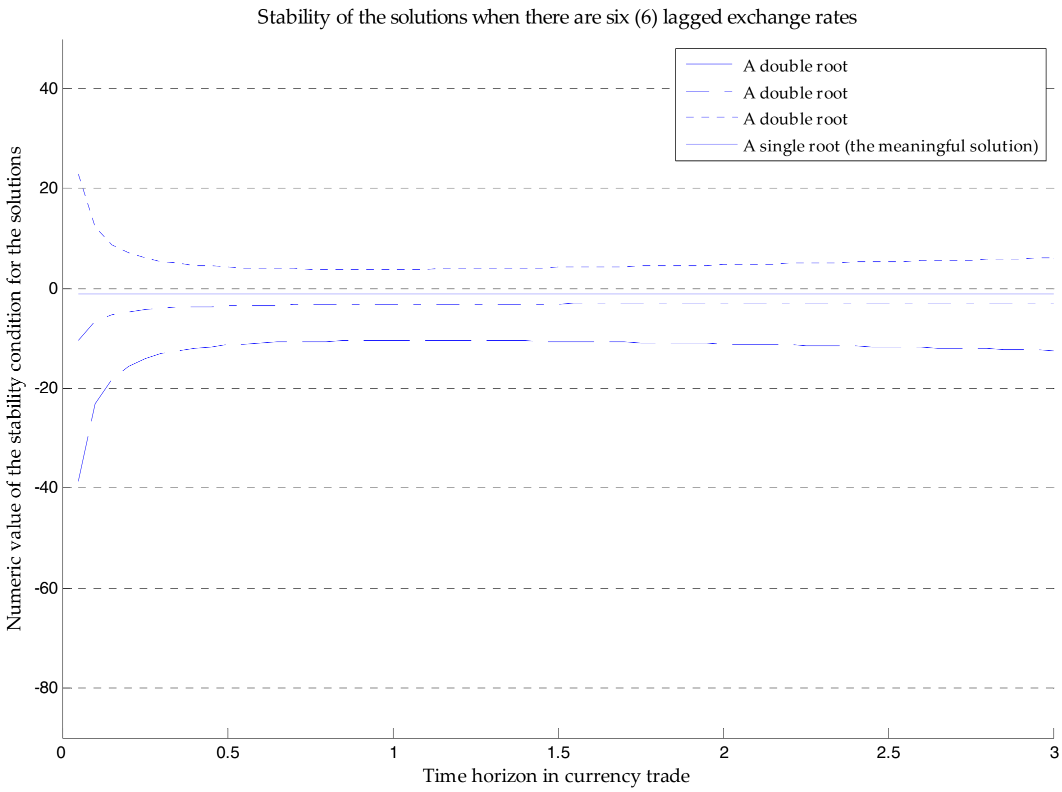

Figure 5: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are six (6) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are five (5) stable solutions and two (2) unstable solutions.

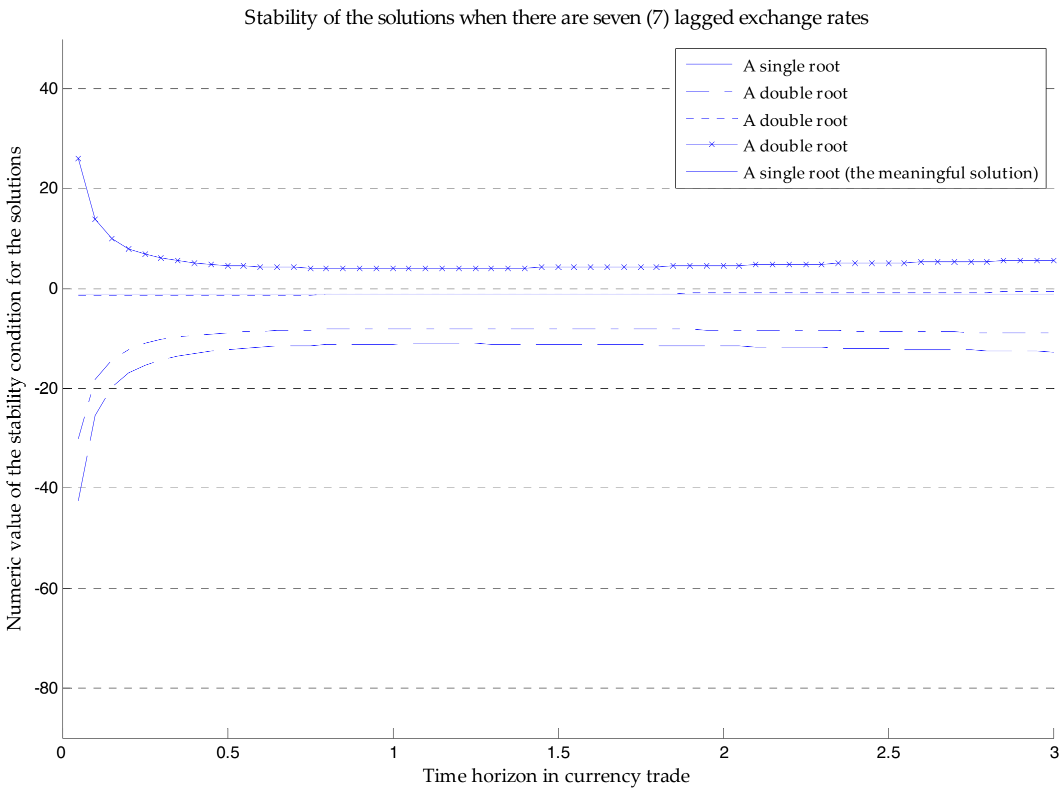

Figure 6: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are seven (7) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are six (6) stable solutions and two (2) unstable solutions.

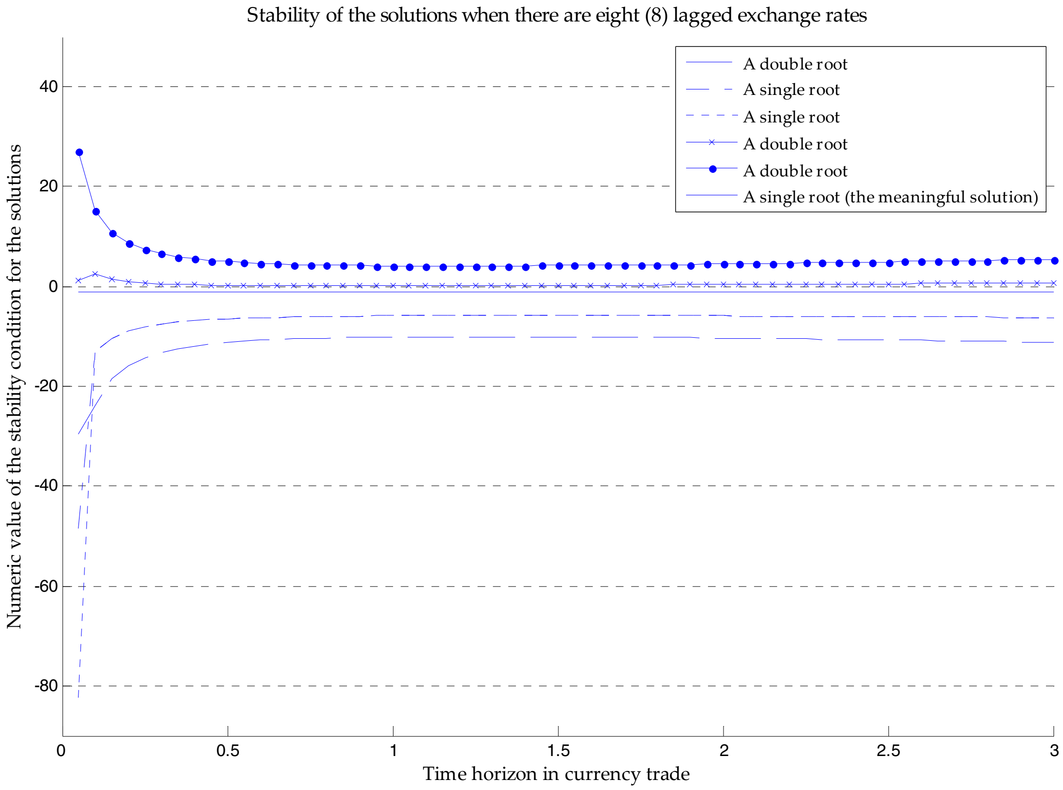

Figure 7: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are eight (8) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are five (5) stable solutions and four (4) unstable solutions.

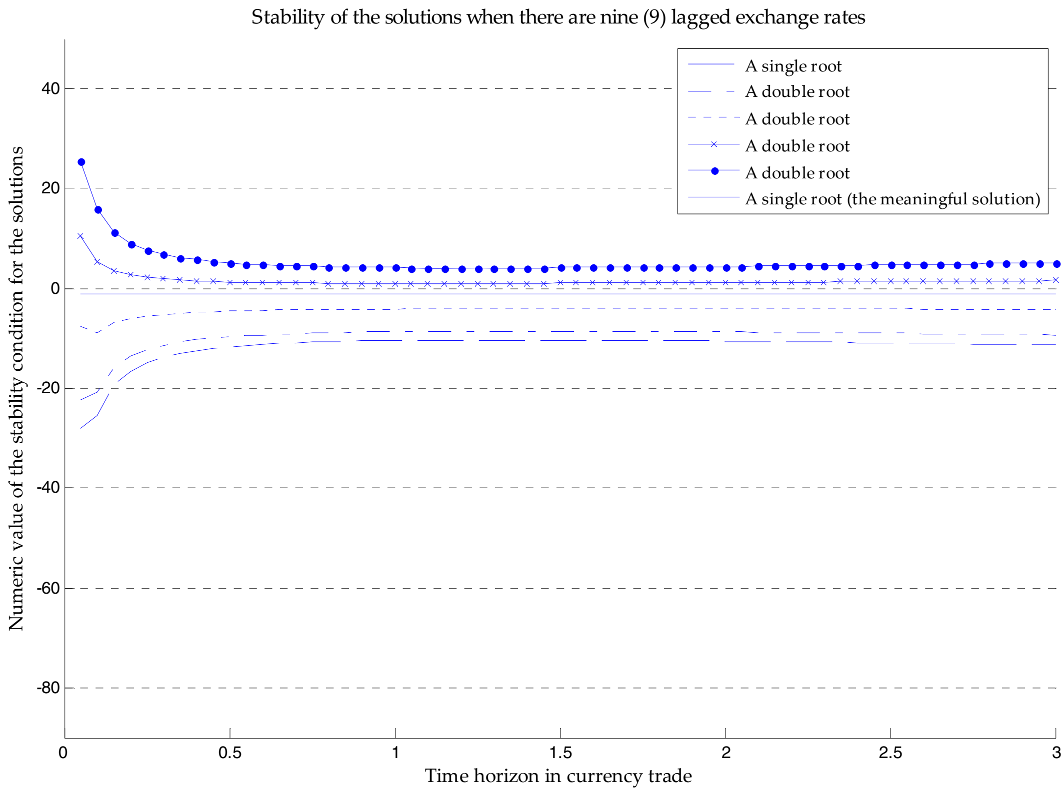

Figure 8: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are nine (9) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are six (6) stable solutions and four (4) unstable solutions.

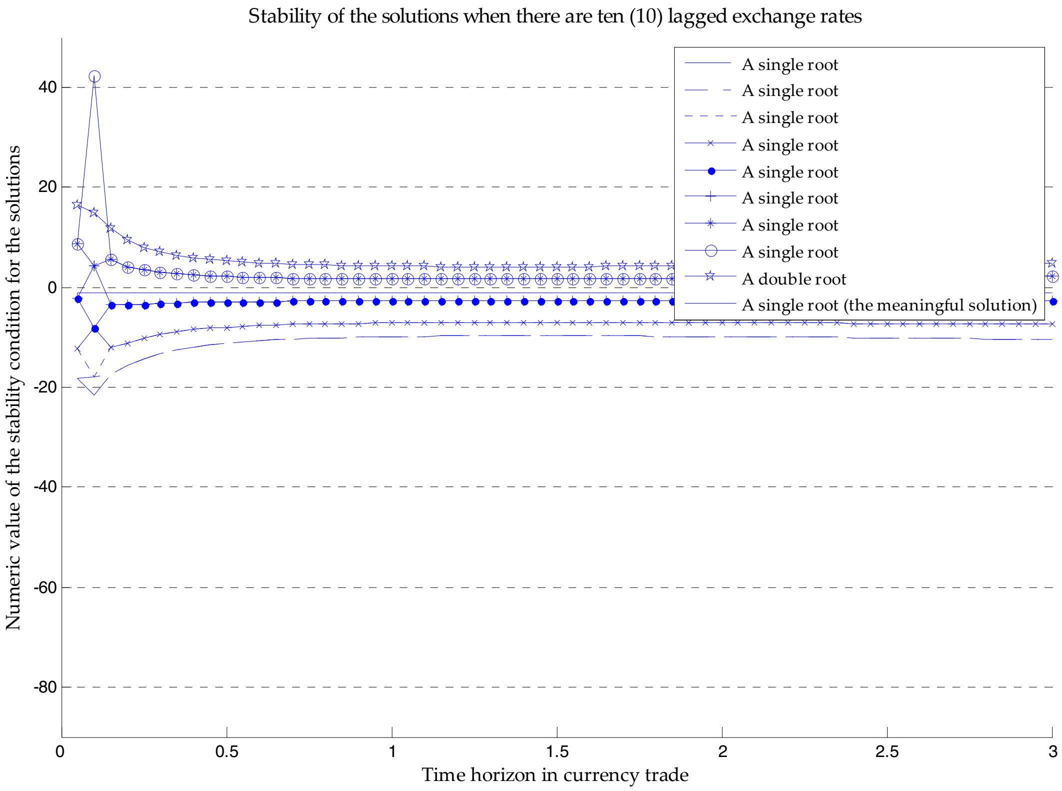

Figure 9: Local asymptotic stability of the solutions under the differential equation in (3.2), when there are ten (10) lagged exchange rates, where all structural parameters in the model have unit values. According to the figure, there are seven (7) stable solutions and four (4) unstable solutions (when the time horizon in currency trade is not too short).

Figures 7–9, respectively. Again, all structural parameters in the model have unit values.

According to these figures, the number of unstable solutions of β_1 has increased to four, meaning that there are five, six and seven stable solutions of β_1, respectively. Thus, in each case, all solutions in (2.20), except four of them, are characterized by least squares learnability.

Now, if we summarize the numerical findings in the cases when $j_{\max} \leq 10$, it turns out that least squares learnability as a selection criterion is not able to reduce the number of attainable REE in a satisfactory way. For example, when there are seven lagged exchange rates in (2.20), meaning that there are eight REE, only two of the equilibria are not adaptively learnable. Thus, as many as six REE are characterized by least squares learnability.

When $j_{\max} > 10$, it is not easy from a practical point of view to determine the number of adaptively learnable and non-learnable REE. However, a qualified guess is that the subsets of least squares learnable REE are too large in these cases. The proposed continuity criterion, however, will solve this problem by single out the unique adaptively learnable REE that is economically meaningful.

4 Continuity as a criterion among REE

The idea behind the proposed selection criterion is that if the model in focus nests another model, then a solution to the general model should have a solution to the nested model as its limit. This means, if we focus on the foreign exchange model investigated in this paper, that the following limit should hold to have an economically meaningful solution

$$\lim_{\tau \to \infty} \mathrm{Re}\left[\frac{\beta_1^0(\tau)\, x_3(\tau)}{1-\beta_1^0(\tau)\, x_3(\tau)}\right] - 1 = -1, \tag{4.1}$$

where β_1^0 is the root that satisfy the continuity criterion, and where it also has been emphasized that β_1^0 and x_3 depend on the time horizon in currency trade.

The limit -1 in (4.1) has been derived by noting that $x_2 = 0$ when the time horizon in currency trade is infinitely long (see (2.15) in Proposition 2.1), which means that $\beta_1 = 0$ (see (2.22) in Proposition 2.2). Moreover, this also implies that all other parameters for past exchange rates vanish (see the two first equations in (2.19)). Consequently, when the limit in (4.1) holds for a specific solution of β_1, the foreign exchange model in (1.1) reduces, in a continuous way, to a 'traditional' foreign exchange model in which past exchange rates are not affecting the current exchange rate. Moreover, since the limit in (4.1) satisfy the condition for a solution in (2.20) to be characterized by least squares learnability, we are also selecting a REE that is attainable for currency traders, which is important.

It is also important to understand that it is not necessarily true that $\beta_1 = 0$ for all solutions of β_1 when the time horizon in currency trade is infinitely long. For example, if we investigate the case when there is only one lagged exchange rate in the general solution in (2.20), there are two roots to (2.22) in

Proposition 2.2 that determines β_1

$$\beta_1^{0,1} = \frac{1}{2x_3} \pm \sqrt{\frac{1}{4x_3^2} + \frac{x_2 \exp(-v)}{x_3}}, \tag{4.2}$$

which have the limits

$$\begin{cases} \lim_{\tau \to \infty} \beta_1^0(\tau) = 0 \\ \lim_{\tau \to \infty} \beta_1^1(\tau) = \frac{1+\beta}{\beta} \end{cases}. \tag{4.3}$$

Thus, it is only one of the two roots in (4.3) that satisfy the continuity criterion in (4.1).

Now, if we return to Figures 1–9 that was discussed above, and focus on 'the meaningful solution' in each figure, it is only this solution of β_1 that satisfy the continuity criterion in (4.1). Recall from the discussion in Section 1 that the other solutions of β_1 are not economically meaningful since the parameters for past exchange rates in (2.20) do not vanish when there is no technical trading in foreign exchange. Moreover, as is clear after inspecting the figures, there is a unique REE that satisfy the aforementioned criterion. Thus, the proposed selection criterion is able to resolve the problem that least squares learnability may select a too large subset of the REE in the foreign exchange model in (1.1).

5 Conclusions

It has been demonstrated in this paper that adaptive learning in least squares sense may be incapable to reduce, in a satisfactory way, the number of attainable equilibria in a rational expectations model. We have, therefore, proposed another selection criterion among the REE that may be used as a complement to the aforementioned selection criterion.

Specifically, in the model investigated in the paper, which was the monetary approach to exchange rate determination that was augmented with technical trading in the currency market, the proposed continuity criterion was able to single out a unique REE that was economically meaningful. The intuition behind the selection criterion was straightforward. Since it is the presence of technical trading that is causing the current exchange rate to depend on past rates, the parameters for these exchange rates must vanish when technical trading is absent in currency trade.

Finally, having in mind the large and growing literature on heterogeneous agents in economics and finance, it is our belief that the proposed continuity criterion is able to pick a unique, or at most a few, REE among several equilibria in a rational expectations model. Of course, it is a matter of future research to investigate this claim.

References

Bask, M (2006) **Announcement Effects on Exchange Rate Movements: Continuity as a Selection Criterion among the REE.** Bank of Finland Discussion Papers No. 6/2006.

Cheung, Y-W – Chinn, M D (2001) **Currency Traders and Exchange Rate Dynamics: A Survey of the US Market.** Journal of International Money and Finance, 20, 439–471.

Evans, G W – Honkapohja, S (2001) **Learning and Expectations in Macroeconomics.** Princeton, New Jersey: Princeton University Press.

Lui, Y-H – Mole, D (1998) **The Use of Fundamental and Technical Analyses by Foreign Exchange Dealers: Hong Kong Evidence.** Journal of International Money and Finance, 17, 535–545.

McCallum, B T (1983) **On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective.** Journal of Monetary Economics, 11, 139–168.

Menkhoff, L (1997) **Examining the Use of Technical Currency Analysis.** International Journal of Finance and Economics, 2, 307–318.

Oberlechner, T (2001) **Importance of Technical and Fundamental Analysis in the European Foreign Exchange Market.** International Journal of Finance and Economics, 6, 81–93.

Oberlechner, T (2004) **The Psychology of the Foreign Exchange Market.** West Sussex, England: Wiley.

Taylor, M P – Allen, H (1992) **The Use of Technical Analysis in the Foreign Exchange Market. Journal of International Money and Finance, 11,** 304–314.

Appendix

Proof of Proposition 2.1

Firstly, substitute the expectations formed by chartism and fundamental analysis in (2.1) and (2.4), respectively, into the market expectations in (2.5)

$$s^e[t+1] = \omega(\tau) E[s[t+1]] + (1-\omega(\tau))(s[t] + \gamma(s[t] - MA[t])). \quad (5.1)$$

Thereafter, substitute the long-period moving average in (2.2) into (5.1)

$$\begin{aligned} & s^e[t+1] \quad (5.2) \\ = \; & \omega(\tau) E[s[t+1]] + \\ & (1-\omega(\tau)) \left(s[t] + \gamma \left(s[t] - (1-\exp(-v)) \sum_{j=0}^{\infty} \exp(-jv)\, s[t-j] \right) \right). \end{aligned}$$

(5.2) is the market expectations summarized in one equation. Secondly, by substituting the market expectations in (5.2) into the baseline model in (2.12), the difference equation that describes the foreign exchange model is derived

$$\begin{aligned} s[t] = \; & \frac{1}{1+\beta} \cdot f[t] + \quad (5.3) \\ & \frac{\beta}{1+\beta} \cdot \left(\begin{array}{l} \omega(\tau) E[s[t+1]] + \\ (1-\omega(\tau)) \left(\begin{array}{l} s[t] + \\ \gamma \left(\begin{array}{l} s[t] - \\ (1-\exp(-v)) \cdot \\ \sum_{j=0}^{\infty} \exp(-jv)\, s[t-j] \end{array} \right) \end{array} \right) \end{array} \right), \end{aligned}$$

where the fundamentals in (2.14) has been used in the derivation of (5.3). Finally, solve (5.3) for the current exchange rate, and substitute the variables in (2.15) as well as the weight function in (2.6) into the resulting equation, and the proof is completed.

Proof of Proposition 2.2

Firstly, let $j_0 = 1$ in the first equation in (2.19)

$$\beta_1 = \frac{\beta_2 x_3 - x_2 \exp(-v)}{1-\beta_1 x_3} = \beta_2 \cdot \frac{x_3}{1-\beta_1 x_3} - x_2 \sum_{j=1}^{1} \frac{x_3^{j-1} \exp(-jv)}{(1-\beta_1 x_3)^j}. \quad (5.4)$$

Secondly, let $j_0 = 2$ in the first equation in (2.19), and substitute this equation into (5.4)

$$\begin{aligned} \beta_1 &= \frac{\beta_3 x_3 - x_2 \exp(-2v)}{1-\beta_1 x_3} \cdot \frac{x_3}{1-\beta_1 x_3} - x_2 \sum_{j=1}^{1} \frac{x_3^{j-1} \exp(-jv)}{(1-\beta_1 x_3)^j} \quad (5.5) \\ &= \beta_3 \cdot \left(\frac{x_3}{1-\beta_1 x_3} \right)^2 - x_2 \sum_{j=1}^{2} \frac{x_3^{j-1} \exp(-jv)}{(1-\beta_1 x_3)^j}, \end{aligned}$$

and repeat this procedure several times

$$\beta_1 = \beta_{j_{\max}} \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^{j_{\max}-1} - x_2 \sum_{j=1}^{j_{\max}-1} \frac{x_3^{j-1}\exp(-jv)}{(1-\beta_1 x_3)^j}. \tag{5.6}$$

Thirdly, substitute the second equation in (2.19) into (5.6)

$$\begin{aligned}\beta_1 &= -\frac{x_2\exp(-j_{\max}v)}{1-\beta_1 x_3} \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^{j_{\max}-1} - \\ &\quad x_2 \sum_{j=1}^{j_{\max}-1} \frac{x_3^{j-1}\exp(-jv)}{(1-\beta_1 x_3)^j} \\ &= -x_2 \sum_{j=1}^{j_{\max}} \frac{x_3^{j-1}\exp(-jv)}{(1-\beta_1 x_3)^j},\end{aligned} \tag{5.7}$$

and (2.22) is proved to hold. Fourthly, (2.22) has $j_{\max}+1$ roots since it is a polynomial of degree $j_{\max}+1$. Finally, that (2.23) must hold is obvious, and the proof is completed.

Proof of Proposition 3.1

Firstly, the first row in (3.7) may be written as

$$\begin{aligned}\frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1} &= \beta_2 \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^2 - \frac{x_2 x_3 \exp(-v)}{(1-\beta_1 x_3)^2} - 1 \\ &= \beta_2 \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^2 - \frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{1} \frac{x_3^{j-1}\exp(-jv)}{(1-\beta_1 x_3)^j} - 1.\end{aligned} \tag{5.8}$$

Secondly, let $j_0 = 2$ in the first equation in (2.19), and substitute this equation into (5.8)

$$\begin{aligned}\frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1} &= \frac{\beta_3 x_3 - x_2\exp(-2v)}{1-\beta_1 x_3} \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^2 - \\ &\quad \frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{1} \frac{x_3^{j-1}\exp(-jv)}{(1-\beta_1 x_3)^j} - 1 \\ &= \beta_3 \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^3 - \frac{x_2 x_3^2 \exp(-2v)}{(1-\beta_1 x_3)^3} - \\ &\quad \frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{1} \frac{x_3^{j-1}\exp(-jv)}{(1-\beta_1 x_3)^j} - 1 \\ &= \beta_3 \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^3 - \frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{2} \frac{x_3^{j-1}\exp(-jv)}{(1-\beta_1 x_3)^j} - 1,\end{aligned} \tag{5.9}$$

and repeat this procedure several times

$$\frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1} = \beta_{j_{\max}} \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^{j_{\max}} - \frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{j_{\max}-1} \frac{x_3^{j-1} \exp\left(-jv\right)}{\left(1-\beta_1 x_3\right)^j} - 1. \tag{5.10}$$

Thirdly, substitute the second equation in (2.19) into (5.10)

$$\begin{aligned} \frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1} &= -\frac{x_2 \exp\left(-j_{\max} v\right)}{1-\beta_1 x_3} \cdot \left(\frac{x_3}{1-\beta_1 x_3}\right)^{j_{\max}} - \\ & \quad \frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{j_{\max}-1} \frac{x_3^{j-1} \exp\left(-jv\right)}{\left(1-\beta_1 x_3\right)^j} - 1 \\ &= -\frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{j_{\max}} \frac{x_3^{j-1} \exp\left(-jv\right)}{\left(1-\beta_1 x_3\right)^j} - 1. \end{aligned} \tag{5.11}$$

Finally, multiply each side of the equation that determines the solutions of β_1, ie, (2.22) in Proposition 2, by $\frac{x_3}{1-\beta_1 x_3}$

$$\frac{\beta_1 x_3}{1-\beta_1 x_3} = -\frac{x_3}{1-\beta_1 x_3} \cdot x_2 \sum_{j=1}^{j_{\max}} \frac{x_3^{j-1} \exp\left(-jv\right)}{\left(1-\beta_1 x_3\right)^j}, \tag{5.12}$$

and substitute (5.12) into (5.11)

$$\frac{d\left(\frac{d\beta_1}{d\tau_a}\right)}{d\beta_1} = \frac{\beta_1 x_3}{1-\beta_1 x_3} - 1. \tag{5.13}$$

Obviously, the parameter β_1 is locally asymptotically stable under (3.2) when (3.12) holds, and the proof is completed.

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Suomen Pankki
Bank of Finland
P.O.Box 160
FI-00101 HELSINKI
Finland

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