BANK OF FINLAND DISCUSSION PAPERS

20 • 2003

Eric Schaling
Research Department
18.8.2003

Learning, inflation expectations and optimal monetary policy

Suomen Pankin keskustelualoitteita
Finlands Banks diskussionsunderlag

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The views expressed are those of the author and do not necessarily reflect the views of the Bank of Finland.

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Learning, inflation expectations and optimal monetary policy

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Abstract

In this paper we analyse disinflation policy in two environments. In the first, the central bank has perfect knowledge, in the sense that it understands and observes the process by which private sector inflation expectations are generated; in the second, the central bank has to learn the private sector inflation forecasting rule. With imperfect knowledge, results depend on the learning scheme that is employed. Here, the learning scheme we investigate is that of least-squares learning (recursive OLS) using the Kalman filter. A novel feature of a learning-based policy – as against the central bank's disinflation policy under perfect knowledge – is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information.

Key words: learning, rational expectations, separation principle, Kalman filter, time-varying parameters, optimal control

JEL classification numbers: C53, E43, E52, F33

Oppiminen, inflaatio-odotukset ja optimaalinen rahapolitiikka

Suomen Pankin keskustelualoitteita 20/2003

Eric Schaling
Tutkimusosasto

Tiivistelmä

Tässä tutkimuksessa vertaillaan inflaationvastaisen rahapolitiikan toimintaa kahdenlaisissa olosuhteissa. Ensimmäisessä tapauksessa keskuspankilla on täydellinen tietämys yleisön inflaatio-odotusten syntymekanismista, ja toisessa tapauksessa keskuspankin on opittava se sääntö, jota yleisö käyttää inflaatio-odotuksia muodostaessaan. Kun keskuspankin tietämys ei ole täydellistä, tulokset riippuvat siitä, miten keskuspankki oppii säännön. Tässä tutkimuksessa tarkastellaan oppimista, joka tapahtuu Kalmanin suotimella rekursiivista pienimmän neliösumman menetelmää käyttäen. Oppimiseen perustuvalle politiikalle on ominaista, että rahapolitiikan akkommodaation aste (eli se, missä määrin politiikka toteuttaa yleisön inflaatio-odotuksia) ei ole vakio, vaan riippuu talouden tilasta. Keskuspankin käyttäytyminen siis muuttuu inflaation hidastuessa ja keskuspankin kerätessä lisää informaatiota. Tämä piirre erottaa oppimiseen perustuvan rahapolitiikan siitä, miten keskuspankki käyttäytyy täydellisen tietämyksen oloissa.

Avainsanat: oppiminen, rationaaliset odotukset, separaatioperiaate, Kalmanin suodin, aikariippuvat parametrit, optimaalinen kontrolli

JEL-luokittelu: C53, E43, E52, F33

Contents

Abstract 3
Tiivistelmä 4

1 Introduction 7

2 The environment 10

3 The perfect knowledge benchmark 13
3.1 Fully credible regime switch 13
3.2 'Endogenous learning' by the private sector 14
3.3 Discretion 15
3.4 Commitment 16
3.5 Comparing discretion and commitment: A calibration 19

4 Imperfect knowledge, filtering and prediction 21
4.1 The Kalman filter 21
4.2 Convergence and limit beliefs 23

5 Central bank learning and optimal monetary policy 25
5.1 Two limiting cases 26
5.2 Optimal learning and the value of experimentation 27
5.3 The case of passive learning 28
5.3.1 Discretion 29
5.3.2 Commitment 31
5.3.3 Incomplete learning and limit beliefs 33

6 Conclusions and suggestions for further research 35

References 36

Appendix 1. Steady state equilibrium 39
Appendix 2. The private sector's optimal predictor for the credibility of the inflation target 40
Appendix 3. The commitment solution 40
Appendix 4. The central bank's optimal filtering problem 46

1 Introduction

As pointed out by Bullard (1991), in the three decades since the publication of the seminal work on rational expectations (RE) in the early 1960s, a steely paradigm was forged in the economics profession regarding acceptable modelling procedures. Simply stated, the paradigm was that economic actors do not persist in making foolish mistakes in forecasting over time.

Since the late 1980s researchers have challenged this paradigm by examining the idea that *how* systematic forecast errors are eliminated may have important implications for macroeconomic policy. Researchers who have focused on this question have been studying what is called 'learning', because any method of expectations formation is known as a learning mechanism. Thus, since the late 1980s a learning literature, or learning paradigm, developed.[1] An excellent introduction to – and survey of – this paradigm is presented in Evans and Honkapohja (2001).

A different strand of literature in the economics profession has been dealing with optimal control or dynamic optimisation. The method of dynamic programming advanced by Bellman has been a main tool for optimisation over time under uncertainty.

In general there are few papers in the literature that combine the themes of *learning* and (optimal) *control*. An exception is recent and important work by Wieland (2000a,b). Wieland (2000a) analyses the situation where a central bank has limited information concerning the transmission channel of monetary policy. Then, the CB is faced with the difficult task of simultaneously *controlling* the policy target and estimating (*learning*) the impact of policy actions. Thus, the so-called separation principle does not hold, and a trade-off between estimation and control arises because policy actions influence estimation (learning) and provide information that may improve future performance. Wieland analyses this trade-off in a simple model with parameter uncertainty and conducts dynamic simulations of the central bank's decision problem.

In this paper we apply the themes of learning and control to the problem of how a central bank should organize a disinflation process, ie *how to reduce inflation*. Thus, our approach follows recent relevant work by Sargent (1999).

Central banks throughout the world are moving to adopt long-run price stability as their primary goal. Thus, there is agreement among central bankers, academics and financial market representatives that low or zero inflation is the appropriate long-run goal of monetary policy. However, there is less agreement on what strategies should be adopted to achieve price stability. For example, on the one hand we have the view that a major cause of rising unemployment in the

[1] Important papers are Lucas (1987) and Marcet and Sargent (1988, 1989a,b,c).

1980s in OECD countries was the tight monetary policy that those countries pursued to reduce inflation. On the other we have the view that a sharp disinflation may be preferable to gradualism because the latter invites speculation about future reversals or U-turns in policy.

The received view in the literature – as expressed by King (1996) at the Kansas City Fed symposium on Achieving Price Stability at Jackson Hole – seems to be for a gradual timetable, with inflation targets consistently set below the public's inflation expectations.

Throughout, King emphasizes the role of *learning* by private agents. He shows how the optimal speed of disinflation depends crucially on whether the private sector immediately believes in the new low inflation regime or not. If they do, the best strategy is to disinflate quickly, since the output costs are zero. Of course, if expectations are slower to adapt, disinflation should be more gradual as well.

King illustrates both these polar cases, labeling the first a 'completely credible regime switch' and the second 'exogenous learning'. He also considers the case where private sector expectations are a weighted average of the central bank's long-run inflation target and the lagged inflation rate. This is termed 'endogenous learning'. Obviously, endogenous learning is a mixture of a completely credible regime switch and exogenous learning. Private sector expectations do not adjust immediately (they depend on actual inflation experience, and hence on the policy choices made during the transition), but are not completely independent of monetary policy decisions either.

In his discussion of *endogenous learning* King says that there are good reasons for the private sector to suppose that in trying to learn about the future inflation rate many of the relevant factors are exogenous to the path of inflation itself. But a central bank may try to convince the private sector of its commitment to price stability by choosing to reduce its inflation target towards the inflation target quickly. King calls this 'teaching by doing'. Then the choice of a particular inflation rate influences the speed at which expectations adjust to price stability.

Teaching by doing effects have recently been analysed by Hoeberichts and Schaling (2000) for simple macro models with both linear and nonlinear (convex) Phillips curves. They also allow the central bank's 'doing' to affect private sector learning. Of course, if the CB recognises its potential for active 'teaching' its incentive structure changes. More specific, it should realize that by disinflating faster, it can reduce the associated output costs by 'teaching' the private sector that it means business. Thus, the dependence of private sector expectations on the actual inflation rate should be part of its optimisation problem. This is in fact what they find: allowing for 'teaching by doing' effects always speeds up the disinflation vis-à-vis the case where this effect is absent. So, their result is that 'speed' in the disinflation process does not necessarily 'kill' in the sense of creating large output losses.

In this paper we analyze disinflation in two environments. One in which the central bank has *perfect knowledge*, in the sense that it understands and observes the process by which private sector inflation expectations are generated, and one in which the central bank has to *learn* the private sector inflation forecasting rule. Here following Evans and Honkapohja (2001), the learning scheme we investigate is that of *least-squares learning* (recursive OLS) using the Kalman filter.

For the case of perfect knowledge we find that the optimal disinflation is faster under commitment than discretion. Next, in the commitment case the disinflation is less gradual, the higher the central bank's rate of time preference and – interestingly – the higher the degree of persistence in inflation expectations.

With imperfect knowledge results depend on the learning scheme that is employed. A novel feature of the passive learning policy – compared to the central bank's optimal disinflation policy under perfect knowledge – is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information.

The remainder of this paper is organized as follows. Section 2 discusses private sector behaviour regarding the credibility of the central bank's inflation target. In section 3 we present the benchmark case of perfect knowledge and contrast discretion and commitment. Imperfect knowlegde and the Kalman filter are introduced in section 4. Section 5 analyses disinflation policies under imperfect knowledge. The plan of the paper is summarised in table 1.1.

Table 1.1 **Classification of cases**

	Perfect knowledge	Imperfect knowledge; Learning via Kalman filter
Static (one-period) optimisation	Discretion; Section 3.3	Optimal learning; Sections 5.2 and 5.3.1
Dynamic optimisation	Commitment; Section 3.4	Optimal and passive learning; Sections 5.2 and 5.3.2

We conclude in section 6. The appendices contain the derivation of steady state relationships, the commitment solution, and the derivation of the Kalman filter equations used in the main text.

2 The environment

In this section we examine the speed of disinflation that would be chosen by a central bank in a world in which monetary policy affects real output in the short run but not in the long run. We use a simple macroeconomic model that combines nominal wage and price stickiness and slow adjustment of expectations to a new monetary policy regime. The model has three key equations – for monetary policy preferences, aggregate supply, and inflation expectations.

The central bank minimizes the following loss function

$$\underset{\{\pi_t\}_{t=0}^{\infty}}{\text{Min}}\, E_t \sum_{t=0}^{\infty} \frac{\delta^t}{2} \left[a(\pi_t - \pi^*)^2 + (z_t - z^*)^2 \right] \tag{2.1}$$

Here π is the inflation (rate) in year t, z is the natural logarithm of output, π^* and z^* are the central bank's inflation and output targets, $0<a<\infty$ represents the central bank's relative weight on inflation stabilization, while the parameter δ (which fulfils $0<\delta<1$) denotes the discount factor (ie a measure of the policy horizon). This expectation is conditional on the central bank's information set in period t. In what follows we set z^* equal to the natural rate of output (which in turn is normalized to zero).

The model is simple. Aggregate supply exceeds the natural rate of output when inflation is higher than was expected by agents when nominal contracts were set. This is captured by equation (2.2) which is a simple short-run Phillips curve.

$$z_t = \pi_t - \hat{E}_{t-1}\pi_t \tag{2.2}$$

where z is the natural logarithm of the output gap.[2] Here the superscript ^ indicates that the expectation of inflation is the subjective expectation (belief) of private agents. This belief does not necessarily coincide with a rational expectation.

Private agents believe that inflation will be reduced from its initial level towards the inflation target, but are not sure by how much. More specific, the public's inflation beliefs are given by

$$\pi_t = \pi^* + u_t \tag{2.3}$$

[2] It would be straightforward to extend the Phillips curve with an aggregate supply shock. Standard assumptions on nominal rigidities would then imply that inflation expectations are set before the shock is observed, while monetary policy would bet set in full knowledge of the shock to output.

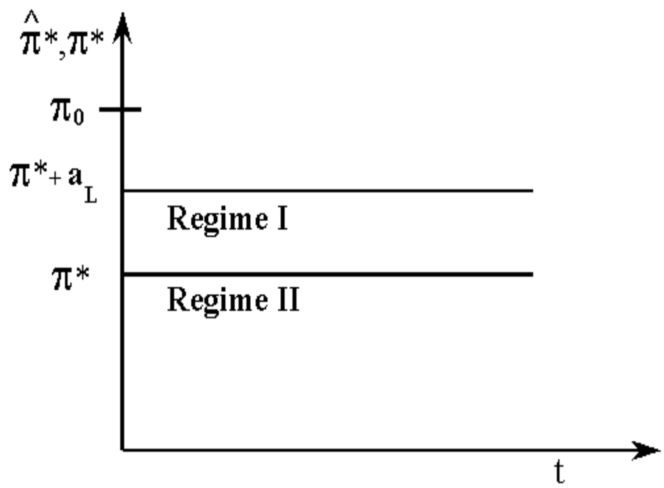
where u is a shock to the inflation rate. So, we assume that equation (2.3) is the *perceived law of motion* of private agents.

In order to study changes in inflation expectations, we extend this system with a stochastic process for u. As in Bullard and Schaling (2001) we use a two-state process defined by

$$u_t = \pi^b \cdot s_t = \begin{cases} \pi^b & \text{if} \quad s_t = 1 \\ 0 & \text{if} \quad s_t = 0 \end{cases} \qquad \text{where } 0 < \pi^b < \pi_0 - \pi^* \tag{2.4}$$

If $\pi^b \to 0$ there is no difference between the two regimes, and so we can think of π^b as scaling the effect of the difference in inflation beliefs in the two regimes. In the case where the parameter $\pi^b \to 0$, there is a completely credible regime switch. Thus, π^b is a measure for the extent to which the public's beliefs (and consequently expectations) about inflation are uncoupled from the intended policy objective.[3] Figure 2.1 illustrates. $\hat{\pi}$ is the private sector's perceived law of motion of the inflation rate, π_0 is the initial inflation rate and $a_L = \pi^b$ is the difference in inflation beliefs in the two regimes.

Figure 2.1 **The perceived law of motion for inflation**



[3] I have borrowed this terminology from Orphanides and Williams (2002). Or, using a term from the older time-consistency literature, it can be seen as a measure for the lack of 'credibility' of the CB's inflation target.

The unobserved state of the system s_t takes on a value of zero or one, and follows a two-state Markov process.[4] There is an associated transition probability matrix $T = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$, where

$$\begin{aligned} &\Pr ob[S_{t+1} = 1 | S_t = 1] = p, \\ &\Pr ob[S_{t+1} = 0 | S_t = 1] = 1 - p, \\ &\Pr ob[S_{t+1} = 0 | S_t = 0] = q, \\ &\Pr ob[S_{t+1} = 1 | S_t = 0] = 1 - q, \end{aligned} \tag{2.5}$$

So the probability of remaining in the high (low) state conditional on being in the high (low) state in the previous period is p(q), and the probability of switching from the high (low) to the low (high) state is 1–q(1–p).

As suggested by Hamilton (1989), the stochastic process for equation (2.5) admits the following AR(1) representation:

$$s_{t+1} = (1 - q) + \gamma s_t + v_{t+1} \tag{2.6}$$

where $\gamma \equiv p + q - 1$, and v_t is a (discrete) white noise process with mean zero and variance σ_v^2.[5]

We want to study the dynamics of the system following a switch to a new regime, which in our model will constitute a switch from one state to the other. We are particularly interested in the effects of this switch on the dynamics of private sector inflation beliefs.

---

[4] We adopt the usual convention that for discrete-valued variables, capital letters denote the random variable and small letters a particular realization. If both interpretations apply we will use small letters.

[5] For the technical details see Appendix 1.

3 The perfect knowledge benchmark

To get some straightforward results, we assume that the central bank understands and observes the process by which private sector inflation expectations are generated. This is the benchmark case of *perfect knowledge*. So, in this section we analyze learning by the private sector, but not by the central bank. We model least-squares learning by the central bank in section 4.

Consider a switch from a monetary policy regime in which inflation has averaged π_0 to a regime of price stability in which inflation equals the inflation target π^*. What is the optimal transition path? That will depend upon how quickly private sector inflation expectations adjust to the new regime. Following King (1996), it is useful to consider two cases: (1) a completely credible policy regime switch: private sector expectations adjust immediately to the new policy reaction function – this is the case of rational or model consistent expectations; (2) 'endogenous learning': the private sector's forecasting rule depends on the policy choices made in the new regime.[6]

3.1 Fully credible regime switch

With a completely credible regime change, private sector inflation expectations are consistent with the new inflation target. From equation (2.4) it can be seen that this is the case where $\pi^b = 0$, so that we have

$$\hat{E}_{t-1}\pi_t = \pi^*$$

Hence output and inflation are given by

$$z_t = 0$$

$$\pi_t = \pi^*$$

Since the level of output is independent of the inflation rate, policy can aim at price stability without any expected output loss. The optimal policy is to move immediately to the inflation target.

[6] We do not consider the case of exogenous learning where expectations are formed independent of the actual policy choices in the new regime.

3.2 'Endogenous learning' by the private sector

This is the more general case. King defines endogenous learning as the case where the private sector's forecasting rule depends on the policy choices made in the new regime.

To see that this in fact the relevant situation in our model, we apply $(1-\gamma L)$ where L is the lag operator $(L^j x_t = x_{t-j})$ to (2.3) and take account of (2.6),

$$(1-\gamma L)\pi_t = \pi^b \cdot (1-\gamma L)s_t + (1-\gamma)\pi^* \tag{3.1}$$

Substituting for $(1-\gamma L)s_t$ from (2.6), we may rewrite equation (3.1) as

$$\pi_t = \gamma\pi_{t-1} + (1-\gamma)\pi^* + \pi^b \cdot (1-q) + a_1 \cdot v_t \tag{3.2}$$

Hence, the combination of the private sector's perceived law of motion (equation (2.3)) and the AR(1) representation of the inflation state (equation (2.6)) gives rise to a first-order stochastic difference equation for inflation.

Taking expectations at time t–1 of equation (3.2), where the expectations operator $\hat{E}$ refers to agents' subjective expectations, we obtain

$$\hat{E}_{t-1}\pi_t = \gamma\pi_{t-1} + (1-\gamma)\pi^* + \pi^b \cdot (1-q) \tag{3.3}$$

Our main finding is that the private sector's optimal inflation forecast – in an environment where the perceived law of motion is one with unobserved regime shifts – involves a lagged inflation term. More precise, at time t–1 private agents' inflation expectations for period t are a linear function of the inflation target, the lagged inflation rate and a constant, where the coefficients are functions of the structural parameters of the Markov switching process, as shown in equation (3.3). Thus, indeed the private sector's forecasting rule depends on the policy choices made in the new monetary policy regime; this can be seen from the presence of the π_{t-1} term.

An important limiting case of equation (3.3) is when $\gamma = \pi^b = 0$. In this case the shock to inflation becomes serially uncorrelated and the private sector's optimal inflation forecast is the inflation target.

3.3 Discretion

Now we solve the model for the case where the central bank does not internalize the constraint (3.3). This is equivalent to the case where $\delta \to 0$. In this discretionary case the minimization problem of the central bank reduces to the static problem

$$\underset{\{\pi_t\}}{\text{Min}}\, E_t \frac{1}{2}\left[a(\pi_t - \pi^*)^2 + (z_t - z^*)^2\right]$$

The associated optimal policy is

$$\pi_t = \frac{1}{1+a} x_t + \frac{a}{1+a} \pi^* \tag{3.4}$$

The optimal transition to price stability is to allow inflation to fall gradually. The inflation rate should decline as a constant proportion of the exogenous expected inflation rate. That proportion depends on the weight a attached to the importance of keeping inflation close to the inflation target relative to keeping output close to its natural rate. The inflation rate moves gradually to the level of the inflation target, but is always below expected inflation. Figure 3.1 shows an example in which expectations decline steadily. Note that inflation adjusts to its long-run value gradually over time.

The optimal path may be contrasted with the two extremes of pursuing price stability from the outset – a 'cold turkey' strategy – and setting the inflation rate to accommodate inflation expectations – an accommodation strategy. The 'cold turkey' strategy is defined by

$$\pi_t = \pi^* \qquad \forall t$$

Price stability is achieved even during the transition period, but only at the cost of an expected cumulative output loss of

$$CZL = -\sum_{t=1}^{T} (x_t - \pi^*)$$

A strategy of full accommodation is defined by

$$\begin{aligned} \pi_t &= x_t \\ &= \gamma \pi_{t-1} + (1-\gamma)\pi^* + \pi^b \cdot (1-q) \end{aligned}$$

It is clear that such a strategy eliminates any output loss, but at the cost of inflation falling only at the exogenous rate of decline of private sector inflation expectations.

3.4 Commitment

In his discussion of *endogenous learning*, King (1996, p. 68) says that there are good reasons for the private sector to suppose that in trying to learn about the future inflation rate many of the relevant factors are exogenous to the path of inflation itself. But a central bank may try to convince the private sector of its commitment to price stability by choosing to reduce its inflation target towards the inflation target quickly. King calls this 'teaching by doing'. Then the choice of a particular inflation rate influences the speed at which expectations adjust to price stability.

In this section we allow the central bank's 'doing' to affect private sector learning. Of course, if the CB recognises its potential for active 'teaching' its incentive structure changes. More specific, it should realise that by disinflating faster, it can reduce the associated output costs by 'teaching' the private sector that it means business. Thus, the dependence of private sector expectations on the actual inflation rate – equation (3.3) above – should be part of its optimisation problem. In what follows we refer to this as the case of 'commitment'.

Now, the central bank's problem is to

$$\underset{\{\pi_t\}_{t=0}^{\infty}}{\mathrm{Max}}\, E_t \sum_{t=0}^{\infty} \frac{\delta^t}{2}\left[-a(\pi_t - \pi^*)^2 - (z_t)^2\right] \tag{3.5}$$

subject to

$$z_t = \pi_t - \hat{E}_{t-1}\pi_t \tag{3.6}$$

and

$$\hat{E}_{t-1}\pi_t = \gamma\pi_{t-1} + (1-\gamma)\pi^* + \pi^b \cdot (1-q) \tag{3.7}$$

It is convenient to define $x_t = \hat{E}_{t-1}\pi_t$ as the state variable and $u_t = \pi_t$ as the control. We solve this problem by the method of Lagrange multipliers.[7] Introduce

[7] For a discussion of the relative merits of the methods of dynamic programming and Lagrange, see Schaling (2001).

the Lagrange multiplier μ_t, and set to zero the derivatives of the Lagrangean expression:

$$L = E_0\left[\sum_{t=0}^{\infty}\left\{\frac{\delta^t}{2}\left[-a(u_t - u^*)^2 - (u_t - x_t)^2\right] - \delta^{t+1}\mu_{t+1}\left[x_{t+1} - \gamma u_t - (1-\gamma)u^* - \pi^b(1-q)\right]\right\}\right] \tag{3.8}$$

In Appendix 3, it is shown that the first-order condition for this problem can be written as[8]

$$\pi_t = C_1\hat{E}_{t-1}\pi_t + C_2 \tag{3.9}$$

The coefficients $\frac{1}{(1+a)+\delta\gamma^2} < C_1 < \frac{\delta\gamma^2+1}{(1+a)+\delta\gamma^2}$ and $0 \leq C_2(C_1) \leq \pi^*$, are given by

$$C_1 = \frac{1}{2}\left\{\left[\frac{(1+a)+\delta\gamma^2}{\delta\gamma^2}\right] - \sqrt{\frac{\left[(1+a)^2+\delta\gamma^2\right]^2 - 4\delta\gamma^2}{\delta^2\gamma^4}}\right\} \tag{3.10}$$

$$C_2 = \frac{\delta\gamma\left[(C_1-1)\pi^b(1-q)\right] + \left[\delta\gamma(C_1-1)(1-\gamma)+a\right]\pi^*}{(1+a) - \delta\gamma\left[(C_1-1)\gamma+1\right]} \tag{3.11}$$

From these equations it can be seen that the optimal values of the coefficients are nonlinear functions of the central bank's weight on inflation stabilization, the discount rate and the degree of persistence in inflation expectations.

We now show:[9]

PROPOSITION 3.1: The higher a the lower the optimal value of the feedback parameter C_1.

For the proof, see Appendix 3. The argument is as follows. A central bank that is more concerned with inflation will be less concerned with output, and hence will accommodation inflation expectations to a lesser extent. To give a numerical example, for our basic parameter configuration (see Table 3.1), $C_1 \approx 0.71$. If we increase a to 0.5, say, C_1 decreases to 0.55.

---

[8] See Bullard and Schaling (2001) and Schaling (2002) for examples of the method of solving for the optimal policy.

[9] Additional results (propositions) are presented in Appendix 3.

We can also derive a result in terms of the central bank's degree of time preference. In Appendix 3 we verify:

PROPOSITION 3.2: If C_1 is smaller than an upper bound $\overline{C}_1$, the higher δ the lower the optimal value of the feedback parameter C_1.

The intuition is that the higher δ, the more concerned the central bank is about the future, ie the longer is its policy horizon (conversely if this parameter is zero, the central bank only 'lives for today'). Under a live for today policy, the central bank is not interested how monetary accommodation today effects inflation expectations for tomorrow. If it becomes more concerned about the future (higher δ) however, it will start paying attention to expected future 'expectations invoices', and accommodate current inflation expectations to a lesser extent, hence the monetary accommodation coefficient C_1 falls. To give a numerical example for our basic parameter configuration (see Table 3.1) and $\delta = 0.225$, $C_1 \approx 0.81$. If we increase δ to 0.9, say, C_1 decreases to 0.71 (see above).

Let us now look how the central bank responds to less faith in its inflation target, as proxied by a higher weight placed on past inflation by private agents in forecasting future inflation. This is the case of more persistence in inflation expectations. It is easy to show:

PROPOSITION 3.3: If C_1 is smaller than an upper bound $\overline{C}_1$, the higher γ the lower the optimal value of the feedback parameter C_1.

The argument is that the higher γ, the better agents 'remember' past inflation rates, and use those to forecast future inflation. If the central bank cares about the future ($\delta \neq 0$), it will try to offset this 'memory effect' by less monetary accommodation. In this way it lets the lower inflation outcomes influence the *level* of expectations to try to offset the higher *persistence* of those expectations. For example, for our basic parameter configuration (see Table 3.1) and $\gamma = 0.2$, $C_1 \approx 0.83$. If we increase γ to 0.9, C_1 decreases to 0.71.

3.5 Comparing discretion and commitment: A calibration

We now turn to a calibrated case to illustrate our results. Table 3.1 summarizes the parameter values used in our calibrated economy.

Table 3.1 **Parameter configuration**[1]

Parameter	Controls	Value
p		0.95
q		0.95
γ	Persistence in PS inflation expectations	0.9
$\pi^b = \underline{\pi}^b$	Difference in PS inflation beliefs	13
π_0	Initial inflation rate	20
π^*	CB's inflation target	2.5
a		0.2
δ		0.9

[1] We illustrate our analytical findings using these calibrations.

Using the above parameter values, Figures 3.1 and 3.2 show the discretionary and commitment disinflation policy respectively.

Figure 3.1 **Disinflation under discretion**

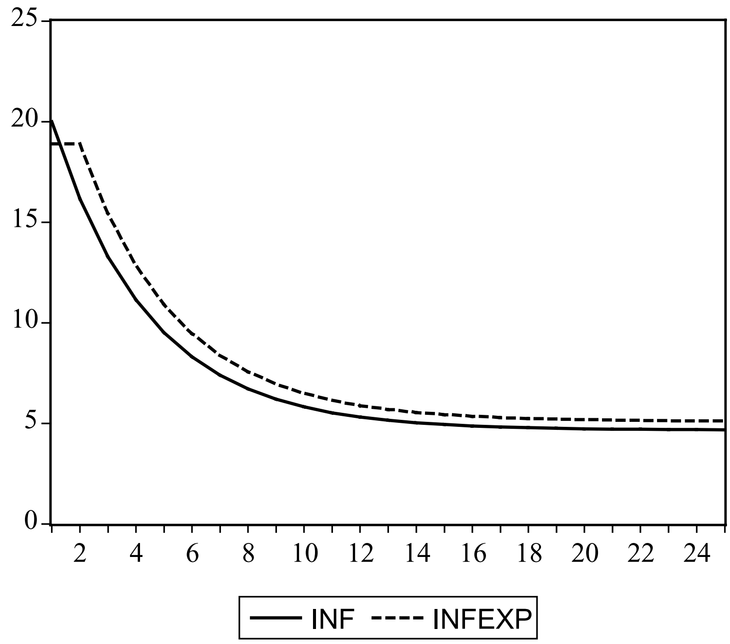
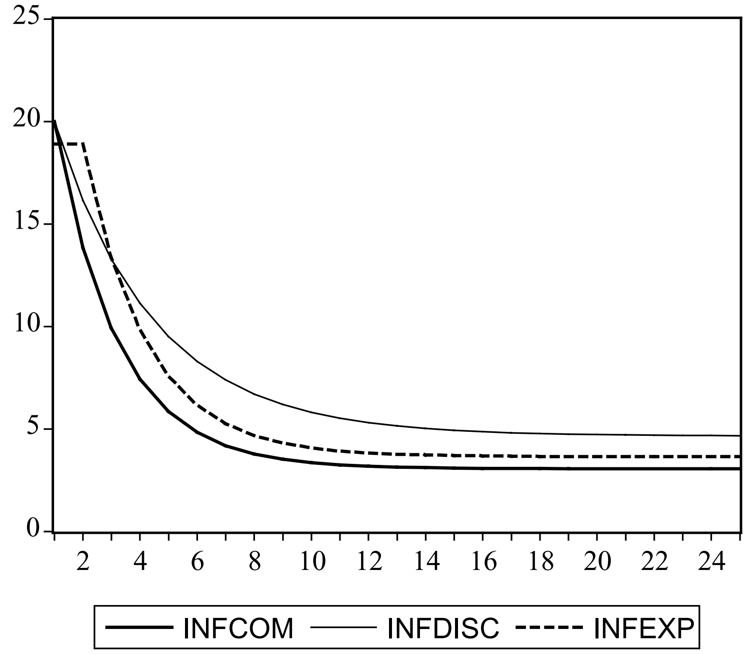


Figure 3.2 **Disinflation under commitment**



From Figure 3.2 it can be seen that disinflation under commitment is always faster than under discretion. As King puts it,

> 'The general prediction of the learning models is that the inflation rate will fall faster in the earlier years of the transition and will always lie below expected inflation.' (1996, p. 70).

The reason is that the choice of a particular inflation rate influences the speed at which expectations adjust. In fact, a quicker disinflation policy 'pays for itself' by speeding up the adjustment of expectations. Of course, the central bank takes this fact into account when deciding on its disinflation program. Another way to think about this, is that central bank credibility – a crucial variable in defining the output loss of the disinflation – here is endogenous. In fact, the central bank's credibility can be increased by the CB by starting off the disinflation by putting its money where its mouth is.

4 Imperfect knowledge, filtering and prediction

The case of perfect knowledge can be represented as follows. First, at time t–1 the central bank sets its expectation (forecast) for private sector inflation expectations. Next, also at time t–1 the private sector sets its forecast, x_t, of inflation for period t. Then, the CB sets inflation at time t based on its own forecast, where – importantly – the forecast turns out to be correct. Figure 4.1 summarizes.

Figure 4.1 **Perfect knowledge: timing of events**

Time t–1		Time t
Stage 1:	Stage 2:	Stage 3:
CB forecasts PS inflation expectations; ie sets $E_{t-1}[x_t]$.	PS forecasts inflation using equation (3.3), ie sets x_t.	CB decides on monetary policy, ie sets $\pi_t(E_{t-1}[x_t]) = \pi_t(x_t)$, on the basis of either discretion or 'commitment'.

Of course, the idea that the CB can forecast or – what is actually equivalent – observe x_t without error is hardly realistic. This assumption will now be relaxed.

4.1 The Kalman filter

Suppose the CB can no longer observe private agents' inflation expectations x_t without error. This means it does not observe x_t at the time it has to set inflation, but that the CB has a noisy forecast (signal) y_t at time t–1 on x_t which it subsequently uses to set the inflation rate π_t at time t.

More specifically, let y_t be decomposed into x_t and ε_t,

$$y_t = x_t + \varepsilon_t \tag{4.1}$$

where y_t is the central bank's forecast (signal) of x_t, and ε_t is its forecast error. We assume the forecast error is white noise, with variance σ_ε^2. So, the central bank's forecast is unbiased, but not without error. An important limiting case of (4.1) is when $\sigma_\varepsilon^2 \to 0$ and we are back to the previous case of perfect knowledge, ie $y_t = x_t$.

To make the problem more tractable we set $\pi^b = [\gamma - (1-\gamma)\pi^*]/1-q \equiv \underline{\pi}^b$. Then, π^b is no longer a free parameter (on those occasions the symbol $\underline{\pi}^b$ is used). This assumption has the advantage of reducing the dimension of the state space in the central bank's optimal filtering problem. In this way we avoid what Ljungqvist and Sargent (2000) call the 'curse of dimensionality'.[10]

Setting $\pi^b = \underline{\pi}^b$ and defining $x_t \equiv \hat{E}_{t-1}\pi_t$, equation (3.7) simplifies to

$$x_t = \gamma(\pi_{t-1} + 1) = \gamma w_{t-1} \qquad \text{where} \quad w_{t-1} \equiv (\pi_{t-1} + 1) \tag{4.2}$$

Note that the situation above can be represented as the case where the CB believes that private sector inflation expectations follow the stochastic process

$$y_t = \gamma w_{t-1} + \varepsilon_t \tag{4.3}$$

corresponding to the true (actual) law of motion of PS inflation expectations, but that γ is unknown to them (this can be seen by substituting the expression for private sector inflation expectations (4.2) into equation (4.1)). Thus, here we assume that the central bank employs a reduced form of the expectations formation process that is correctly specified.[11]

So, we assume that equation (4.3) is the *perceived law of motion* of the central bank and that the policymaker attempts to estimate γ. Following Evans and Honkapohja (2001), this is our key bounded rationality assumption: we back away from the rational expectations assumption, replacing it with the assumption that, in forecasting private sector inflation expectations, the central bank acts like an econometrician.

The central bank's estimates will be updated over time as more information is collected. Letting c_{t-1} denote its estimate through time t–1, the central bank's one-step-ahead forecast at t–1, is given by

$$E_{t-1}[y_t] = c_{t-1} w_{t-1} \tag{4.4}$$

Under this assumption we have the following model of the evolution of the economy. Let Ω_t be the central bank's information set for time t. Suppose that at time t–1 the central bank has data on the economy from periods $\tau = t-1, \ldots, t-n$. Thus the time t–1 information set is $\Omega_{t-1} = \{y_\tau, w_\tau\}_{\tau=t-n}^{t-1}$. Imagine that we have already calculated the ordinary least squares estimate c_{t-1} of γ in the model $(y_{t-1}; w_{t-2}\gamma, \sigma_\varepsilon^2)$. Given the new information, which is provided by the observations y_t, w_{t-1}, we wish to form a revised or updated estimate of γ. Using

[10] For the technical details see Appendix 4. In addition, we then choose the parameter π_0 in such a way that the inequality $0 < \pi^b < \pi_0 - \pi^*$ (see equation (2.4)) remains satisfied.

[11] Instead – as pointed out by Orphanides and Williams (2002) – the learner may be uncertain of the correct from and estimate a more general specification, for example, in our case a linear regression with additional lags of expected inflation which nests (4.3).

data through period t, the least squares regression parameter for equation (4.3) can be written in recursive form (see Appendix 4 for details)

$$c_t = c_{t-1} + \kappa_t (y_t - w_{t-1} c_{t-1}) \tag{4.5}$$

$$p_t = p_{t-1} - \kappa_t w_{t-1} p_{t-1} \tag{4.6}$$

The method by which the revised estimate of γ is obtained may be described as a filtering process, which maps the sequence of prediction errors into a sequence of revisions; and $\kappa_t = p_t w_{t-1} (\sigma_\varepsilon^2)^{-1}$ may be described as the gain of the filter, ie the Kalman gain.[12] It is notable that, as the value of t increases, the value of κ_t will decrease. Thus, the impact of successive prediction errors upon the values of the estimate of γ will diminish as the amount of information already incorporated in the estimate increases.

4.2 Convergence and limit beliefs

We now have a fully specified dynamic system defined by equations (4.2), (4.4), (4.5) and (4.6). The question of interest is now whether $c_t \to \gamma$ as $t \to \infty$. In fact, as time goes by the Kalman gain will go to zero (see Appendix 4 for technical details), so that indeed the estimated parameter converges to the true parameter. However, this result only goes through for the case of an exogenous inflation (data) sequence; that is the case where the separation principle holds.

For the relevant case here, where inflation sequence is not exogenous, but is chosen to be the outcome of an optimization problem, the separation principle does not hold and standard convergence results are not applicable. Thus, following WieIand (2000b) it remains to discuss the asymptotic properties of estimates and forecasts on the one hand, and policy actions on the other. In what follows we use the term *beliefs* as shorthand for estimates and forecasts. This also has the advantage of following the terminology in the literature, especially Easley and Kiefer (1988) and Kiefer and Nyarko (1989) (KN hereafter).

Standard convergence results are not applicable, because along any sample path for which the parameter estimate converges, the sequence of policy actions also converges. If actions converge too rapidly, they may not generate enough information for identifying the unknown parameter and the limit distribution representing the central bank's limit belief need not be centred on the true parameter value. KN show that the process of posterior beliefs converges to a limit belief $\bar{c}$ for any multiple linear regression process under minimal

[12] Equations (4.5) and (4.6) are known as the updating, or smoothing equations.

distributional assumptions. However, this convergence result does not pin down the limit belief itself. There may exist multiple limit beliefs that are outcomes of optimal policy but do not coincide with the true parameter value. Incorrect beliefs may be self-reinforcing because learning is costly (and the more so, the higher the value of the parameter a) and actions that would be sub-optimal under the truth, may be optimal under these subjective incorrect beliefs.

For the case of a simple regression with known error distribution (as is employed here), KN show that all limit belief and policy pairs $(\bar{c},\bar{\pi})$, whether correct or not, share three properties, *belief invariance, one-period optimization and mean prediction*, which can be used to describe the set of possible limit beliefs. In our model these pairs are given in Figure 4.2.

Figure 4.2 **Limit beliefs and policy pairs**

Belief invariance: $\bar{c} = \dfrac{\bar{y}}{(1+\bar{\pi})}$

One-period optimization: $\bar{\pi} = \dfrac{\bar{c}}{(1+a)-\bar{c}} + \dfrac{a}{(1+a)-\bar{c}}\pi^*$

Mean prediction: $E[\gamma|\bar{c}](1+\bar{\pi}) = \gamma(1+\bar{\pi})$

First, *belief invariance* simply follows from the convergence result. For a belief to be a limit belief it needs to be self-reinforcing, ie given the limit action $\bar{\pi}$, updating and predicting according to the Kalman filter equations should again generate the limit belief $\bar{c}$. Thus, limit belief and policy pairs define fixed points of the Kalman filter equations. Second, with invariant beliefs the dynamic optimization problem reduces to the static problem of minimizing the expected one-period loss. Thus, *one-period optimization* refers to the fact that the limit action $\bar{\pi}$ minimizes the expected one-period reward conditional on the limit belief $\bar{c}$.[13] Third, if the control variable is held constant at $\bar{\pi}$ forever, the central bank will at minimum learn the associated mean value of private sector inflation expectations in the limit, which constitutes the *mean prediction property* of limit beliefs.

[13] Note that the solution for one-period optimization follows from the discretionary solution (5.11) if c_{t-1} is replaced with $\bar{c}$, and π_t with $\bar{\pi}$

5 Central bank learning and optimal monetary policy

We now examine how the nature of monetary policy is affected by learning considerations. Under imperfect knowledge the central bank minimizes

$$\underset{\{\pi_t\}_{t=0}^{\infty}}{\text{Min}}\, E_t \sum_{t=0}^{\infty} \frac{\delta^t}{2}\left[a(\pi_t - \pi^*)^2 + (\pi_t - y_t + \varepsilon_t)^2\right] \tag{5.1}$$

The minimization is subject to the Kalman filter equations (4.4)–(4.6) and (A4.11):

$$E_{t-1}[y_t] = c_{t-1}(1+\pi_{t-1}) \tag{5.2}$$

$$c_t = c_{t-1} + \kappa_t(y_t - E_{t-1}y_t) \qquad (5.3)^{14}$$

$$p_t = p_{t-1} - \kappa_t(1+\pi_{t-1})p_{t-1} \tag{5.4}$$

$$\kappa_t = \frac{1}{1+\pi_{t-1}}\left[1 + \frac{\sigma_\varepsilon^2}{p_{t-1}(1+\pi_{t-1})^2}\right]^{-1} = \frac{1}{1+\pi_{t-1}}\left[1+\sigma\right]^{-1} \tag{5.5}$$

Now the timing of events is as follows. First, at time t–1 the central bank sets its expectation (forecast) for private sector inflation expectations according to equation (5.2). Here the ordinary least squares estimate c_{t-1} of γ in the model $(y_{t-1}; w_{t-2}, \gamma, \sigma_\varepsilon^2)$ has been calculated on the basis of (5.3) and (5.4); that is using values for $c_{t-2}, \kappa_{t-1}, y_{t-1}, w_{t-2}$ (where I have used w_t as shorthand for $(1+\pi_{t-1})$). Next, also at time t–1 private sector inflation expectations for period t, x_t, are determined by w_{t-1} and the true parameter γ according to equation (4.2).

Then, the CB sets inflation at time t based on its own forecast. Also, in period t given the new information provided by y_t, w_{t-1}, the central bank forms a revised or updated estimate of γ. Figure 5.1 summarizes.

[14] Where I have substituted $E_{t-1}y_t$ for $c_{t-1}(1+\pi_{t-1})$ using equation (5.2).

Figure 5.1 **Imperfect knowledge: timing of events**

Time t–1		Time t	
Stage 1:	Stage 2:	Stage 3:	Stage 4:
CB forecasts PS inflation expectations using C_{t-1} and w_{t-1}, ie sets $E_{t-1}[x_t] = E_{t-1}[y_t]$.	PS forecasts inflation using w_{t-1} and γ, ie sets x_t.	3a) CB decides on monetary policy, ie sets $\pi_t(E_{t-1}[y_t])$. 3b) Nature chooses, ε_t, and $y_t = x_t + \varepsilon_t$ realizes. 3c) CB observes y_t and forms a revised estimate c_t.	Back to stage 1, for time t = t+1 etc.

Note that the Kalman gain (5.5) is a nonlinear function of the central bank's control variable. Hence, the updating equations (5.3) and (5.4) are also nonlinear in the inflation rate. These updating equations represent the *learning channel*, through which the current policy choice π_t affects next period's parameter estimate c_{t+1} and the associated prediction $E_t y_{t+1}$.

5.1 Two limiting cases

To obtain some intuition, we now take a closer look at the term $\sigma \equiv \frac{\sigma_\varepsilon^2}{p_{t-1}(1+\pi_{t-1})^2}$ in the Kalman gain κ_t. Here $p_{t-1}(1+\pi_{t-1})^2$ is the portion of the prediction error variance due to uncertainty in c_{t-1} and σ_ε^2 is the portion of the prediction error variance due to the random shock ε_t. Thus, it is nothing else than the inverse of the *signal to noise ratio*. We can easily see that $\left|\frac{\partial \kappa_t}{\partial \sigma}\right| < 0$, suggesting that as the amount of noise in the signal y_t increases, relatively less weight is given to new information in the prediction error, $(y_t - E_{t-1}y_t)$. This is quite intuitive, since an increase in the noise may be interpreted as a deterioration of the information content of $(y_t - E_{t-1}y_t)$ relative to c_{t-1}. Similarly, if the amount of noise that is contaminating the signal diminishes, more weight will be given to new information relative to the previous estimate c_{t-1}.

The model has two important limiting cases. One limiting case is the one where the noise to signal ratio goes to infinity. Then, new observations are so noisy that they are essentially useless, and play no role in updating the previous estimate. This is the case where the Kalman gain goes to zero, or

$$\lim_{\sigma\to\infty}\frac{1}{1+\pi_{t-1}}\left[\frac{1}{1+\sigma}\right]=\lim_{\sigma\to\infty}\kappa_t=0$$

Substituting this expression into the updating equation (5.3) gives $c_t = c_{t-1} = c_0$. Hence, this is the case where the central bank engages in forecasting or *prediction*,

$$E_{t-1}[y_t]=c_0(1+\pi_{t-1}) \tag{5.6}$$

but not in updating. Note that in this case the *separation principle* holds as the central bank's optimal estimation of the state γ, no longer depends on policy outcomes. Note that the model can then be solved for either discretion or commitment.

The mirror image of the previous situation is the case where the new observations are not polluted by any noise. Then, the central bank should just set policy based on its most recent observation y_t. In this case the relevant limit of the Kalman gain is given by

$$\lim_{\sigma\to\infty}\frac{1}{1+\pi_{t-1}}\left[\frac{1}{1+\sigma}\right]=\lim_{\sigma\to\infty}\kappa_t=\frac{1}{1+\pi_{t-1}}$$

Substituting this expression into the updating equation (5.3) gives

$$c_t=\frac{y_t}{1+\pi_{t-1}}=\frac{x_t}{1+\pi_{t-1}}=\frac{\gamma(1+\pi_{t-1})}{1+\pi_{t-1}}=\gamma \tag{5.7}$$

In this case at time t–1 the central bank would be able to forecast private sector inflation expectations perfectly, and then at time t set policy based on its forecast. So, this case is nothing else but the case of perfect knowledge analyzed in section 3.

5.2 Optimal learning and the value of experimentation

We now turn to the case where estimation and control are not separated. Of course, estimation and control cannot be separated because parameter updates and forecasts depend on past monetary policy choices. The effect of policy on future estimates and forecasts is also apparent from the Bellman equation associated with this dynamic programming problem:

$$V_t(x_t^{'}) = \max_{\pi_t}\{(r(\pi_t, x_t^{'})) + \delta E_t V_{t+1}(x_t^{'})\} \tag{5.8}$$

Where the vector of state variables is $x_t^{'} = (E_t y_{t+1}, c_t, \kappa_t, y_t, p_t)$, π is the control and r is the one-period return function.

Following Wieland (2000b), it can easily be seen that the two terms on the right hand side characterize the tradeoff between current control and estimation (which here is used as an umbrella term to include prediction). The first term is current expected reward, while the second term is the expected continuation value in the next period, which reflects the expected improvement in future payoffs due to better information about the unknown parameter.

Note that if $\delta \rightarrow 0$ the central bank only 'lives for today', and is not interested to consider the effects of its policy actions on future payoffs. Then there is no horse race, and the optimal policy is simply to maximize the one-period return function. In that case the optimal policy is simply the discretionary solution, which under imperfect knowledge is presented in section 5.3.1 below.

As shown by Easley and Kiefer (1988) and Kiefer and Nyarko (1989) an optimal feedback rule exists and the value function is continuous and satisfies the Bellman equation.[15] Policy and value functions can be obtained using an iterative algortihm based on the Bellman equation starting with an initial guess about V(.).[16] However, analytical solutions are not feasible because the dynamic contraint of the optimization problem associated with the Kalman filter is highly nonlinear. As pointed out by Wieland (2000b), there are many examples, including Wieland (2000a,b), Ellison and Valla (2001) for which no analytical solutions have been found even though the unknown stochastic process is linear and the return function is quadratic.

5.3 The case of passive learning

In order to get some analytical results, we now consider the case of *passive learning*. This is the case where the central bank disregards the effect of current policy actions on future estimation and prediction. In this case the policy maker

[15] As pointed out by Wieland (2000b), one can use standard dynamic programming methods and show that Blackwell's sufficiency condition – monotonicity and discounting – are satisfied. Thus, equation (5.8) has a fixed point in the space of continuous functions, which is the value function V(x′).

[16] A typical algorithm is described in Ljunqvist and Sargent (2000), Chapter 3.

treats control and estimation separately. As pointed out by Bertocchi and Spagat (1993) in this case learning is *passive* in the sense that there is no experimentation.

The central bank will first choose π_t to minimise the expected loss based on its current parameter estimate. After observing y_t, the central bank will proceed by updating its estimate and selecting next period's control. As pointed out by Wieland (2002b) this behaviour is myopic since it disregards the effect of current policy actions on future predictions and estimates.

5.3.1 Discretion

As before first we consider the case where the central bank does not internalise internalize its 'teaching by doing', that is the case where $\delta \to 0$.

Time t: Control

In this discretionary case the central bank will choose π_t to minimise the expected one-period loss

$$\begin{aligned} &E_{t-1}\frac{1}{2}\left[a(\pi_t - \pi^*)^2 + (\pi_t - y_t + \varepsilon_t)^2\right] \\ &= \left[\frac{a}{2}(\pi_t - \pi^*)^2 + \frac{1}{2}\pi_t^2 - \pi_t E_{t-1}y_t + \frac{1}{2}\sigma_\varepsilon^2 + \frac{1}{2}\left((E_{t-1}y_t)^2 + f_{t|t-1}\right)\right] \end{aligned} \tag{5.9}$$

Where I have decomposed y_t as $y_t = E_{t-1}y_t + (y_t - E_{t-1}y_t)$ and noted that the prediction variance $E_{t-1}(y_t - E_{t-1}y_t)^2 = f_{t|t-1}$ (see Appendix 4). The optimal policy rule is the one where the central bank should partially accommodate its *forecast* of private sector inflation expectations.

$$\pi_t = \frac{1}{1+a}E_{t-1}y_t + \frac{a}{1+a}\pi^* \tag{5.10}$$

Using equation (5.6) in the expression above, this policy can also be expressed in terms of a response to the determinants of this forecast, namely past inflation and the central bank's initial estimate of the degree of persistence of inflation expectations.

$$\pi_t = \frac{c_{t-1}}{1+a}(\pi_{t-1} + 1) + \frac{a}{1+a}\pi^* \tag{5.11}$$

Which is identical to the discretionary policy under perfect knowledge if the parameter γ is replaced by its estimate c_{t-1}.

A novel feature of the passive learning policy – compared to the central bank's discretionary disinflation policy under perfect knowledge – is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information. This can be easily seen from equation (5.11) where the monetary accommodation coefficient is now time-varying.

<u>Time t: Prediction and estimation</u>

Having solved its control problem, after observing y_t the central bank will replace its previous estimate c_{t-1} with c_t using the smoothing equations (5.3) and (5.4) and the Kalman gain (5.5). Next, its forecast at time t for private sector inflation expectations at time t + 1, is given by

$$E_t[y_{t+1}] = c_t\left[1 + \frac{c_{t-1}}{1+a}(\pi_{t-1} + 1) + \frac{a}{1+a}\pi^*\right] \tag{5.12}$$

<u>Time t + 1: Control</u>

Then, the central bank sets policy at time $t+1$ based on the state $E_t y_{t+1}$. The solution is similar to (5.11).

<u>Time t + 1: Prediction and estimation</u>

Having solved its control problem, after observing y_{t+1} the central bank will replace its previous estimate c_t with c_{t+1} using the smoothing equations and the Kalman gain.

$$\kappa_{t+1} = \frac{1}{1+\pi_t}\left[1+\frac{\sigma_\varepsilon^2}{p_t(1+\pi_t)^2}\right]^{-1}$$

$$= \frac{1}{1+\frac{c_{t-1}}{1+a}(\pi_{t-1}+1)+\frac{a}{1+a}\pi^*}\left[1+\frac{\sigma_\varepsilon^2}{p_t\left(1+\frac{c_{t-1}}{1+a}(\pi_{t-1}+1)+\frac{a}{1+a}\pi^*\right)^2}\right]^{-1} \quad (5.13)$$

Note that this gain depends on the inflation rate chosen by the monetary authority two periods earlier.

5.3.2 Commitment

Now we turn to the case where the central bank internalizes the effects of today's monetary accommodation on tomorrow's inflation expectations.

Time t: Control

To solve this problem, first we derive the central bank's policy rule $\pi_t(E_{t-1}y_t, \pi^*)$, which selects an action based on the current state $E_{t-1}y_t$. Given a specification of the central bank's forecast, given by the Kalman filter equations this policy rule can be derived analytically. Formally, the central bank's control problem is now to

$$\underset{\{\pi_t\}_{t=0}^{\infty}}{\text{Min}}\, E_t \sum_{t=0}^{\infty} \frac{\delta^t}{2}[a(\pi_t - \pi^*)^2 + (\pi_t - y_t + \varepsilon_t)^2$$

subject to

$$E_{t-1}[y_t] = c_{t-1}(1+\pi_{t-1}) \quad (5.14)$$

It is convenient to define $x_t = E_{t-1}[y_t]$ as the state variable and $u_t = \pi_t$ as the control. We solve this problem by the method of Lagrange multipliers. Introduce the Lagrange multiplier μ_t, and set to zero the derivatives of the Lagrangean expression:

$$L = E_0\left[\sum_{t=0}^{\infty}\left\{\frac{\delta^t}{2}\left[-a(u_t-u^*)^2-(u_t-x_t)^2\right]-\delta^{t+1}\mu_{t+1}\left[x_{t+1}-c_t u_t - c_t\right]\right\}\right] \quad (5.15)$$

Following a line of reasoning similar to the corresponding case of perfect knowledge (for details see Appendix 3), it can easily be shown that the first-order condition can be written as

$$
\begin{aligned}
\pi_t &= C_{1,t-1}E_{t-1}[y_t] + C_{2,t-1} \\
&= C_{1,t-1}c_{t-1}(1+\pi_{t-1}) + C_{2,t-1}
\end{aligned}
\tag{5.16}
$$

The coefficients $\frac{1}{(1+a)+\delta c_{t-1}^2} < C_{1,t-1} < \frac{\delta c_{t-1}^2 + 1}{(1+a)+\delta c_{t-1}^2}$ and $0 \le C_{2,t-1}(C_{1,t-1}) \le \pi^*$, are given by

$$
C_{1,t-1} = \frac{1}{2}\left\{\left[\frac{(1+a)+\delta c_{t-1}^2}{\delta c_{t-1}^2}\right] - \sqrt{\frac{\left[(1+a)^2+\delta c_{t-1}^2\right]^2 - 4\delta c_{t-1}^2}{\delta^2 c_{t-1}^4}}\right\}
\tag{5.17}
$$

$$
C_{2,t-1} = \frac{\delta c_{t-1}\left[(C_{1,t-1}-1)\underline{\pi}^b(1-q)\right] + \left[\delta c_{t-1}(C_{1,t-1}-1)(1-c_{t-1}) + a\right]\pi^*}{(1+a) - \delta c_{t-1}\left[(C_{1,t-1}-1)c_{t-1}+1\right]}
\tag{5.18}
$$

Again, a novel feature of the passive learning policy – compared to the central bank's optimal disinflation policy under perfect knowledge – is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information. This can be easily seen from equation (5.16) where the coefficients are now time-varying.

Time t: Prediction and estimation

Having solved its control problem, after observing y_t the central bank will replace its previous estimate c_{t-1} with c_t using the smoothing equations (5.3) and (5.4) and the Kalman gain (5.5). Next, its forecast at time t for private sector inflation expectations at time t + 1, is given by

$$
E_t[y_{t+1}] = c_t\left[1 + C_{1,t-1}c_{t-1}(1+\pi_{t-1}) + C_{2,t-1}\right]
\tag{5.19}
$$

Time t + 1: Control

Then, the central bank sets policy at time $t+1$ based on the state $E_t y_{t+1}$. The solution is similar to (5.16).

Time t + 1: Prediction and estimation

Having solved its control problem, after observing y_{t+1} the central bank will replace its previous estimate c_t with c_{t+1} using the smoothing equations and the Kalman gain.

$$\begin{aligned}\kappa_{t+1} &= \frac{1}{1+\pi_t}\left[1+\frac{\sigma_\varepsilon^2}{p_t(1+\pi_t)^2}\right]^{-1} \\ &= \frac{1}{1+C_{1,t-1}c_{t-1}(1+\pi_{t-1})+C_{2,t-1}}\left[1+\frac{\sigma_\varepsilon^2}{p_t\left(1+C_{1,t-1}c_{t-1}(1+\pi_{t-1})+C_{2,t-1}\right)^2}\right]^{-1}\end{aligned} \tag{5.20}$$

Note that this gain depends on the inflation rate chosen by the monetary authority two periods earlier.

5.3.3 Incomplete learning and limit beliefs

As pointed out by Wieland (2000a), under a policy of passive learning – here for the cases of either discretion or commitment – there is the real possibility of *incomplete learning* that has been investigated in the theoretical literature on Bayesian learning. The basic intuition behind this possibility is simply the following: if the policy instrument – the lagged inflation rate – which is the right-hand side variable in the regression (4.3), does not exhibit enough variation, it may not be possible to correctly identify the unknown parameter γ.

For the case of our simple regression (4.3) Kiefer and Nyarko (1989) have shown that that all limit belief and policy pairs $(\bar{c},\bar{\pi})$, whether correct or not, share three properties, *belief invariance, one-period optimization* and *mean prediction*, which can be used to describe the set of possible limit beliefs. These properties imply a system of equations with multiple solutions that is summarized in Table 5.1. Each solution represents a belief that is *self-reinforcing* under the passive learning policy. In this case the passive learning policy induces many observations on private sector inflation expectations that are not very informative and do not lead the central bank to revise his incorrect beliefs. An example of such an incorrect limit belief and policy pair for our calibrated economy is given in Table 5.1.

Table 5.1 **An incorrect limit belief and policy pair**

Parameter	Value(s)
$\bar{c}$	0.5
π^*	2.5
a	0.2
$\bar{y}$	1.2143
$\bar{\pi}$	1.4285

In his sensitivity analysis, Wieland (2000a) shows that under passive learning policies

1. the frequency with which a sustained policy bias (ie persistent deviations of the state variable from its target) occurs depends on how close the initial belief is to a self-reinforcing incorrect limit belief.
2. the more confident the central bank becomes about its initial (incorrect) parameter estimate, less weight is given to new data in revising his beliefs. Therefore the likelihood of a sustained policy bias due to incorrect beliefs increases.

Regarding the second point, in terms of our model the relevant concept is the noise to signal ratio $\sigma \equiv \frac{\sigma_\varepsilon^2}{p_{t-1}(1+\pi_{t-1})^2}$ in the Kalman gain κ_t. Here $p_{t-1}(1+\pi_{t-1})^2$ is the portion of the prediction error variance due to uncertainty in c_{t-1} and σ_ε^2 is the portion of the prediction error variance due to the random shock ε_t. Thus, it is nothing else than the inverse of the *signal to noise ratio*. We already know that as the amount of noise in the signal y_t increases, relatively less weight is given to new information in the prediction error, $(y_t - E_{t-1}y_t)$. Similarly, if the amount of noise that is contaminating the signal diminishes, more weight will be given to new information relative to the previous estimate c_{t-1}.

Thus, under a passive learning policy a lower signal to noise ratio – other things equal – increases the likelihood of a sustained policy bias (ie persistent deviations of the state variable from its target) due to incorrect beliefs.

Finally, Wieland (2000b) shows that the optimal extent of experimentation, ie using the 'active' rather than the passive learning policy is largest in the neighbourhood of the above incorrect limit beliefs.

6 Conclusions and suggestions for further research

In this paper we have analyze disinflation in two environments. One in which the central bank has *perfect knowledge*, in the sense that it understands and observes the process by which private sector inflation expectations are generated, and one in which the central bank has to *learn* the private sector inflation forecasting rule.

For the case of perfect knowledge we found that the optimal disinflation is faster under commitment than discretion. Next, in the commitment case the disinflation is less gradual, the higher the central bank's rate of time preference and the higher the degree of persistence in inflation expectations.

With imperfect knowledge results depend on the learning scheme that is employed. A novel feature of the passive learning policy – compared to the central bank's optimal disinflation policy under perfect knowledge – is that the degree of monetary accommodation (the extent to which the central bank accommodates private sector inflation expectations) is no longer constant across the disinflation, but becomes state-dependent. This means that the central bank's behaviour changes during the disinflation as it collects more information.

There are a number of ways the paper can be extended. One limitation of the present analysis is that there is no rational learning of private agents about the monetary policy regime. It would be more plausible if agents also update their beliefs about the evolution of inflation following observations about actual monetary policy choices.[17]

An example of a paper that looks at the case where the private sector is learning about central bank behavior is Andolfatto, Hendry and Moran (2002). Using a standard monetary dynamic stochastic general equilibrium model, they embed a learning mechanism regarding the interest-rate-targeting rule that the monetary authorities follow. There the learning mechanism enables optimizing economic agents to distinguish between transitory shocks to the policy rule and occasional shifts in the inflation target of the monetary policy authorities. We see this as one potential avenue for further work.

[17] This is modelled by Hoeberichts and Schaling (2000), using Bayesian learning.

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Appendix 1

Steady state equilibrium

The innovation sequence $\{V_t\}$ in equation (2.6) satisfies

$$\begin{aligned} &\Pr ob\left[V_{t+1} = (1-p)\middle|S_t = 1\right] = p, \\ &\Pr ob\left[V_{t+1} = -p\middle|S_t = 1\right] = 1-p, \\ &\Pr ob\left[V_{t+1} = -(1-q)\middle|S_t = 0\right] = q, \\ &\Pr ob\left[V_{t+1} = q\middle|S_t = 0\right] = 1-q \end{aligned} \qquad \text{(A1.1)}$$

with $E_t V_{t+1} = 0$ and $\sigma_v^2 = \hat{E}(V_t^2) = p(1-p)\overline{p} + q(1-q)(1-\overline{p})$. (where I have used that $\overline{p} \equiv (1-q)/(1-p+1-q)$)[18]

From (A1.1) we see that $\hat{E}_0 V_t = 0$ for all $t > 0$. Using this fact, and iterating (2.6) into the future, we can write

$$\hat{E}_0 S_t = \gamma^t \hat{E}_0 S_0 + \frac{(1-q)(1-\gamma^t)}{(1-\gamma)} \qquad \text{(A1.2)}$$

where $\hat{E}_0$ denotes the private sector expectation conditional on information available at date zero (which need not include observation of s_0). Observing that $\hat{E}_0 S_t$ can be interpreted as the probability that $S_t = 1$ given information at time zero (denoted $P_0[S_t = 1]$), (A1.2) can be rewritten

$$P_0[S_t = 1] = \overline{p} + \gamma^t(\overline{p}_0 - \overline{p}) \qquad \text{(A1.3)}$$

where $p_0 \equiv P_0[S_0 = 1]$.

From equation (A1.3) we can see that for large t the economy is expected to be in the high inflation state (state 1) with probability $\overline{p}$, in which case u would be π^b. Similarly, the economy will be in the low inflation state (state 0) with probability $1-\overline{p}$, in which case u would be zero. Hence, the *expected* long-run level of u (denoted as $\overline{u}$) is

$$\overline{u} = \overline{p} \cdot \pi^b \qquad \text{(A1.4)}$$

[18] For more details see Hamilton (1989, 360–363).

From equation (2.3) it then follows that the (unconditional mean) steady state level of inflation $(\overline{\pi})$, is

$$\overline{\pi} = \pi^* + \overline{p} \cdot \pi^b \tag{A1.5}$$

Appendix 2

The private sector's optimal predictor for the credibility of the inflation target

Taking expectations at time $t-1$ of equation (3.2), where the expectations operator $\hat{E}$ refers to agents' subjective expectations, we obtain

$$\hat{E}_{t-1}\pi_t = \gamma\pi_{t-1} + (1-\gamma)\pi^* + \pi^b \cdot (1-q) \tag{A2.1}$$

However, from equation (2.3) it follows that

$$\hat{E}_{t-1}\pi_t = \pi^* + \hat{E}_{t-1}u_t \tag{A2.2}$$

Hence, consistency requires that $\gamma\pi_{t-1} + (1-\gamma)\pi^* + \pi^b \cdot (1-q) = \pi + \hat{E}_{t-1}u_t$ or

$$\hat{E}_{t-1}u_t = \gamma(\pi_{t-1} - \pi^*) + \pi^b \cdot (1-q) \tag{A2.3}$$

Appendix 3

The commitment solution

The central bank's problem is to

$$\underset{\{\pi_t\}_{t=0}^{\infty}}{\operatorname{Max}} E_t \sum_{t=0}^{\infty} \frac{\delta^t}{2}[-a(\pi_t - \pi^*)^2 - (z_t)^2] \tag{A3.1}$$

subject to

$$z_t = \pi_t - \hat{E}_{t-1}\pi_t \tag{A3.2}$$

and

$$\hat{E}_{t-1}\pi_t = \gamma\pi_{t-1} + (1-\gamma)\pi^* + \pi^b \cdot (1-q) \quad \text{(A3.3)}$$

It is convenient to define $x_t = \hat{E}_{t-1}\pi_t$ as the state variable and $u_t = \pi_t$ as the control. We solve this problem by the method of Lagrange multiplier. Introduce the Lagrange multiplier μ_t, and set to zero the derivatives of the Lagrangean expression:

$$L = E_0\left[\sum_{t=0}^{\infty}\left\{\frac{\delta^t}{2}\left[-a(u_t - u^*)^2 - (u_t - x_t)^2\right]\right.\right.$$
$$\left.\left. - \delta^{t+1}\mu_{t+1}\left[x_{t+1} - \gamma u_t - (1-\gamma)u^* - \pi^b(1-q)\right]\right\}\right] \quad \text{(A3.4)}$$

The central bank's first-order conditions take the form

$$-a(u_t - u^*) - (u_t - x_t) + \delta\gamma E_t\mu_{t+1} = 0 \quad \text{(A3.5)}$$

$$\mu_t = (u_t - x_t) \quad \text{(A3.6)}$$

First, we find an expression for $E_t\mu_{t+1}$. Leading (A3.6) by one period and taking expectations we get:

$$E_t\mu_{t+1} = (E_t u_{t+1} - E_t x_{t+1}) \quad \text{(A3.7)}$$

Substituting (A3.7) into (A3.5), we can derive the Euler equation

$$-a(u_t - u^*) - (u_t - x_t) + \delta\gamma\left(E_t u_{t+1} - E_t x_{t+1}\right) = 0 \quad \text{(A3.8)}$$

In the case of a policy of strict inflation reduction, the rule would be

$$u_t = u^* \quad \text{(A3.9)}$$

Similarly, in the case of full accommodation of expectations, the rule would be

$$u_t = x_t \quad \text{(A3.10)}$$

Thus, it appears that in case of flexible inflation targeting the rule will be a linear combination of (A3.9) and (A3.10), that is $u_t = cx_t + (1-c)u^*$, where $0 \le c \le 1$. Or alternatively,

$$u_t = C_1 x_t + C_2 \tag{A3.11}$$

which is equation (3.9) in the main text (where I have substituted $x_t = \hat{E}_{t-1}\pi_t$ and $u_t = \pi_t$).

Here the coefficients C_1 and C_2 remain to be determined, and the prior is that $0 \le C_1 \le 1$ and $0 \le C_2(C_1) \le u^*$. Now we identify the coefficients C_1 and C_2.

Expectations for the state at period t + 1 follow from the constraint in (A3.4), combining the latter with the decision rule for , we can write:

$$E_t x_{t+1} = \gamma C_1 x_t + \gamma C_2 + (1-\gamma)u^* + \pi^b(1-q) \tag{A3.12}$$

From (A3.11) it follows that

$$E_t u_{t+1} = C_1 E_t x_{t+1} + C_2 = C_1[\gamma C_1 x_t + \gamma C_2 + (1-\gamma)u^* + \pi^b(1-q)] + C_2 \tag{A3.13}$$

Substituting (A3.12) and (A3.13) into the Euler equation (A3.8) above, and equating constant terms and coefficients on the state variables yields the following expressions for C_1 and C_2 in terms of the structural parameters of the model

$$C_1 = \frac{1}{(1+a)+\delta\gamma^2}\left[\delta\gamma^2 C_1^2 + 1\right] \tag{A3.14}$$

$$C_2 = \frac{\delta\gamma\left[(C_1-1)\gamma+1\right]}{1+a}C_2 + \frac{\delta\gamma\left[\pi^b(1-q)+(1-\gamma)u^*\right](C_1-1)+au^*}{1+a} \tag{A3.15}$$

Equation (A3.14) implicitly defines the value of C_1. It can be written as $C_1 = F(C_1)$. Note that the function $F(C_1)$ on the right hand side with domain <0,1> is monotonically increasing in C_1, that $\lim_{C_1 \to 0} F(C_1) = \frac{1}{(1+a)+\delta\gamma^2}$, $\lim_{C_1 \to 1} F(C_1) = \frac{\delta\gamma^2+1}{(1+a)+\delta\gamma^2}$. We realize that there is a unique positive solution C_1, which fulfills $\frac{1}{(1+a)+\delta\gamma^2} < C_1 < \frac{\delta\gamma^2+1}{(1+a)+\delta\gamma^2}$. It can be solved analytically:

$$C_1 = \frac{1}{2}\left\{\left[\frac{(1+a)+\delta\gamma^2}{\delta\gamma^2}\right] - \sqrt{\frac{\left[(1+a)^2+\delta\gamma^2\right]^2 - 4\delta\gamma^2}{\delta^2\gamma^4}}\right\} \tag{A3.16}$$

Similarly, Equation (A3.15) implicitly defines the value of C_2. It can be written as $C_2 = G(C_2)$. Note that the function $G(C_2)$ on the right hand side with domain $<0,u^*>$ is monotonically increasing in C_2. Again, we realize that there is a unique positive solution C_2^*. It can be solved analytically:

$$C_2 = \frac{\delta\gamma\left[(C_1-1)\pi^b(1-q)\right]+\left[\delta\gamma(C_1-1)(1-\gamma)+a\right]u^*}{(1+a)-\delta\gamma\left[(C_1-1)\gamma+1\right]} \tag{A3.17}$$

Moreover, it can be easily established that $\lim_{a\to 0} C_2 = 0$ and that $\lim_{a\to\infty} C_2 = u^*$.

We are now ready to prove:

PROPOSITION A3.1: The higher a the lower the optimal value of the feedback parameter C_1.

Proof: $\frac{\partial F}{\partial a} = -\frac{(\delta\gamma^2 C_1^2+1)}{\left[(1+a)+\delta\gamma\right]^2} < 0$, this implies that when a goes up, the function $F(C_1)$ shifts downward. As a consequence, the equilibrium value of C_1 decreases.

PROPOSITION A3.2: If C_1 is smaller than an upper bound $\overline{C}_1$, the higher δ the lower the optimal value of the feedback parameter C_1.

Proof: $\frac{\partial F}{\partial \delta} = \frac{\gamma^2\left[C_1^2(1+a)-1\right]}{\left[(1+a)+\delta\gamma^2\right]^2}$. Note that the nominator of this expression is negative if the above condition is satisfied. It can be written as $C_1 < \overline{C}_1$, where $\overline{C}_1 = \sqrt{\frac{1}{1+a}}$.

Numerical results indicate that for our basic parameter configuration (see Table 3.1), the above condition is satisfied for the entire range of inflation aversion preferences ($0 < a < \infty$).

PROPOSITION A3.3: If C_1 is smaller than an upper bound $\overline{C}_1$, the higher γ the lower the optimal value of the feedback parameter C_1.

Proof: $\frac{\partial F}{\partial \gamma} = \frac{2\delta\gamma\left[C_1^2(1+a)-1\right]}{\left[(1+a)+\delta\gamma^2\right]^2}$. Note that the nominator of this expression is negative if the above condition is satisfied. For more details see Proposition A3.2 above.

Numerical results indicate that for our basic parameter configuration (see Table 3.1), the above condition is satisfied for the entire range of inflation aversion preferences ($0 < a < \infty$).

PROPOSITION A3.4: If $a > -\delta\gamma(C_1 - 1)(1-\gamma)$, the higher u* the higher the value of the constant C_2.

Proof: $\frac{\partial C_2}{\partial u^*} = \frac{\left[\delta\gamma(C_1-1)(1-\gamma)+a\right]}{(1+a)-\delta\gamma\left[(C_1-1)\gamma+1\right]}$. The nominator of this expression is positive if the above condition is satisfied.

PROPOSITION A3.5: The higher a the higher the value of the constant C_2.

Proof: $\frac{\partial C_2}{\partial a} = \frac{(1-\delta\gamma C_1)u^* - \delta\gamma\left[(C_1-1)\pi^b(1-q)\right]}{\left\{(1+a)-\delta\gamma\left[(C_1-1)\gamma\right]+1\right\}^2} > 0$.

To give a numerical example, for our basic parameter configuration (see Table 3.1), $C_2 \approx 0.48$. If we increase a to 0.5, say, C_2 increases to 0.90.

PROPOSITION A3.6: The higher π^b the lower the value of the constant C_2.

Proof: $\frac{\partial C_2}{\partial \pi^b} = \frac{\delta\gamma(C_1-1)(1-q)}{(1+a)-\delta\gamma\left[(C_1-1)\gamma+1\right]} < 0$.

PROPOSITION A3.7: If C_2 is greater than a lower bound C_2, the higher δ the higher the value of the constant C_2.

Proof: $$\frac{\partial G}{\partial \delta} = \frac{\gamma\{[\pi^b(1-q)+(1-\gamma)u^*+\gamma C_2][(C_1-1)+\delta(\partial C_1/\partial\delta)]+C_2\}}{1+a}.$$ The nominator of this expression is positive if the above condition is satisfied. It can be written as $C_2 > \underline{C}_2$, where $\underline{C}_2 = \frac{-[\pi^b(1-q)+(1-\gamma)u^*][(C_1-1)+\delta(\partial C_1/\partial\delta)]}{\{\gamma[(C_1-1)+\delta(\partial C_1/\partial\delta)]+1\}}$. For plausible parameter values this condition is likely to be satisfied.

To give a numerical example for our basic parameter configuration (see Table 3.1) and $\delta = 0.225$, $C_2 \approx 0.45$. If we increase δ to 0.9, say, C_2 increases to 0.48.

PROPOSITION A3.8: If C_2 is greater than a lower bound C_2, the higher γ the higher the value of the constant C_2.

Proof:
$$\frac{\partial G}{\partial \gamma} = \frac{\delta[\gamma C_2+\pi^b(1-q)+u^*(1-\gamma)]\{(C_1-1)+\gamma(\partial C_1/\partial\gamma)\}+\delta\gamma(C_1-1)[C_2-u^*]+\delta C_2}{1+a}.$$
The nominator of this expression is positive if the above condition is satisfied. It can be written as $C_2 > \underline{C}_2$, where $\underline{C}_2 = \frac{-\{(C_1-1)+\gamma(\partial C_1/\partial\gamma)\}\delta[\pi^b(1-q)+u^*(1-\gamma)]+\delta\gamma u^*(C_1-1)}{\delta\{2\gamma(C_1-1)+\gamma^2(\partial C_1/\partial\gamma)+1\}}$. For plausible parameter values this condition is likely to be satisfied.

For example, for our basic parameter configuration (see Table 3.1) and $\gamma = 0.2$, $C_2 \approx 0.27$. If we increase γ to 0.9, C_2 increases to 0.48.

Appendix 4

The central bank's optimal filtering problem

In this appendix we derive the central bank's optimal forecasting rule for private sector inflation expectations by applying the Kalman filter.

State space form

The policymaker's estimation problem can be put into state-space form by defining the state vector as the parameter γ. Then the (state) *transition equation* is

$$\gamma_t = \gamma_{t-1} \tag{A4.1}$$

However, the state is not observed directly. Instead the state of the system is conveyed by an observed variable (signal) y_t, which is subject to contamination by noise (measurement error) ε_t. Thus, the *measurement equation* is

$$y_t = \gamma_t z_t' + \varepsilon_t \tag{A4.2}$$

where the scalar $z_t' \equiv w_{t-1} = (1 + \pi_{t-1})$, and ε_t is a serially uncorrelated disturbance with mean zero and variance σ_ε^2, that is $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = \sigma_\varepsilon^2$.[19]

The Kalman filter

The technique of the Kalman filter depends on the system that consists of (A4.1) and (A4.2) and its aim is to find unbiased estimates of the sequence of the state γ_t via a recursive process of estimation.[20]

The process starts at time t = 1 say; and it is assumed that prior information on the previous state vector γ_0 is available in the form of an unbiased estimate c_0, which has been drawn from a distribution with a mean of γ_0 and variance p_0. Depending on the uncertainty surrounding the initial estimate, large (small) values should be attributed to p_0 to reflect the low (high) precision of the initial estimate.

[19] Note that (A4.2) is in the form of an ordinary regression equation.

[20] A process of estimation which keeps pace with the data by generating an estimate of the current state variable with each new observation y_t is described as *filtering*. The retrospective enhancement of a state estimate, using data – which has arisen subsequently – is described as *smoothing*. The estimation of a future state variable is described as *prediction*.

In the terminology of Bayesian statistics, this is a matter of attributing a diffuse prior distribution to γ_0.

The basic filter is described by four equations governing prediction, (namely equations (A4.3), (A4.5), (A4.6) and (A4.7)), and two for updating/smoothing (namely equations (A4.10) and (A4.13)). These equations are derived below.

In each time period, new information on the system is provided by the variable y_t; and estimates of γ_t may be formed both before and after the receipt of this information. The estimate of the state at time t formed without knowledge of y_t will be denoted by $c_{t|t-1}$; the estimate that incorporates the information of y_t will be denoted by c_t.

In the absence of information of y_t, the estimate $c_{t|t-1}$ of γ_t comes directly from equation (A4.1) where γ_{t-1} is replaced by c_{t-1}. Thus

$$c_{t|t-1} = c_{t-1} \qquad \text{(A4.3)}$$

Equation (A4.3) is the *state prediction equation*.

The mean-square error of this estimator will be denoted by $p_{t|t-1} = E\{(\gamma_t - c_{t|t-1})^2\}$, whilst that of the updated estimator c_t will be denoted by $p_t = E\{(\gamma_t - c_t)^2\}$. To derive the expression for $p_{t|t-1}$ in terms of p_{t-1}, we subtract equation (A4.3) from equation (A4.1) to give

$$\gamma_t - c_{t|t-1} = \gamma_{t-1} - c_{t-1} \qquad \text{(A4.4)}$$

It follows that the *prediction variance* is

$$p_{t|t-1} = E\{(\gamma_t - c_{t|t-1})^2\} = E\{(\gamma_{t-1} - c_{t-1})^2\} = p_{t-1} \qquad \text{(A4.5)}$$

Before learning its value, we may predict y_t from equation (A4.2) by replacing γ_t by its estimate $c_{t|t-1}$ and replacing ε_t by $E(\varepsilon_t) = 0$. This gives the *observation prediction* equation

$$E_{t-1}y_t = z_t' c_{t|t-1} \qquad \text{(A4.6)}$$

The mean-square-error of this prediction is $f_{t|t-1} = E\{(y_t - E_{t-1}y_t)^2\}$. To express $f_{t|t-1}$ in terms of $p_{t|t-1}$, we subtract equation (A4.6) from equation (A4.2) to give the prediction error $e_t = y_t - E_{t-1}y_t = z_t'(\gamma_t - c_{t|t-1}) + \varepsilon_t$. Then, since $(\gamma_t - c_{t|t-1})$ and ε_t are statistically independent, and since $Var(\varepsilon_t) = \sigma_\varepsilon^2$, it follows that the *prediction variance* is

$$f_{t|t-1} = (z_t')^2 p_{t|t-1} + \sigma_\varepsilon^2 \tag{A4.7}$$

The business of incorporating the new information provided by y_t into the estimate of the state variable may be regarded as a matter of estimating the parameter γ_t in the system

$$\begin{bmatrix} c_{t|t-1} \\ y_t \end{bmatrix} = \begin{bmatrix} 1 \\ z_t' \end{bmatrix} \gamma_t + \begin{bmatrix} \varsigma_t \\ \varepsilon_t \end{bmatrix}$$

where $\varsigma_t = c_{t|t-1} - \gamma_t$.

By applying the method of generalised least squares (see eg Pollock (1999)), we obtain an estimating equation for γ_t in the form of

$$\begin{aligned} c_t &= (p_{t|t-1}^{-1} + (z_t')^2(\sigma_\varepsilon^2)^{-1})^{-1}(p_{t|t-1}^{-1} c_{t|t-1} + z_t'(\sigma_\varepsilon^2)^{-1} y_t) \\ &= p_t(p_{t|t-1}^{-1} c_{t|t-1} + z_t'(\sigma_\varepsilon^2)^{-1} y_t), \end{aligned} \tag{A4.8}$$

where

$$p_t = (p_{t|t-1}^{-1} + (z_t')^2(\sigma_\varepsilon^2)^{-1})^{-1} \tag{A4.9}$$

is the variance of the estimator.

To give equation (A4.8) a form, which is amenable to a recursive procedure, we consider the identity

$$\begin{aligned} p_{t|t-1}^{-1} c_{t|t-1} + z_t'(\sigma_\varepsilon^2)^{-1} y_t &= (p_t^{-1} - (z_t')^2(\sigma_\varepsilon^2)^{-1}) c_{t|t-1} + z_t'(\sigma_\varepsilon^2)^{-1} y_t \\ &= p_t^{-1} c_{t|t-1} + z_t'(\sigma_\varepsilon^2)^{-1}(y_t - z_t' c_{t|t-1}) \end{aligned}$$

Using this on the RHS of equation (A4.8), and noting that from (A4.3) $c_{t|t-1} = c_{t-1}$, gives

$$\begin{aligned} c_t &= c_{t-1} + p_t z_t'(\sigma_\varepsilon^2)^{-1}(y_t - z_t' c_{t-1}) \\ &= c_{t-1} + \kappa_t(y_t - z_t' c_{t-1}) \\ &= c_{t-1} + \kappa_t(y_t - w_{t-1} c_{t-1}) \\ &= (1 - \kappa_t w_{t-1}) c_{t-1} + \kappa_t y_t \end{aligned} \tag{A4.10}$$

where $\kappa_t = p_t z_t^{'}(\sigma_\varepsilon^2)^{-1} = p_{t|t-1} z_t^{'} f_{t|t-1}^{-1} = p_{t-1} z_t^{'} f_{t|t-1}^{-1} = p_{t-1} w_{t-1} f_{t|t-1}^{-1}$ is commonly described as the Kalman gain. Equation (A4.10) is equation (4.5) in the main text.

Using (A4.9), we can show that

$$\begin{aligned}\kappa_t &= p_t z_t^{'}(\sigma_\varepsilon^2)^{-1} \\ &= (p_{t|t-1}^{-1} + (z_t^{'})^2(\sigma_\varepsilon^2)^{-1})^{-1} z_t^{'}(\sigma_\varepsilon^2)^{-1} \\ &= p_{t|t-1} z_t^{'}((z_t^{'})^2 p_{t|t-1} + \sigma_\varepsilon^2)^{-1} \\ &= \frac{1}{z_t^{'}}\left[\frac{p_{t|t-1}(z_t^{'})^2}{p_{t|t-1}(z_t^{'})^2 + \sigma_\varepsilon^2}\right]\end{aligned} \qquad \text{(A4.11)}$$

where $p_{t|t-1}(z_t^{'})^2$ is the portion of the prediction error variance due to uncertainty in $c_{t|t-1}$ and σ_ε^2 is the portion of the prediction error variance due to the random shock ε_t. We can easily see that

$$\left|\frac{\partial \kappa_t}{\partial(p_{t|t-1}(z_t^{'})^2)}\right| > 0,$$

suggesting that as uncertainty with $c_{t|t-1}$ increases, relatively more weight is given to new information in the prediction error, $y_t - z_t^{'} c_{t-1}$. This is quite intuitive, since an increase in uncertainty in $c_{t|t-1}$ may be interpreted as a deterioration of the information content of $c_{t|t-1}$, relative to that of $y_t - z_t^{'} c_{t-1}$.

Equation (A4.9) can be rewritten as

$$p_t = p_{t|t-1} - p_{t|t-1} z_t^{'}((z_t^{'})^2 p_{t|t-1} + \sigma_\varepsilon^2)^{-1} z_t^{'} p_{t|t-1} \qquad \text{(A4.12)}$$

Combining equation (A4.12) with (A4.11), and noting that from (A4.5) $p_{t|t-1} = p_{t-1}$, the so-called Ricatti equation – which provides a means for generating the variance of the state prediction – can be written in recursive form as

$$p_t = p_{t-1} - \kappa_t z_t^{'} p_{t-1} = p_{t-1} - \kappa_t w_{t-1} p_{t-1} \qquad \text{(A4.13)}$$

which is equation (4.6) in the main text.

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