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Abstract

We propose simulation-based forecasting methods for the noncausal vector autoregressive model proposed by Lanne and Saikkonen (2012). Simulation or numerical methods are required because the prediction problem is generally nonlinear and, therefore, its analytical solution is not available. It turns out that different special cases of the model call for different simulation procedures. Simulation experiments demonstrate that gains in forecasting accuracy are achieved by using the correct noncausal VAR model instead of its conventional causal counterpart. In an empirical application, a noncausal VAR model comprised of U.S. inflation and marginal cost turns out superior to the best-fitting conventional causal VAR model in forecasting inflation.

Keywords: Noncausal vector autoregression, forecasting, simulation, importance sampling, inflation.

JEL codes: C32, C53, E31.

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1 Introduction

The conventional vector autoregressive (VAR) model has become a standard tool in various fields of applications. In economics and finance the VAR model is typically used in structural analysis to study the dynamics and interrelationships between variables of interest. Another application of the VAR model is forecasting. For instance, economic decision makers, such as central banks and investors in financial markets, aim to forecast key macroeconomic and financial time series to assess the future state of the economy and investment opportunities.

The conventional causal VAR model has a moving average representation in terms of its present and past error terms. A characteristic feature of this model is that its error terms are not predictable by past values of the involved time series. In contrast, the moving average representation of the noncausal VAR model recently considered by Davis and Song (2010) and Lanne and Saikkonen (2012) also involve future error terms. In addition to theoretical advancements these authors demonstrate the practical usefulness of the noncausal VAR model in economic and financial applications. As discussed by Lanne and Saikkonen (2012), an important economic application of the noncausal VAR model is checking the validity of widely used test procedures based on the causal VAR model in testing economic hypotheses, especially in models involving expectations.

As yet, the development of the noncausal VAR model is at its early stages and even the literature of univariate noncausal autoregressive models is scant (see Breidt et al. (1991), Rosenblatt (2000), Davis and Song (2010), Lanne and Saikkonen (2011, 2012) and the references therein). The object of this paper is to devise forecasting techniques for the noncausal VAR model of Lanne and Saikkonen (2012). In addition to computing forecasts these techniques are also needed in computing impulse response functions, and hence in conducting structural analysis within the noncausal VAR model. Thus, our contribution should widen the applicability of the noncausal VAR model in empirical research.

In the causal VAR model, forecasting is simple in that explicit formulas are available. In the noncausal VAR model the situation is different because the pre-

diction problem is, in general, nonlinear and, consequently, forecasts cannot be obtained without resorting to numerical methods. Further discussion on this point is provided by Lanne, Luoto, and Saikkonen (2012b) who develop a simulation-based forecasting method for the univariate noncausal AR model proposed by Lanne and Saikkonen (2011). It turns out that forecasts of the considered noncausal VAR model can be computed analogously only when a suitable condition on the structure of the model holds. One case where the required condition always holds is the purely noncausal VAR model whose moving average representation only involves present and future error terms. In general, the required condition states that a certain parameter matrix involving the autoregressive coefficients of the model is nonsingular. Due to estimation errors this nonsingularity always holds in practice but, to avoid potential problems with nearly singular cases, we develop a forecasting technique which does not depend on the structure of the model. To achieve this robustness, more demanding computations based on importance sampling are needed (see, e.g., Geweke (1996) for a general discussion of importance sampling).

We examine the properties of our forecasting techniques by means of Monte Carlo simulations which also provide guidance for some user-chosen quantities needed in the application of these techniques. The simulations conducted demonstrate that our forecasting techniques perform well and that the correct noncausal VAR model outperforms its causal counterpart in forecast accuracy.

An empirical application to inflation forecasting illustrates the practical usefulness of our forecasting techniques. This application is partly inspired by the work of Lanne, Luoma, and Luoto (2012a) and Lanne et al. (2012b) who find that the univariate noncausal AR model outperforms its conventional causal counterpart in forecasting U.S. inflation. We consider a bivariate system consisting of inflation and the real marginal cost that has often been employed in monetary economics, especially in studies related to the New Keynesian Phillips Curve (see, e.g., Gali and Gertler (1999), Nason and Smith (2008), and the references therein). Our results are similar to those obtained by Lanne et al. (2012a, 2012b) in that a non-

causal VAR model provides the best in-sample fit and outperforms the best-fitting causal VAR model in out-of-sample forecasting.

The rest of the paper is structured as follows. Section 2 describes the noncausal VAR model of Lanne and Saikkonen (2012) and briefly discusses statistical inference. Section 3 develops the forecasting techniques of the paper, while Section 4 illustrates their performance by means of Monte Carlo simulations. Section 5 presents the empirical application. Section 6 concludes. Finally, some technical details are collected in three appendices.

2 Noncausal VAR model

In this section, we first describe the noncausal VAR model of Lanne and Saikkonen (2012) and then discuss briefly parameter estimation and statistical inference. Unless otherwise indicated, all vectors will be treated as column vectors and, for notational convenience, we shall write $x = (x_1, ..., x_n)$ for the (column) vector x where the components x_i may be either scalars or vectors (or both).

2.1 Model

Following Lanne and Saikkonen (2012) we consider the n-dimensional stochastic process y_t $(t = 0, \pm 1, \pm 2, ...)$ generated by

$$\Pi(B)\,\Phi\left(B^{-1}\right) y_t = \epsilon_t, \tag{1}$$

where ϵ_t $(n \times 1)$ is a sequence of independent, identically distributed random vectors with zero mean and finite positive definite covariance matrix, and $\Pi(B) = I_n - \Pi_1 B - \cdots - \Pi_r B^r$ and $\Phi(B^{-1}) = I_n - \Phi_1 B^{-1} - \cdots - \Phi_s B^{-s}$ are $n \times n$ matrix operators with B the usual backward shift operator, that is, $B^k y_t = y_{t-k}$ $(k = 0, \pm 1, ...)$. Moreover, the matrix polynomials $\Pi(z)$ and $\Phi(z)$ $(z \in \mathbb{C})$ have their zeros outside the unit disc, so that

$$\det \Pi(z) \neq 0, \quad |z| \leq 1, \quad \text{and} \quad \det \Phi(z) \neq 0, \quad |z| \leq 1. \tag{2}$$

These conditions guarantee the validity of various moving average representations to be used in our subsequent developments.

If $\Phi_j \neq 0$ for some $j \in \{1, .., s\}$, equation (1) defines a noncausal vector autoregression referred to as purely noncausal when $\Pi_1 = \cdots = \Pi_r = 0$ (or $r = 0$). When $\Phi_1 = \cdots = \Phi_s = 0$ (or $s = 0$) the conventional causal model is obtained. Then the former condition in (2) guarantees the stationarity of the model. In the general set-up of model (1) the same is true for the process

$$u_t = \Phi\left(B^{-1}\right) y_t. \tag{3}$$

Specifically, there exists a $\delta_1 > 0$ such that $\Pi(z)^{-1}$ has a well defined power series representation $\Pi(z)^{-1} = \sum_{j=0}^{\infty} M_j z^j = M(z)$ for $|z| < 1 + \delta_1$. Consequently, the process u_t has the causal moving average representation

$$u_t = M(B)\,\epsilon_t = \sum_{j=0}^{\infty} M_j \epsilon_{t-j}, \tag{4}$$

where $M_0 = I_n$ and the coefficient matrices M_j decay to zero at a geometric rate as $j \to \infty$.

Write $\Pi(z)^{-1} = \det(\Pi(z))^{-1}\,\Xi(z) = M(z)$, where $\Xi(z)$ is the adjoint polynomial matrix of $\Pi(z)$. Then, $\det(\Pi(B))\,u_t = \Xi(B)\,\epsilon_t$ (see (4)) and, by the definition of u_t in (3),

$$\Phi\left(B^{-1}\right) w_t = \Xi(B)\,\epsilon_t,$$

where, setting $\det(\Pi(z)) = a(z) = 1 - a_1 z - \cdots - a_{nr} z^{nr}$,

$$w_t = \det(\Pi(B))\,y_t = a(B) y_t. \tag{5}$$

Note that $\Xi(z)$ is a matrix polynomial of degree at most $(n-1)\,r$ and, because $\Pi(0) = I_n$, we also have $\Xi(0) = I_n$. By the latter condition in (2) one can find a $0 < \delta_2 < 1$ such that $\Phi\left(z^{-1}\right)^{-1}\Xi(z)$ has a well defined power series representation

$$\Phi\left(z^{-1}\right)^{-1}\Xi(z) = \sum_{j=-(n-1)r}^{\infty} N_j z^{-j} = N\left(z^{-1}\right) \tag{6}$$

for $|z| > 1 - \delta_2$. Thus, the process w_t has the moving average representation

$$w_t = \sum_{j=-(n-1)r}^{\infty} N_j \epsilon_{t+j}, \tag{7}$$

where the coefficient matrices N_j decay to zero at a geometric rate as $j \to \infty$. Using the equalities in (6) one can solve these matrices recursively as functions of the parameters Π_j $(j = 1, ..., r)$ and Φ_j $(j = 1, ..., s)$ (see Appendix A.1).

Finally, from (2) one obtains the moving average representation

$$y_t = \sum_{j=-\infty}^{\infty} \Psi_j \epsilon_{t-j}, \tag{8}$$

where Ψ_j $(n \times n)$ is the coefficient matrix of z^j in the Laurent series expansion of $\Psi(z) \overset{def}{=} \Phi(z^{-1})^{-1} \Pi(z)^{-1}$ which exists for $1 - \delta_2 < |z| < 1 + \delta_1$ with Ψ_j decaying to zero at a geometric rate as $|j| \to \infty$. The representation (8) implies that y_t is a stationary and ergodic process with finite second moments.

Model (1) is referred to as the VAR(r, s) model. In the conventional causal case the abbreviation VAR(r) is also used. In the next section, we present the joint distribution of an observed time series generated by the VAR(r, s) process. This joint distribution is needed to develop our forecasting methods and it also facilitates our discussion on parameter estimation and statistical inference.

2.2 Joint distribution of the VAR(r, s) process

It is well-known that causal and noncausal autoregressions cannot be distinguished by second-order properties or the Gaussian likelihood (see Lanne and Saikkonen (2011, 2012) and the references therein). Therefore, it is necessary to assume that the error term ϵ_t is non-Gaussian. The theoretical results of Lanne and Saikkonen (2012) assume that the distribution of ϵ_t is of a fairly general elliptical form. However, an inspection of the arguments used in Section 3.1 of that paper reveals that this assumption is not needed to derive the distribution of the observed data and, therefore, it is not necessary for our forecasting methods. Thus, unless otherwise indicated we only assume that the (non-Gaussian) distribution of ϵ_t is continuous with density function $f(\cdot)$, whose possible dependence on (unknown) parameters is not made explicit.

A detailed derivation of the joint distribution of the observed data can be found in Lanne and Saikkonen (2012), so here we only describe the final result. To this

end, define the $n \times 1$ vectors

$$v_{k,T-s+k} = w_{T-s+k} - \sum_{j=-(n-1)r}^{-k} N_j \epsilon_{T-s+k+j}, \quad k = 1, ..., s, \tag{9}$$

where the sum is interpreted as zero when $k > (n-1)\,r$, that is, when the lower bound exceeds the upper bound (this convention will also be used later). Note also that, by (1) and (7), $v_{k,T-s+k}$ can be expressed as a function of the observed data $y_1, ..., y_T$ and that the representation $v_{k,T-s+k} = \sum_{j=-k+1}^{\infty} N_j \epsilon_{T-s+k+j}$ holds, showing that $v_{k,T-s+k}$, $k = 1, ..., s$, are independent of ϵ_t, $t \leq T - s$. We also introduce the vector $\boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3)$ where $\boldsymbol{z}_1 = (u_1, ..., u_r)$, $\boldsymbol{z}_2 = (\epsilon_{r+1}, ..., \epsilon_{T-s})$, and $\boldsymbol{z}_3 = (v_{1,T-s+1}, ..., v_{s,T})$ are independent in view of the preceding discussion and (4). These vectors can be expressed as functions of the observed data (and parameters), and in what follows we use a tilde to make this functional dependence explicit. Thus, the components of the vectors $\tilde{\boldsymbol{z}}_1$ and $\tilde{\boldsymbol{z}}_2$ are $\tilde{u}_t = \Phi\left(B^{-1}\right) y_t$, $t = 1, ..., r$, (see (3)) and $\tilde{\epsilon}_t = \Pi\left(B\right) \Phi\left(B^{-1}\right) y_t$, $t = r+1, ..., T-s$, (see (1)), respectively. Furthermore, the components of $\tilde{\boldsymbol{z}}_3$, $\tilde{v}_{k,T-s+k}$, are defined by replacing w_{T-s+k} and $\epsilon_{T-s+k+j}$ on the right hand side of (9) by $a\left(B\right) y_{T-s+k}$ (see (5)) and $\tilde{\epsilon}_{T-s+k+j}$, $j = -\left(n-1\right) r, ..., -k$, $k = 1, ..., s$, respectively.

It is shown in Section 3.1 of Lanne and Saikkonen (2012) that the random vector $\boldsymbol{z}$ is related to the data vector $\boldsymbol{y} = (y_1, ..., y_T)$ according to $\boldsymbol{z} = \boldsymbol{H}_3 \boldsymbol{H}_2 \boldsymbol{H}_1 \boldsymbol{y}$, where $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, and $\boldsymbol{H}_3$ $(T \times T)$ are nonsingular transformation matrices that depend on the parameters Π_j $(j = 1, ..., r)$ and Φ_j $(j = 1, ..., s)$ with $\boldsymbol{H}_2$ and $\boldsymbol{H}_3$ having unit determinant. Thus, it follows that the joint density function of the data vector $\boldsymbol{y}$ is given by (assuming T large enough)

$$p\left(\boldsymbol{y}\right) = h_{\boldsymbol{z}_1}\left(\tilde{\boldsymbol{z}}_1\right) \cdot \prod_{t=r+1}^{T-s} f\left(\tilde{\epsilon}_t\right) \cdot h_{\boldsymbol{z}_3}(\tilde{\boldsymbol{z}}_3) \cdot \left|\det\left(\boldsymbol{H}_1\right)\right|. \tag{10}$$

For our subsequent developments the explicit expression of the matrix $\boldsymbol{H}_1$ is not relevant because the determinant term $\left|\det\left(\boldsymbol{H}_1\right)\right|$ will vanish from our forecasting formulas. In the purely noncausal case the joint density function $p\left(\boldsymbol{y}\right)$ can be simplified by replacing the first factor $h_{\boldsymbol{z}_1}\left(\tilde{\boldsymbol{z}}_1\right)$ by unity, setting $r = 0$ and $\tilde{\epsilon}_t = \Phi\left(B^{-1}\right) y_t$ in the second factor, and $\tilde{\boldsymbol{z}}_3 = (y_{T-s+1}, ..., y_T)$ in the third factor.

We shall now briefly discuss parameter estimation and statistical inference in the VAR(r, s) model (1). Following Lanne and Saikkonen (2012) we here assume that the error term ϵ_t has an elliptical distribution and use the second factor of the right hand side of (10) to obtain a computationally feasible approximation for the likelihood function. Maximizing this function over the permissible parameter space yields an (approximate) maximum likelihood (ML) estimator. Lanne and Saikkonen (2012) show that, under appropriate regularity conditions, the resulting (local) ML estimator is consistent and asymptotically normally distributed and that conventional methods to compute standard errors for estimated parameters and to construct likelihood-based tests apply.

The preceding discussion assumes that the orders r and s of the VAR(r, s) model (1) are known. As in Lanne and Saikkonen (2012) we specify these orders as follows. First, using least squares or Gaussian ML we find a causal VAR(p) model that adequately describes the autocorrelation structure of the data with the order p determined by using conventional procedures such as model selection criteria and diagnostic checks. Then we check the residuals of this causal VAR(p) model for Gaussianity and, only when we detect deviations from Gaussianity, we consider noncausal VAR models. Next we choose a non-Gaussian error distribution, such as the multivariate t–distribution used in Lanne and Saikkonen (2012), and estimate all causal and noncausal VAR(r, s) models with the orders r and s summing to the selected order p. Finally, of these alternative models we choose the one that maximizes the likelihood function and evaluate its adequacy with conventional diagnostic tools.

3 Forecasting

In this section, we consider forecasting future observations y_{T+h} $(h \geq 1)$ and, unless otherwise stated, we shall assume that the model is not causal and not univariate, so that $s > 0$ and $n > 1$. We let $\mathsf{E}_T(\cdot)$ signify the conditional expectation operator given the observed data $\boldsymbol{y} = (y_1, ..., y_T)$.

Our starting point is equation (7) which we make operational by approximating

the infinite sum therein by a finite sum. Specifically, from equations (5) and (7) we obtain the approximation

$$\mathsf{E}_T\left(y_{T+h}\right) \approx a_1 \mathsf{E}_T\left(y_{T+h-1}\right) + \cdots + a_{nr} \mathsf{E}_T\left(y_{T+h-nr}\right) + \mathsf{E}_T\left(\sum_{j=-(n-1)r}^{M-h} N_j \epsilon_{T+h+j}\right), \tag{11}$$

where $M > 0$ is supposed to be "large". As $\mathsf{E}_T\left(y_{T+h-j}\right) = y_{T+h-j}$ for $j \geq h$, (approximate) forecasts can be computed recursively starting from $h = 1$ if the last conditional expectation on the right hand side of (11) can be computed for every $h \geq 1$. In the univariate case ($n = 1$) considered by Lanne et al. (2012b) this conditional expectation depends on the error terms $\epsilon_{T+1}, ..., \epsilon_{T+M}$ only. However, except for the purely noncausal case ($r = 0$) this does not happen in our multivariate case where the error terms $\epsilon_{T+1-(n-1)r}, ..., \epsilon_T$ are also involved and the fact that $\epsilon_{T-s+1}, ..., \epsilon_T$ ($s > 0$) cannot be expressed as functions of the observed data (see (1)) causes complications. In the purely noncausal case these error terms vanish from the right hand side of (11), simplifying the situation and allowing a straightforward extension of the forecasting method of Lanne et al. (2012b). Therefore, and also to help understand the difficulties in the general case ($r > 0$, $s > 0$), we shall first consider forecasting in the purely noncausal case. The general case requires a more delicate treatment provided in Section 3.2.

3.1 Purely noncausal case

In the purely noncausal case ($r = 0$) the approximation (11) reduces to

$$\mathsf{E}_T\left(y_{T+h}\right) \approx \mathsf{E}_T\left(\sum_{j=0}^{M-h} N_j \epsilon_{T+h+j}\right), \quad N_0 = I_n. \tag{12}$$

To compute the conditional expectation on the right hand side we follow Lanne et al. (2012b) and derive the conditional density of $\boldsymbol{\epsilon}^+ = (\epsilon_{T+1}, ..., \epsilon_{T+M})$ given the data vector $\boldsymbol{y}$. Recall that now $\tilde{\epsilon}_t = \Phi\left(B^{-1}\right) y_t$ and $\tilde{\boldsymbol{z}}_3 = (y_{T-s+1}, ..., y_T)$. Using the expression of the density function $p\left(\boldsymbol{y}\right)$ in (10) and the preceding discussion one can check that the joint density function of $(\boldsymbol{y}, \boldsymbol{\epsilon}^+)$ can be written as

$$p\left(\boldsymbol{y}, \boldsymbol{\epsilon}^+\right) = \prod_{t=1}^{T-s} f\left(\tilde{\epsilon}_t\right) \cdot h_{\boldsymbol{z}_3, \boldsymbol{\epsilon}^+}(\boldsymbol{y}_3, \boldsymbol{\epsilon}^+) \cdot \left|\det\left(\boldsymbol{H}_1\right)\right|, \tag{13}$$

where $h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)$ is the joint density function of $(\boldsymbol{z}_3,\boldsymbol{\epsilon}^+)$ and $\boldsymbol{y}_3=(y_{T-s+1},...,y_T)$ (in this section we replace $\tilde{\boldsymbol{z}}_3$ by the more typical notation $\boldsymbol{y}_3$). From (10) (specialized to the present case) and (13) we find that the conditional density function of $\boldsymbol{\epsilon}^+$ given $\boldsymbol{y}$ is

$$p\left(\boldsymbol{\epsilon}^+ \mid \boldsymbol{y}\right)=\frac{h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)}{h_{\boldsymbol{z}_3}(\boldsymbol{y}_3)}=\frac{h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)}{\int h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)d\boldsymbol{\epsilon}^+}.$$

The right hand side of (12) can thus be written as

$$\mathsf{E}_T\left(\sum_{j=0}^{M-h}N_j\epsilon_{T+h+j}\right)=\frac{\int\sum_{j=0}^{M-h}N_j\epsilon_{T+h+j}\cdot h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)d\boldsymbol{\epsilon}^+}{\int h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)d\boldsymbol{\epsilon}^+}. \tag{14}$$

As in Lanne et al. (2012b), we now derive a feasible approximation for the density function $h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)$. As $y_t=\sum_{j=0}^{\infty}N_j\epsilon_{t+j}$ and $N_0=I_n$, we have the approximate relation

$$\begin{bmatrix} I_n & N_1 & \cdots & \cdots & \cdots & \cdots & N_{M+s-1}\\ 0 & \ddots & \ddots & & & & \vdots\\ \vdots & \ddots & I_n & N_1 & \cdots & \cdots & N_M\\ \vdots & & \ddots & I_n & 0 & & 0\\ \vdots & & & \ddots & \ddots & \ddots & \vdots\\ \vdots & & & & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & I_n \end{bmatrix}\begin{bmatrix}\epsilon_{T-s+1}\\ \vdots\\ \epsilon_T\\ \epsilon_{T+1}\\ \vdots\\ \epsilon_{T+M}\end{bmatrix}\approx\begin{bmatrix}y_{T-s+1}\\ \vdots\\ y_T\\ \epsilon_{T+1}\\ \vdots\\ \epsilon_{T+M}\end{bmatrix}$$

or briefly $\boldsymbol{A}\boldsymbol{\epsilon}^{++}\approx\boldsymbol{\upsilon}$. As the matrix $\boldsymbol{A}$ is nonsingular with unit determinant this yields $\boldsymbol{\epsilon}^{++}\approx\boldsymbol{A}^{-1}\boldsymbol{\upsilon}$ or

$$(\epsilon_{T-s+1},...,\epsilon_T,\epsilon_{T+1},...,\epsilon_{T+M})\approx\left(\tilde{\epsilon}_{T-s+1}\left(\boldsymbol{\epsilon}^+\right),...,\tilde{\epsilon}_T\left(\boldsymbol{\epsilon}^+\right),\epsilon_{T+1},...,\epsilon_{T+M}\right),$$

where $\tilde{\epsilon}_{T-s+1}\left(\boldsymbol{\epsilon}^+\right),...,\tilde{\epsilon}_T\left(\boldsymbol{\epsilon}^+\right)$ $(n\times 1)$ are the first s (vector) components of the vector $\boldsymbol{A}^{-1}\boldsymbol{\upsilon}$, and hence dependent on $y_{T-s+1},...,y_T$. Thus, it follows that the density function $h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)$ can be approximated as

$$h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^+}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^+)\approx\prod_{j=1}^{s}f\left(\tilde{\epsilon}_{T-s+j}\left(\boldsymbol{\epsilon}^+\right)\right)\cdot\prod_{t=T+1}^{T+M}f\left(\epsilon_t\right). \tag{15}$$

As in Lanne et al. (2012b), we have to compute the values of the two integrals on the right hand side of (14). More generally, for any function of $\boldsymbol{\epsilon}^+$, say $q\left(\boldsymbol{\epsilon}^+\right)$,

we can use (15) to obtain

$$\int q\left(\boldsymbol{\epsilon}^{+}\right)\cdot h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^{+}}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^{+})d\boldsymbol{\epsilon}^{+}\approx\int q\left(\boldsymbol{\epsilon}^{+}\right)\cdot\prod_{j=1}^{s}f\left(\tilde{\epsilon}_{T-s+j}\left(\boldsymbol{\epsilon}^{+}\right)\right)\cdot\prod_{t=T+1}^{T+M}f\left(\epsilon_t\right)d\boldsymbol{\epsilon}^{+}.$$

(Here as well as in similar subsequent instances existence and finiteness of the stated expectations are assumed.) The last expression can be interpreted as the expectation of the product of the first two factors in the integrand with respect to the distribution of $\boldsymbol{\epsilon}^{+}=(\epsilon_{T+1},...,\epsilon_{T+M})$. Using Monte Carlo simulation, this expectation can therefore be approximated as

$$\int q\left(\boldsymbol{\epsilon}^{+}\right)\cdot h_{\boldsymbol{z}_3,\boldsymbol{\epsilon}^{+}}(\boldsymbol{y}_3,\boldsymbol{\epsilon}^{+})d\boldsymbol{\epsilon}^{+}\approx\frac{1}{m}\sum_{i=1}^{m}q\left(\boldsymbol{\epsilon}^{+(i)}\right)\cdot\prod_{j=1}^{s}f\left(\tilde{\epsilon}_{T-s+j}\left(\boldsymbol{\epsilon}^{+(i)}\right)\right),\qquad(16)$$

where $\boldsymbol{\epsilon}^{+(i)}=(\epsilon_{T+1}^{(i)},...,\epsilon_{T+M}^{(i)})$, $i=1,...,m$, are mutually independent simulated realizations from the distribution of $\boldsymbol{\epsilon}^{+}$ so that $\epsilon_{T+1}^{(i)},...,\epsilon_{T+M}^{(i)}$ are independent random vectors for every i. As $m\to\infty$, the right hand side of (16) converges almost surely and provides an approximation for the left hand side that can be made arbitrarily accurate by choosing m and M large enough.

Applying (16) with $q\left(\boldsymbol{\epsilon}^{+}\right)=\sum_{j=0}^{M-h}N_j\epsilon_{T+h+j}$ and $q\left(\boldsymbol{\epsilon}^{+}\right)=1$ to the numerator and denominator on the right hand side of (14), respectively, we obtain approximations for the involved integrals, and hence for $\mathsf{E}_T\left(y_{T+h}\right)$. Thus, we get the (approximate) forecast

$$\hat{\mathsf{E}}_T\left(y_{T+h}\right)=\frac{\sum_{i=1}^{m}\sum_{j=0}^{M-h}N_j\epsilon_{T+h+j}^{(i)}\cdot\prod_{j=1}^{s}f\left(\tilde{\epsilon}_{T-s+j}\left(\boldsymbol{\epsilon}^{+(i)}\right)\right)}{\sum_{i=1}^{m}\prod_{j=1}^{s}f\left(\tilde{\epsilon}_{T-s+j}\left(\boldsymbol{\epsilon}^{+(i)}\right)\right)},\ h\geq 1,$$

which, for m and M large enough, approximates the true forecast $\mathsf{E}_T\left(y_{T+h}\right)$ arbitrarily closely. Appendix A.1 shows how to compute the coefficient matrices N_j recursively as functions of the parameters Π_j $(j=1,...,r)$ and Φ_j $(j=1,...,s)$. Choosing the values of the integers m and M will be discussed in Section 4.

3.2 General case

As already indicated, the general noncausal case seems to require techniques more burdensome than those in the purely noncausal case (or in the general univariate

noncausal case). To demonstrate this, consider the joint density of the augmented data vector $(\boldsymbol{y}, \boldsymbol{\epsilon}^+)$ and conclude from the discussion leading to the density function $p(\boldsymbol{y})$ in (10) that the joint density of $(\boldsymbol{y}, \boldsymbol{\epsilon}^+)$, and hence the conditional density of $\boldsymbol{\epsilon}^+$ given $\boldsymbol{y}$, involves the joint density of $(\boldsymbol{z}_3, \boldsymbol{\epsilon}^+)$. For simplicity, suppose that $s = 1$ so that $\boldsymbol{z}_3 = v_{1,T} = \sum_{j=0}^{\infty} N_j \epsilon_{T+j}$ and $\boldsymbol{z}_3 \approx \sum_{j=0}^{M} N_j \epsilon_{T+j}$ for M large (see (9) and the subsequent discussion). In the purely noncausal case we have $N_0 = I_n$, but this does not hold in the general case and it is even possible that the matrix N_0 is singular.[1] When this happens the random vectors $\boldsymbol{z}_3$ and $\boldsymbol{\epsilon}^+ = (\epsilon_{T+1}, ..., \epsilon_{T+M})$ are approximately linearly dependent so that, apart from the approximation error, the joint distribution of $\boldsymbol{z}_3$ and $\boldsymbol{\epsilon}^+$ is singular. This makes the conventional use of the joint density of $\boldsymbol{z}_3$ and $\boldsymbol{\epsilon}^+$, employed in the purely noncausal case, inappropriate.

To get an idea how the difficulty described above can be overcome, infer from equation (9) that, when $s = 1$, we have

$$\tilde{v}_{1,T} - \mathsf{E}_T \left(\sum_{j=1}^{\infty} N_j \epsilon_{T+j} \right) = N_0 \mathsf{E}_T (\epsilon_T), \tag{17}$$

where $\tilde{v}_{1,T} = w_T - \sum_{j=-(n-1)r}^{-1} N_j \tilde{\epsilon}_{T+j}$ is a function of the observed data. Now, suppose the matrix N_0 is nonsingular and that we can compute the conditional expectation of (a truncated version of) the infinite sum in (17) so that, by the nonsingularity of N_0, we can also compute $\mathsf{E}_T(\epsilon_T)$. Consider the approximate relation (11) with $h = 1$ and note that of the error terms $\epsilon_{T+1-(n-1)r}, ..., \epsilon_T$ all except ϵ_T are functions of the observed data (this is because $s = 1$; see (1)). Thus, if we can also compute the conditional expectation of $\sum_{j=0}^{\infty} N_j \epsilon_{T+1+j}$ (or its truncated version) we can compute the last conditional expectation on the right hand side of (11) with $h = 1$, and hence (an approximation for) $\mathsf{E}_T(y_{t+1})$. However, when the matrix N_0 is singular this approach does not apply as such but needs to be modified. The modification to be developed in the next section is generally applicable but requires the use of importance sampling not needed in the purely noncausal case considered in the preceding section. In Section 3.2.2 we show how a simpler

[1] For example, when $r = 1$ and $s = 1$, one can infer from Appendix A.1 that the matrix N_0 is singular when $\Pi_1 = \begin{bmatrix} 0 & 0 \\ -3/4 & 3/4 \end{bmatrix}$ and $\Phi_1 = \begin{bmatrix} 2/3 & 2/3 \\ 0 & 0 \end{bmatrix}$.

technique, similar to that derived in the purely noncausal case, can be obtained when a suitable condition about the structure of the model holds. When $s = 1$ this condition requires that the matrix N_0 is nonsingular.

3.2.1 Importance-sampling-based forecasting

Consider the general case with $s \geq 1$ and $r \geq 1$ where the counterpart of equation (17) is (see (7) and (9))

$$\begin{bmatrix} \tilde{v}_{1,T-s+1} \\ \tilde{v}_{2,T-s+2} \\ \vdots \\ \tilde{v}_{s,T} \end{bmatrix} - \mathsf{E}_T \begin{bmatrix} \sum_{j=s}^{\infty} N_j \epsilon_{T-s+1+j} \\ \sum_{j=s-1}^{\infty} N_j \epsilon_{T-s+2+j} \\ \vdots \\ \sum_{j=1}^{\infty} N_j \epsilon_{T+j} \end{bmatrix} = \mathsf{E}_T \begin{bmatrix} \sum_{j=0}^{s-1} N_j \epsilon_{T-s+1+j} \\ \sum_{j=-1}^{s-2} N_j \epsilon_{T-s+2+j} \\ \vdots \\ \sum_{j=-s+1}^{0} N_j \epsilon_{T+j} \end{bmatrix}.$$

Write the vector on the right hand side as

$$\begin{bmatrix} \sum_{j=0}^{s-1} N_j \epsilon_{T-s+1+j} \\ \sum_{j=-1}^{s-2} N_j \epsilon_{T-s+2+j} \\ \vdots \\ \sum_{j=-s+1}^{0} N_j \epsilon_{T+j} \end{bmatrix} = \begin{bmatrix} N_0 & N_1 & \cdots & N_{s-1} \\ N_{-1} & N_0 & \cdots & N_{s-2} \\ \vdots & & \ddots & \vdots \\ N_{-s+1} & N_{-s+2} & \cdots & N_0 \end{bmatrix} \begin{bmatrix} \epsilon_{T-s+1} \\ \epsilon_{T-s+2} \\ \vdots \\ \epsilon_T \end{bmatrix}.$$

Let $\boldsymbol{N}_0, ..., \boldsymbol{N}_{s-1}$ $(sn \times n)$ signify the s column blocks of the first factor on the right and, more generally, define the $sn \times n$ matrices

$$\boldsymbol{N}_j = \begin{bmatrix} N_j \\ \vdots \\ N_{j-s+1} \end{bmatrix}, \quad j = 0, 1, ...$$

Furthermore, define the matrix

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{N}_0 & \cdots & \boldsymbol{N}_{sn-1} \\ \boldsymbol{K}_0 & \cdots & \boldsymbol{K}_{sn-1} \end{bmatrix}, \quad sn^2 \times sn^2,$$

where the $sn\,(n-1) \times n$ matrices $\boldsymbol{K}_0, ..., \boldsymbol{K}_{sn-1}$ are chosen in such a way that this matrix is nonsingular. It is demonstrated in Appendix A.1 that the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$ $(sn \times sn^2)$ is of full row rank, implying that the mentioned choice of the matrices $\boldsymbol{K}_0, ..., \boldsymbol{K}_{sn-1}$ is possible and guarantees the nonsingularity of a

transformation matrix to be defined below. One possibility that always applies is to choose the rows of $[\boldsymbol{K}_0 \ \cdots \ \boldsymbol{K}_{sn-1}]$ $(sn\,(n-1) \times sn^2)$ as basis vectors of the orthogonal complement of the space spanned by the rows of $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$. The purpose of the matrix $\boldsymbol{Q}$ is to remove the difficulties discussed above. To illustrate this, note that when $s = 1$ we have $\boldsymbol{N}_0 = N_0$ and the upper block of rows of $\boldsymbol{Q}$ becomes $[N_0 \ \cdots \ N_{n-1}]$ $(n \times n^2)$, a matrix of full row rank (this particular case and the example in Footnote 1 also show that the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{s-1}]$ $(sn \times sn)$ can be singular).

To obtain a generally workable analog of equation (17) first define the vector

$$\begin{bmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1} \\ \boldsymbol{K}_0 \ \cdots \ \boldsymbol{K}_{sn-1} \end{bmatrix} \begin{bmatrix} \epsilon_{T-s+1} \\ \vdots \\ \epsilon_{T-s+sn} \end{bmatrix}, \quad sn^2 \times 1, \tag{18}$$

where $\boldsymbol{\zeta}_1$ is $sn \times 1$ and $\boldsymbol{\zeta}_2$ is $sn\,(n-1) \times 1$. As $\boldsymbol{z}_3 = (v_{1,T-s+1}, ..., v_{s,T})$ with $v_{k,T-s+k} = \sum_{j=-k+1}^{\infty} N_j \epsilon_{T-s+k+j}$ (see the discussion following (9)) we find from the definition of $\boldsymbol{N}_j$ that $\boldsymbol{z}_3 = \sum_{j=0}^{\infty} \boldsymbol{N}_j \epsilon_{T-s+1+j}$, or equivalently,

$$\boldsymbol{z}_3 - \sum_{j=sn}^{\infty} \boldsymbol{N}_j \epsilon_{T-s+j+1} = [\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}] \begin{bmatrix} \epsilon_{T-s+1} \\ \vdots \\ \epsilon_{T-s+sn} \end{bmatrix} = \boldsymbol{\zeta}_1,$$

where the latter equality is due to (18). Thus, using this fact in (18) and taking conditional expectations yields the (approximate) relation

$$\begin{bmatrix} \tilde{\boldsymbol{z}}_3 - \mathsf{E}_T\left(\sum_{j=sn}^{M+s-1} \boldsymbol{N}_j \epsilon_{T-s+j+1}\right) \\ \mathsf{E}_T(\boldsymbol{\zeta}_2) \end{bmatrix} \approx \begin{bmatrix} \boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1} \\ \boldsymbol{K}_0 \ \cdots \ \boldsymbol{K}_{sn-1} \end{bmatrix} \begin{bmatrix} \mathsf{E}_T(\epsilon_{T-s+1}) \\ \vdots \\ \mathsf{E}_T(\epsilon_{T-s+sn}) \end{bmatrix} \tag{19}$$

If we can forecast $\sum_{j=sn}^{M+s-1} \boldsymbol{N}_j \epsilon_{T-s+j+1}$ and $\boldsymbol{\zeta}_2$ on the left, the nonsingularity of the matrix on the right enables us to obtain forecasts for $\epsilon_{T-s+1}, ..., \epsilon_{T-s+sn}$ and, furthermore, for y_{T+h}, $h \geq 1$, as will be seen. To obtain forecasts for $\sum_{j=sn}^{M+s-1} \boldsymbol{N}_j \epsilon_{T-s+j+1}$ and $\boldsymbol{\zeta}_2$ we consider the extended data vector $(\boldsymbol{y},\boldsymbol{\xi})$, where $\boldsymbol{\xi} = (\boldsymbol{\zeta}_2, \boldsymbol{e}^+)$ with $\boldsymbol{e}^+ = (\epsilon_{T-s+sn+1}, ..., \epsilon_{T+M})$, and derive the conditional density of $\boldsymbol{\xi}$ given $\boldsymbol{y}$.

From the expression of the density function of $\boldsymbol{y}$ in (10) and the discussion preceding it we find that the joint density function of $(\boldsymbol{y},\boldsymbol{\xi})$ is

$$p(\boldsymbol{y},\boldsymbol{\xi}) = h_{\boldsymbol{z}_1}(\tilde{\boldsymbol{z}}_1) \cdot \left(\prod_{t=r+1}^{T-s} f(\tilde{\epsilon}_t) \right) \cdot h_{\boldsymbol{z}_3,\boldsymbol{\xi}}(\tilde{\boldsymbol{z}}_3,\boldsymbol{\xi}) \cdot |\det(\boldsymbol{H}_1)|,$$

where now $\tilde{\epsilon}_t = \Pi(B)\,\Phi(B^{-1})\,y_t$ and $h_{\boldsymbol{z}_3,\boldsymbol{\xi}}(\tilde{\boldsymbol{z}}_3,\boldsymbol{\xi})$ signifies the joint density function of $\boldsymbol{z}_3$ and $\boldsymbol{\xi}$ (note that here independence of $(\boldsymbol{z}_1,\boldsymbol{z}_2)$ and $(\boldsymbol{z}_3,\boldsymbol{\xi})$ has also been used). Dividing both sides of the preceding equation by the density function of $\boldsymbol{y}$ (see (10)) shows that the conditional density function of $\boldsymbol{\xi}$ given $\boldsymbol{y}$ is

$$p(\boldsymbol{\xi} \mid \boldsymbol{y}) = \frac{h_{\boldsymbol{z}_3,\boldsymbol{\xi}}(\tilde{\boldsymbol{z}}_3,\boldsymbol{\xi})}{h_{\boldsymbol{z}_3}(\tilde{\boldsymbol{z}}_3)} = \frac{h_{\boldsymbol{z}_3,\boldsymbol{\xi}}(\tilde{\boldsymbol{z}}_3,\boldsymbol{\xi})}{\int h_{\boldsymbol{z}_3,\boldsymbol{\xi}}(\tilde{\boldsymbol{z}}_3,\boldsymbol{\xi}) d\boldsymbol{\xi}}.$$

Thus, we need to derive the joint density of $\boldsymbol{z}_3 = (v_{1,T-s+1},...,v_{s,T})$ and $\boldsymbol{\xi} = (\boldsymbol{\zeta}_2,\boldsymbol{e}^+)$. It is shown in Appendix A.2 that this problem can be reduced to the derivation of $h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2)$, the joint density function of $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$. Specifically, we have

$$h_{\boldsymbol{z}_3,\boldsymbol{\xi}}(\tilde{\boldsymbol{z}}_3,\boldsymbol{\xi}) \approx h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+),\boldsymbol{\zeta}_2) \cdot \prod_{t=T-s+sn+1}^{T+M} f(\epsilon_t), \tag{20}$$

where

$$\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+) = \tilde{\boldsymbol{z}}_3 - \sum_{j=sn}^{M+s-1} \boldsymbol{N}_j \epsilon_{T-s+j+1}$$

with the $(n \times 1$ vector) components $\tilde{\zeta}_{1,k}(\boldsymbol{e}^+) = \tilde{v}_{k,T-s+k} - \sum_{j=sn}^{M+s-1} N_{j-k+1}\epsilon_{T-s+j+1}$ $(k = 1,...,s)$. To derive the joint density of $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$, define the matrix

$$\boldsymbol{R} = \begin{bmatrix} R_{1,1} & \cdots & R_{1,sn} \\ \vdots & & \vdots \\ R_{sn,1} & \cdots & R_{sn,sn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{N}_0 & \cdots & \boldsymbol{N}_{sn-1} \\ \boldsymbol{K}_0 & \cdots & \boldsymbol{K}_{sn-1} \end{bmatrix}^{-1} = \boldsymbol{Q}^{-1}, \quad sn^2 \times sn^2,$$

where $R_{j,k}$ $(j,k = 1,...,n)$ is of order $n \times n$. From (18) it is seen that $\boldsymbol{R}$ is the matrix of the linear transformation $(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) \rightarrow (\epsilon_{T-s+1},..,\epsilon_{T-s+sn})$ so that

$$h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) = \prod_{j=1}^{sn} f\left(\sum_{k=1}^{s} R_{j,k}\zeta_{1,k} + \sum_{k=s+1}^{sn} R_{j,k}\zeta_{2,k} \right) \cdot |\det(\boldsymbol{R})|, \tag{21}$$

where $\zeta_{i,k}$ is the kth (vector) component of $\boldsymbol{\zeta}_i$ $(i = 1,2)$.

The preceding derivations can be used to obtain an approximation for the conditional expectation $\mathsf{E}_T(q(\boldsymbol{\xi}))$ with $q(\boldsymbol{\xi})$ a function of $\boldsymbol{\xi}$. Specifically, we have

$$\mathsf{E}_T(q(\boldsymbol{\xi})) \approx \frac{\int q(\boldsymbol{\xi}) \cdot h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2) \cdot \prod_{t=T-s+sn+1}^{T+M} f(\epsilon_t)\, d\boldsymbol{\xi}}{\int h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2) \cdot \prod_{t=T-s+sn+1}^{T+M} f(\epsilon_t)\, d\boldsymbol{\xi}} \tag{22}$$

The integrals in (22) can be computed numerically but techniques more complicated than in the preceding section or in Lanne et al. (2012b) seem to be required. As in Breidt and Hsu (2005), where an analogous forecasting procedure for (univariate) noninvertible moving average models is developed, one can employ an importance sampling technique (see, e.g., Sec. 4.3 of Geweke (1996)). To this end, let $\varphi(\cdot)$ be an $sn(n-1)$-dimensional density function whose support contains the support of the distribution of $\boldsymbol{\zeta}_2$. Then write the numerator in (22) as

$$\begin{aligned} &\int q(\boldsymbol{\xi}) \cdot h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2) \cdot \prod_{t=T-s+sn+1}^{T+M} f(\epsilon_t)\, d\boldsymbol{\xi} \qquad (23) \\ = &\int q(\boldsymbol{\xi}) \cdot W(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2) \cdot \varphi(\boldsymbol{\zeta}_2) \cdot \prod_{t=T-s+sn+1}^{T+M} f(\epsilon_t)\, d\boldsymbol{\xi}, \end{aligned}$$

where

$$W(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2) = \frac{h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2)}{\varphi(\boldsymbol{\zeta}_2)}.$$

Similarly, write the denominator in (22) as

$$\begin{aligned} &\int h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2) \cdot \prod_{t=T-s+sn+1}^{T+M} f(\epsilon_t)\, d\boldsymbol{\xi} \qquad (24) \\ = &\int W(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2) \cdot \varphi(\boldsymbol{\zeta}_2) \cdot \prod_{t=T-s+sn+1}^{T+M} f(\epsilon_t)\, d\boldsymbol{\xi}. \end{aligned}$$

Clearly, the right hand side of (23) is the expectation of $q(\boldsymbol{\xi}) \cdot W(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2)$ with respect to a distribution with density $\varphi \times f \times \cdots \times f$ ($M - sn + s$ copies of f) and the right hand side of (24) is the expectation of $W(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^+), \boldsymbol{\zeta}_2)$ with respect to the same distribution. Thus, the conditional expectation in (22) can be approximated via Monte Carlo simulation as

$$\hat{\mathsf{E}}_T(q(\boldsymbol{\xi})) = \frac{\sum_{i=1}^m q(\boldsymbol{\xi}^{(i)}) \cdot W(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^{+(i)})}{\sum_{i=1}^m W(\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^{+(i)}))}, \tag{25}$$

where $\boldsymbol{\xi}^{(i)} = (\boldsymbol{\zeta}_2^{(i)}, \epsilon_{T-s+sn+1}^{(i)}, ..., \epsilon_{T+M}^{(i)})$, $i = 1, ..., m$, are mutually independent simulated realizations from a distribution with density $\varphi \times f \times \cdots \times f$ (regarding $\tilde{\boldsymbol{\zeta}}_1(\boldsymbol{e}^{+(i)})$, see equation (20)). Thus, $\boldsymbol{\zeta}_2^{(i)}$ $(sn\,(n-1) \times 1)$ is drawn from a distribution with density φ and $\epsilon_{T-s+sn+1}^{(i)}, ..., \epsilon_{T+M}^{(i)}$ $(n \times 1)$ are drawn independently of $\boldsymbol{\zeta}_2^{(i)}$ from a distribution with density f and, similarly to $\epsilon_{T-s+sn+1}, ..., \epsilon_{T+M}$, the random vectors $\epsilon_{T-s+sn+1}^{(i)}, ..., \epsilon_{T+M}^{(i)}$ are independent for every i.

Forecasts of y_{T+h} $(h \geq 1)$ can now be obtained recursively as follows.

Step 1. Apply (25) with $q(\boldsymbol{\xi}) = \sum_{j=sn}^{M+s-1} \boldsymbol{N}_j \epsilon_{T-s+j+1}$ and $q(\boldsymbol{\xi}) = \boldsymbol{\zeta}_2$ to obtain $\hat{\mathsf{E}}_T\left(\sum_{j=sn}^{M+s-1} \boldsymbol{N}_j \epsilon_{T-s+j+1}\right)$ and $\hat{\mathsf{E}}_T(\boldsymbol{\zeta}_2)$, and furthermore (see (19))

$$\begin{bmatrix} \hat{\mathsf{E}}_T(\epsilon_{T-s+1}) \\ \vdots \\ \hat{\mathsf{E}}_T(\epsilon_{T-s+sn}) \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} \tilde{\boldsymbol{z}}_3 - \hat{\mathsf{E}}_T\left(\sum_{j=sn}^{M+s-1} \boldsymbol{N}_j \epsilon_{T-s+j+1}\right) \\ \hat{\mathsf{E}}_T(\boldsymbol{\zeta}_2) \end{bmatrix}.$$

Step 2. Apply (25) with $q(\boldsymbol{\xi}) = \sum_{j=sn-s-h+1}^{M-h} N_j \epsilon_{T+h+j}$, $h \geq 1$, and compute recursively (see (11))

$$\begin{aligned} \hat{\mathsf{E}}_T\left(y_{T+h}\right) = a_1 \hat{\mathsf{E}}_T\left(y_{T+h-1}\right) + \cdots + a_{nr} \hat{\mathsf{E}}_T\left(y_{T+h-nr}\right) + \sum_{j=-(n-1)r}^{-s-h} N_j \tilde{\epsilon}_{T+h+j} \\ + N_{-s-h+1} \hat{\mathsf{E}}_T(\epsilon_{T-s+1}) + \cdots + N_{sn-s-h} \hat{\mathsf{E}}_T(\epsilon_{T-s+sn}) \\ + \hat{\mathsf{E}}_T\left(\sum_{j=sn-s-h+1}^{M-h} N_j \epsilon_{T+h+j}\right), \quad h = 1, 2, ..., \end{aligned}$$

where $\hat{\mathsf{E}}_T\left(y_{T+h-k}\right) = y_{T+h-k}$ for $k \geq h$ and $N_j = 0$ for $j < -(n-1)\,r$. Thus, $\sum_{j=-(n-1)r}^{-s-h} N_j \tilde{\epsilon}_{T+h+j} = 0$ for $s + h > (n-1)\,r$ and $N_{-s-h+1} \hat{\mathsf{E}}_T(\epsilon_{T-s+1}) = \cdots = N_{sn-s-h} \hat{\mathsf{E}}_T(\epsilon_{T-s+sn}) = 0$ for $h + s - sn > (n-1)\,r$.

In addition to choosing values for the integers m and M (to be discussed in Section 4) the application of the preceding procedure requires two choices. First, one has to choose the matrix $[\boldsymbol{K}_0 \ \cdots \ \boldsymbol{K}_{sn-1}]$ $(sn\,(n-1) \times sn^2)$ whose rows we here assume to be formed of the basis vectors of the orthogonal complement of the space spanned by the rows of $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$. Second, one has to choose the $sn(n-1)$-dimensional auxiliary density function $\varphi(\boldsymbol{\zeta}_2)$. As $\boldsymbol{\zeta}_2 = \sum_{j=0}^{sn-1} \boldsymbol{K}_j \epsilon_{T-s+1+j}$, a potentially reasonable choice might be based on the chosen error distribution. In

the bivariate special case with $s = 1$ the random vector $\boldsymbol{\zeta}_2$ is also bivariate, and one could choose $\varphi(\boldsymbol{\zeta}_2)$ as the density function of the error term ϵ_t. In general, as the dimension of $\boldsymbol{\zeta}_2$ is $s(n-1)$ times the dimension of ϵ_t, one could similarly choose $\varphi(\boldsymbol{\zeta}_2)$ as the density function of $(\epsilon_{T-s+1}, ..., \epsilon_{T-s+sn})$, that is, $f \times \cdots \times f$ ($sn\,(n-1)$ copies). This choice is probably not optimal but, due to its simplicity, will be used in our subsequent numerical illustrations where the error term is assumed to have a multivariate t–distribution. Breidt and Hsu (2005) use a somewhat similar importance sampler in their forecasting procedure.

3.2.2 Forecasting without importance sampling

It is possible to simplify the preceding simulation method if suitable knowledge of the structure of the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$ is available. In particular, as we shall show below, it is possible to avoid the use of importance sampling if the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{s-1}]$ $(sn \times sn)$ is nonsingular, for then we can choose

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{N}_0 & \cdots & \boldsymbol{N}_{sn-1} \\ \boldsymbol{K}_0 & \cdots & \boldsymbol{K}_{sn-1} \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{0} & I_{sn(n-1)} \end{bmatrix},$$

where $\boldsymbol{Q}_{11} = [\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{s-1}]$ and $\boldsymbol{Q}_{12} = [\boldsymbol{N}_s \ \cdots \ \boldsymbol{N}_{sn-1}]$. In the purely noncausal case considered in Section 3.1, this choice is always possible because then $N_j = 0$, $j < 0$, and $N_0 = I_n$, implying that the matrix $\boldsymbol{Q}_{11}$ is upper triangular with unit diagonal elements. However, in general this need not be the case (see the beginning of Section 3 and Footnote 1). On the other hand, in practice the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{s-1}]$ is unknown and has to replaced by an estimate which, due to estimation errors, is necessarily nonsingular (with probability one). Moreover, a simulated example provided in the next section suggests that, even when the assumed nonsingularity does not hold, the forecasting procedure to be derived in this section performs well compared to its robust but computationally more demanding alternative developed in the previous section. Note also that in practice one can assess the potential singularity of the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{s-1}]$ by examining, for example, the eigenvalues or determinant of its estimate.

When the matrix $\boldsymbol{Q}$ is as above, we have $\boldsymbol{\zeta}_2 = (\epsilon_{T+1}, ..., \epsilon_{T-s+sn})$ (see (18)) and

$\boldsymbol{R} = \boldsymbol{Q}^{-1}$ is of the same form as $\boldsymbol{Q}$ or, specifically,

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{0} & I_{sn(n-1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}_{11}^{-1} & -\boldsymbol{Q}_{11}\boldsymbol{Q}_{12} \\ \boldsymbol{0} & I_{sn(n-1)} \end{bmatrix}.$$

Thus, in this case the joint density function of $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ becomes (see (21))

$$h_{\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2}(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) = \prod_{j=1}^{s} f\left(\sum_{k=1}^{s} R_{jk}\zeta_{1,k} + \sum_{k=s+1}^{sn} R_{jk}\epsilon_{T-s+k}\right) \cdot \prod_{j=s+1}^{sn} f\left(\epsilon_{T-s+j}\right) \cdot \left|\det\left(\boldsymbol{R}\right)\right|,$$

and the approximate relation (22) can be written as

$$\mathsf{E}_T\left(q\left(\boldsymbol{\xi}\right)\right) \approx \frac{\int q\left(\boldsymbol{\xi}\right) \cdot \prod_{j=1}^{s} f\left(\sum_{k=1}^{s} R_{jk}\tilde{\zeta}_{1,k}(\boldsymbol{e}^{+}) + \sum_{k=s+1}^{sn} R_{jk}\epsilon_{T-s+k}\right) \cdot \prod_{t=T+1}^{T+M} f\left(\epsilon_t\right) d\boldsymbol{\xi}}{\int \prod_{j=1}^{s} f\left(\sum_{k=1}^{s} R_{jk}\tilde{\zeta}_{1,k}(\boldsymbol{e}^{+}) + \sum_{k=s+1}^{sn} R_{jk}\epsilon_{T-s+k}\right) \cdot \prod_{t=T+1}^{T+M} f\left(\epsilon_t\right) d\boldsymbol{\xi}},$$

where $\tilde{\zeta}_{1,k}(\boldsymbol{e}^{+})$ is defined below (20). Thus, as now $\boldsymbol{\xi} = (\boldsymbol{\zeta}_2, \boldsymbol{e}^{+}) = (\epsilon_{T+1}, ..., \epsilon_{T+M})$, the integral in the numerator is the expectation of

$$q\left(\boldsymbol{\xi}\right) \cdot \prod_{j=1}^{s} f\left(\sum_{k=1}^{s} R_{j,k}\tilde{\zeta}_{1,k}(\boldsymbol{e}^{+}) + \sum_{k=s+1}^{sn} R_{j,k}\epsilon_{T-s+k}\right)$$

with respect to a distribution with density $f \times \cdots \times f$ (M copies) whereas the integral in the denominator is the expectation of

$$\prod_{j=1}^{s} f\left(\sum_{k=1}^{s} R_{j,k}\tilde{\zeta}_{1,k}(\boldsymbol{e}^{+}) + \sum_{k=s+1}^{sn} R_{j,k}\epsilon_{T-s+k}\right)$$

with respect to the same distribution.

The preceding discussion shows that we can approximate the conditional expectation $\mathsf{E}_T\left(q\left(\boldsymbol{\xi}\right)\right)$ via Monte Carlo simulation as

$$\hat{\mathsf{E}}_T\left(q\left(\boldsymbol{\xi}\right)\right) = \frac{\sum_{i=1}^{m} q(\boldsymbol{\xi}^{(i)}) \cdot \prod_{j=1}^{s} f\left(\sum_{k=1}^{s} R_{j,k}\tilde{\zeta}_{1,k}(\boldsymbol{e}^{+(i)}) + \sum_{k=s+1}^{sn} R_{j,k}\epsilon_{T-s+k}^{(i)}\right)}{\sum_{i=1}^{m} \prod_{j=1}^{s} f\left(\sum_{k=1}^{s} R_{j,k}\tilde{\zeta}_{1,k}(\boldsymbol{e}^{+(i)} + \sum_{k=s+1}^{sn} R_{j,k}\epsilon_{T-s+k}^{(i)}\right)}, \tag{26}$$

where $\boldsymbol{\xi}^{(i)} = (\epsilon_{T+1}^{(i)}, ..., \epsilon_{T+M}^{(i)}) = \boldsymbol{\epsilon}^{+(i)}$, $i = 1, ..., m$, are independent draws from a distribution with density $f \times \cdots \times f$ (M copies). Forecasts can be obtained by modifying the two steps in the forecasting procedure of the previous section as follows. The vector $\boldsymbol{\zeta}_2$ in Step 1 is defined as $\boldsymbol{\zeta}_2 = (\epsilon_{T+1}, ..., \epsilon_{T-s+sn})$ and the simulation scheme (26) is used in place of (25) in both steps. This simulation procedure is similar to that derived in the purely noncausal case in Section 3.1 to which it, in fact, reduces in that special case, as demonstrated in Appendix A.3.

4 Simulation study

4.1 Simulated processes

In this section, we examine the performance of our forecasting techniques by using Monte Carlo simulations and data generation processes (DGPs) based on bivariate models estimated for real data. The same data, comprised of quarterly U.S. inflation and the real marginal cost, is also used in the next section to provide an illustration of our forecasting techniques. As mentioned in the Introduction, inflation and the real marginal cost are variables extensively studied in the previous literature on inflation (see, e.g., Gali and Gertler (1999), Canova (2007), Nason and Smith (2008), and the references therein).

Our quarterly data set, from the Federal Reserve Bank of St. Louis FRED databank, covers the period from 1955:1 to 2010:3**.** Inflation is computed as the log-difference of the seasonally adjusted GDP implicit price deflator and the real marginal cost is approximated by the real unit labor cost (for details, see Lanne and Luoto (2012)). We use the period from 1955:1 to 1989:4 to estimate VAR(r, s) models that will serve as DGPs in the subsequent Monte Carlo simulations.[2]

To specify a potentially noncausal VAR model we proceed along the lines discussed in Section 2.2 and first consider a Gaussian VAR(p) model. The conventional model selection criteria AIC and BIC and autocorrelation functions of the residuals suggested the order $p = 2$. However, the assumption of Gaussian errors could be rejected by the Q–Q plots of the residuals and, given uncorrelated residuals, by the clear autocorrelation in the squared residuals of the inflation equation. Thus, we consider second-order models, that is, VAR(r, s) models with $r + s = 2$ and, to capture the fat tails of the residual distribution, we choose the (bivariate) t–distribution for the errors.

Of the second-order models the VAR$(0, 2)$ model maximizes the likelihood function but only marginally compared to the VAR$(1, 1)$ model, whereas in terms of

[2]GAUSS 10 and its BHHH optimization routine in the CMLMT package are employed in estimation, simulation, and forecasting.

residual diagnostics the VAR$(1,1)$ model performs slightly better, as the residuals of the VAR$(0,2)$ model appear conditionally heteroskedastic. In the VAR$(1,1)$ model, the estimates of the parameters $\Pi_{1,12}$ and $\Phi_{1,12}$ appear small compared to their standard errors and the same applies to the estimates of the parameters $\Phi_{1,12}$ and $\Phi_{2,12}$ in the VAR$(0,2)$ model (we use $\Phi_{k,ij}$ to signify the (i,j) element of the matrix Φ_k with a similar notation used for Π_k). Restricting these parameters to zero also seems reasonable according to the likelihood ratio test (p–values 0.271 and 0.083 in the VAR$(1,1)$ and VAR$(0,2)$ models, respectively) and, in the case of the VAR$(0,2)$ model, their imposition considerably improves the rather poor estimation accuracy of the degrees-of-freedom parameter of the t–distribution. The restrictions have no marked effect on the residual diagnostics of the two models but, interestingly, the maximum value of the likelihood function of the restricted VAR$(1,1)$ model turns out to be slightly greater than that of the VAR$(0,2)$ model. All in all, both of these restricted models perform reasonably well and they will be used as DGPs in our simulation experiments and in the forecasting exercise of the next section. The estimation results are presented in Table 1. For comparison, we shall also consider the conventional causal VAR(2) model and, to see how our forecasting procedures work in a higher order case, a fourth-order model will be briefly discussed later.

It may be worth noting that the restrictions employed in the noncausal models in Table 1 are imposed on purely statistical grounds. As they imply that neither leads nor lags of the marginal cost (y_{2t}) appear in the equation of inflation (y_{1t}), one might think that the marginal cost has no effect on inflation forecasts. However, one should be cautious about making such an interpretation. To see the reason for this, consider the VAR$(0,2)$ model whose moving average representation is such that y_{1t} (inflation) depends on $\epsilon_{1,t+j}$, whereas y_{2t} (marginal cost) depends on both $\epsilon_{1,t+j}$ and $\epsilon_{2,t+j}$ ($j \geq 0$). Thus, as $\epsilon_{1,t+j}$ affects both inflation and the marginal cost, one cannot rule out the possibility that the marginal cost can help forecast $\epsilon_{1,t+j}$ and thereby inflation (see the (approximate) forecasting formula (12)). A similar argument applies to the VAR$(1,1)$ model.

4.2 Simulation set-up

We simulate 10 000 realizations of length $T+8$ from the DGPs defined by in Table 1 (100 observations are discarded from the beginning and end of the simulated series to eliminate the impact of initialization effects). We estimate a causal VAR(2) model as well as the correct noncausal VAR(1,1) or VAR(0,2) models from the first T observations in each realization. Note that the estimated models are unrestricted, i.e., the restrictions $\Phi_{1,12} = \Phi_{2,12} = 0$ and $\Pi_{1,12} = \Phi_{1,12} = 0$ discussed above are not taken into account. Next, point forecasts 1–8 periods ahead are constructed as described in Section 3. The sample size T is set to 300, and the number of simulated realizations m employed in the noncausal forecasting procedures ranges from $m = 10\ 000$ to $m = 500\ 000$. Based on the findings of Lanne et al. (2012b), the value of the truncation parameter M is set at 50 (essentially the same results are obtained with $M = 30$ and $M = 100$).

When forecasts are based on the noncausal VAR$(1, 1)$ model and importance sampling is used we have to choose the auxiliary density function $\varphi(\boldsymbol{\zeta}_2)$. Following the discussion at the end of Section 3.2.1, our choice is the density function of $(\epsilon_T, \epsilon_{T+1})$ with the independent ϵ_T and ϵ_{T+1} having the bivariate t–distribution shown in Table 1 (Panel B). In the case of the forecasting procedure derived in Section 3.2.2 the assumed nonsingularity boils down to the nonsingularity of the matrix N_0 (see the beginning of Section 3.2.2 and note that now $n = 2$ and $s = 1$). Using the estimates in Table 1 and formulas in Appendix A.1 we find that the determinant of N_0 is 0.173, showing that the required nonsingularity holds.

4.3 Results

Table 2 presents the mean squared forecast errors (MSFEs) of the VAR$(0, 2)$ model shown in Table 1 when the forecast horizon ranges from 1 to 8 periods. For each forecast horizon, we report the MSFEs separately for both variables as well as the determinant of the MSFE matrix (cf., e.g., Athanasopoulos and Vahid (2008)). The results show that there is a clear improvement in forecasting accuracy when the number of simulated realizations m increases from 10 000 to 100 000 or 200

000. This can be seen especially in the first variable (y_{1t}). The improvement is much smaller when m increases from 200 000 up to 500 000.

Tables 3 and 4 report results similar to those in Table 2 obtained for the VAR$(1,1)$ model with the two forecasting procedures developed in Sections 3.2.1 and 3.2.2. Whether importance sampling is used (Table 3) or not (Table 4) has only a minor effect on the MSFEs and the determinants of the MSFE matrices. By and large, forecasts based on the correct assumption of the nonsingularity of the matrix N_0 are slightly more accurate. As in the case of the VAR$(0,2)$ model, a clear improvement in forecasting accuracy is achieved by increasing the number of simulated realizations from 10 000 to 100 000 or 200 000 whereas further increases in the value of this parameter have only a marginal impact. Altogether the results of Tables 2–4 suggest that, in practice, $m = 200\ 000$ is a reasonable choice. This is much more than needed in the univariate case where Lanne et al. (2012b) found the choice $m = 10\ 000$ to be sufficient.

Table 5 shows the relative MSFEs obtained by dividing the MSFEs of the (correct) VAR$(0,2)$ or VAR$(1,1)$ model in Tables 2–4, respectively, by those of a (misspecified) causal VAR(2) model with Gaussian errors (using t–distributed errors instead of Gaussian errors yields very similar results). In addition to the forecasts of the VAR$(1,1)$ model based on importance sampling (indicated by IS), we also report those obtained by (correctly) assuming the nonsingularity of the matrix N_0. The number of simulated realizations employed is $m = 200\ 000$. The relative determinants of the MSFE matrices are always below unity, demonstrating that gains in the forecasting accuracy of the two variables can be achieved by using the correct noncausal model instead of its causal representation. However, an inspection of the MSFEs of each variable indicates that gains are mainly achieved in forecasting the first variable y_{1t}, whose relative MSFEs are below unity, whereas those of the second variable y_{2t} lie around unity ranging between 0.995 and 1.004.

As a small illustration of the potential consequences of (incorrectly) using the forecasting procedure of Section 3.2.2 when the matrix N_0 is singular we consider the bivariate VAR$(1,1)$ model with the coefficient matrices given in Footnote 1

(see the beginning of Section 3.2). Table 6 reports the relative MSFEs between the two forecasting procedures with the number of simulated realizations $m = 200\ 000$. The results show that the differences between the two procedures are minor (the figures range between 0.993 and 1.003). This admittedly very limited simulation experiment suggests that, at least in the first-order case ($r = s = 1$), falsely relying on the nonsingularity assumption and employing the forecasting procedure of Section 3.2.2 is not critical. More evidence on this matter is needed, however, before any far-reaching conclusions can be drawn.

We also examined a fourth-order model to see how the two forecasting procedures derived in Section 3.2 perform in a higher-order case. The DGPs are again estimated from the same data (AIC suggests order four for causal models with t–distributed errors). Of the (Gaussian) fourth-order models, a VAR$(1, 3)$ model maximizes the likelihood function. However, according to estimation results, this model appears overparameterized and does not perform well in terms of residual diagnostics. As the parameters $\Pi_{1,12}$ and $\Phi_{j,12}$, $j = 1, 2, 3$, are rather imprecisely estimated we restrict them to zero. These restrictions correspond to those used in the VAR$(0, 2)$ and VAR$(1, 1)$ models above, and when they are imposed a reasonable fit is obtained. Thus, we used this restricted VAR$(1, 3)$ model as a DGP in the higher-order case. The auxiliary density function $\varphi(\boldsymbol{\zeta}_2)$ needed in importance sampling was chosen as described at the end of Section 3.2.2 (in this case, four times the density function of the bivariate t–distributed error term ϵ_t). Qualitatively the simulation results were similar to those obtained with the VAR$(1, 1)$ model (details are available upon request). In particular, whether importance sampling was used or not had no substantial effect on the forecasting accuracy, and compared to the causal VAR(4) model the forecasts were more accurate.

5 Empirical illustration

In this section, we consider out-of-sample forecasting with the bivariate noncausal VAR models introduced in Table 1 as well as their causal counterparts. An issue of special interest is whether U.S. inflation forecasts can also in the VAR framework

be improved by allowing for noncausality, in accordance with the findings of Lanne et al. (2012a, 2012b) based on univariate AR models. Their results may reflect the fact that omitted factors predictable by lagged values of inflation are contained in the error term of a univariate AR model and the error term of the noncausal AR model is predictable unlike its causal counterpart. As the real marginal cost included in our bivariate model could be such an omitted factor, it is of interest to see how inflation forecasts behave when the real marginal cost is explicitly included in the model.

We compute forecasts by using an expansive window of observations such that the models are re-estimated at each date with the estimation period augmented by one observation. Following Lanne et al. (2012b), the starting point of the out-of-sample forecasting period is set to 1990:1 and the last forecasts are computed for 2010:3, so that forecasts are computed for 83 quarters. Based on the simulation results of the previous section, the number of simulated realizations m used in forecasting with noncausal VAR(r, s) models ($s \geq 1$) is set at $m = 200\ 000$.

Table 7 presents the MSFEs and determinants of the MSFE matrices for the causal VAR(2) models with Gaussian (VAR(2)-N) and t–distributed errors (VAR(2)-t), and for the noncausal VAR$(1, 1)$ and VAR$(0, 2)$ models (see Table 1). Note that now the restrictions $\Phi_{1,12} = \Phi_{2,12} = 0$ and $\Pi_{1,12} = \Phi_{1,12} = 0$ are imposed on the VAR$(1, 1)$ and VAR$(0, 2)$ models, respectively. In the causal VAR(2) model no restrictions are employed, as in model selection reasonable restrictions were not found (this particularly applies to the restrictions $\Pi_{1,12} = \Pi_{2,12} = 0$).

First consider the inflation forecasts that we are mostly interested in. Table 7 shows that the VAR$(1, 1)$ model yields the smallest MSFEs except for the two-quarter horizon where it is slightly outperformed by the VAR$(0, 2)$ model. Moreover, irrespective of the forecast horizon, the VAR$(1, 1)$ model outperforms the two causal VAR(2) models of which the VAR(2)-N model performs better and it also performs quite well in comparison with the VAR$(1, 1)$ model when the forecast horizon is short. However, when the forecast horizon is four quarters or more the VAR$(1, 1)$ model is clearly superior. In line with the simulation results of the

previous section, the differences between the two forecasting methods in the case of the VAR$(1,1)$ model are negligible.

As far as forecasting the marginal cost is concerned, especially the Gaussian VAR(2) model performs slightly better than the noncausal models with the exception of one-quarter forecasts where the VAR$(0,2)$ yields the smallest MSFE. However, the determinants of the MSFE matrices reported in Table 7 show that the noncausal models produce the best overall forecasts. In particular, in terms of this criterion, the purely noncausal VAR$(0,2)$ model yields the most accurate forecasts for one and two quarters ahead whereas the VAR$(1,1)$ model is the best when the forecast horizon is longer.

To sum up, the results show that the noncausal models produce more accurate forecasts for U.S. inflation than their causal alternatives, and this also holds for the bivariate system consisting of inflation and the marginal cost. However, causal models, especially the Gaussian model, perform slightly better than the noncausal models in forecasting the marginal cost.

6 Conclusion

In this paper, we have developed forecasting methods for the noncausal VAR model of Lanne and Saikkonen (2012). To our knowledge, this is the first attempt to make forecasting in noncausal VAR models practically feasible. Due to the nonlinear nature of the prediction problem explicit formulas to compute forecasts are not available and, therefore, our forecasting methods exploit simulation-based techniques. The needed techniques turned out to be more complex than in the univariate case of Lanne et al. (2012b) with the extent of complexity depending on the structure of the model. However, according to the simulation experiments conducted, the proposed forecasting methods perform quite well even in the most complicated case, where importance sampling is employed. They also appear feasible in practice, as illustrated by our empirical application where noncausal VAR models performed well in comparison with their causal counterparts.

By making forecasting in the noncausal VAR model of Lanne and Saikkonen

(2012) feasible in practice this paper has paved the way for developing methods for structural analysis within these models, including the computation of impulse response functions. Lanne and Saikkonen (2012) have also pointed out that noncausality is closely related to possible nonfundamental solutions of theoretical economic and financial models such as Dynamic Stochastic General Equilibrium (DSGE) models. As nonfundamentalness implies dependence on future error terms, it would be interesting to use the noncausal VAR model instead of the causal VAR model as a benchmark in assessing forecasting ability of DSGE models (cf. Rubaszek and Skrzypczynski, 2008).

Appendix: Technical details

A.1: Structure of the matrices N_j in (7)

In this appendix, we demonstrate that the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$ $(sn \times sn^2)$ is of full row rank sn. First, conclude from the identity $\Phi(z^{-1})^{-1}\Xi(z) = N(z^{-1})$ that

$$\begin{aligned}
N_{-(n-1)r} &= -\Xi_{(n-1)r} \\
N_{-(n-1)r+1} &= \Phi_1 N_{-(n-1)r} - \Xi_{(n-1)r-1} \\
&\vdots \\
N_{-(n-1)r+s} &= \Phi_1 N_{-(n-1)r+s-1} + \cdots + \Phi_s N_{-(n-1)r} - \Xi_{(n-1)r-s} \\
&\vdots \\
N_{-1} &= \Phi_1 N_{-2} + \cdots + \Phi_s N_{-s} - \Xi_1.
\end{aligned}$$

Here, as well as elsewhere, $N_k = 0$ for $k < -(n-1)r$. Furthermore, the matrices N_k, $k \geq 0$, satisfy

$$\begin{aligned}
N_0 &= \Phi_1 N_{-1} + \cdots + \Phi_s N_{-s} + I_n \\
N_k &= \Phi_1 N_{k-1} + \cdots + \Phi_s N_{-s}, \quad k \geq 1.
\end{aligned}$$

Note that in the pure noncausal case only the matrices N_j, $j \geq 0$, are relevant and the preceding equations apply with $N_j = 0$, $j < 0$. Because the matrices

Ξ_j, $j = 1, ..., (n-1)\,r$, are functions of the parameters $\Pi_1, ..., \Pi_r$ the preceding equations show how the coefficient matrices N_j can be computed as functions of the autoregressive parameters.

Define the matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{s-1} & \Phi_s \\ I_n & 0 & & & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_n & 0 \end{bmatrix} \quad (sn \times sn)\,.$$

Then, using the definition of the matrix $\boldsymbol{N}_k$ (see the beginning of Section 3.2.1) we have

$$\boldsymbol{N}_k = \boldsymbol{\Phi}\boldsymbol{N}_{k-1} = \boldsymbol{\Phi}^k \boldsymbol{N}_0, \quad k \geq 1.$$

First we demonstrate that the rows of the infinite dimensional matrix $[\boldsymbol{N}_0\ \boldsymbol{N}_1\ \cdots]$ are linearly independent. As the spectral density matrix of y_t is positive definite there can be no exact linear dependences between the components of the data vector $\boldsymbol{y}$. Thus, as the vector $\boldsymbol{z}$ is obtained from $\boldsymbol{y}$ by a nonsingular linear transformation (see the discussion preceding (10)) it follows that there can be no exact linear dependences between the components of $\boldsymbol{z}$. Hence, the same is true for $\boldsymbol{z}_3 = (v_{1,T-s+1}, ..., v_{s,T})$ and, as $v_{k,T-s+k} = \sum_{j=-k+1}^{\infty} N_j \epsilon_{T-s+k+j}$, we have

$$\begin{bmatrix} v_{1,T-s+1} \\ \vdots \\ v_{s,T} \end{bmatrix} = [\boldsymbol{N}_0\ \cdots\ \boldsymbol{N}_{s-1}\ \boldsymbol{N}_s\ \cdots] \begin{bmatrix} \epsilon_{T-s+1} \\ \vdots \\ \epsilon_T \\ \epsilon_{T+1} \\ \vdots \end{bmatrix}.$$

From this it follows that the rows of the infinite dimensional matrix $[\boldsymbol{N}_0\ \boldsymbol{N}_1\ \cdots]$ are linearly independent.

Now we can proceed as in Hannan and Deistler (1988, p. 44-45). By the Caley-Hamilton theorem, the matrix $\boldsymbol{\Phi}$ satisfies its characteristic equation $\det(\mu I_{sn} - \boldsymbol{\Phi}) = 0$, which is of degree sn, so that $\boldsymbol{\Phi}^{sn} = c_1 I_{sn} + c_2\boldsymbol{\Phi} + \cdots + c_{sn-1}\boldsymbol{\Phi}^{sn-1}$ for

some scalars $c_1, ..., c_{sn-1}$. Thus, as $\boldsymbol{N}_k = \boldsymbol{\Phi}^k \boldsymbol{N}_0$, $k \geq 1$, we also have $\boldsymbol{N}_{sn} = c_1\boldsymbol{N}_0 + c_2\boldsymbol{N}_1 + \cdots + c_{sn-1}\boldsymbol{N}_{sn-1}$, implying that the columns of the matrix $\boldsymbol{N}_{sn}$ can be expressed as linear combinations the columns of the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$. This fact can be extended inductively to the columns of any $\boldsymbol{N}_k$, $k \geq sn$. Thus, the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$ must be of full row rank sn because otherwise we could find a vector c $(sn \times 1)$ such that $c'[\boldsymbol{N}_0 \ \boldsymbol{N}_1 \ \cdots] = 0$.

Note that the preceding discussion also shows that the matrix $\boldsymbol{N}_0$ must be nonzero because otherwise we would have $N_k = 0$ for all $k \geq 0$, implying that the matrix $[\boldsymbol{N}_0 \ \cdots \ \boldsymbol{N}_{sn-1}]$ is zero.

A.2: Joint density in (20)

In this appendix, we derive the joint density of $\boldsymbol{z}_3 = (v_{1,T-s+1}, ..., v_{s,T})$ and $\boldsymbol{\xi} = (\boldsymbol{\zeta}_2, \boldsymbol{e}^+)$ needed in Section 3.2.1. First recall that $\boldsymbol{\zeta}_1 = \boldsymbol{z}_3 - \sum_{j=sn}^{\infty} \boldsymbol{N}_j \epsilon_{T-s+j+1}$ (see the discussion following equation (18). Thus, as $\boldsymbol{\epsilon}^+ = (\epsilon_{T-s+sn+1}, ..., \epsilon_{T+M})$, we get the approximate relation

$$\begin{bmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \\ \boldsymbol{\epsilon}^+ \end{bmatrix} \approx \boldsymbol{C} \begin{bmatrix} \boldsymbol{z}_3 \\ \boldsymbol{\zeta}_2 \\ \boldsymbol{\epsilon}^+ \end{bmatrix},$$

where

$$\boldsymbol{C} = \begin{bmatrix} I_{sn} & 0 & -\boldsymbol{N}_{sn} & -\boldsymbol{N}_{sn+1} & \cdots & -\boldsymbol{N}_{M+s-1} \\ 0 & I_{sn(n-1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & I_n \end{bmatrix}.$$

The matrix $\boldsymbol{C}$ is clearly nonsingular with unit determinant. Thus, it follows that, to a close approximation, the joint density function of $\boldsymbol{z}_3$ and $\boldsymbol{\xi} = (\boldsymbol{\zeta}_2, \boldsymbol{e}^+)$ is as given in (20) (note that here independence of $(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)$ and $\boldsymbol{e}^+ = (\epsilon_{T-s+sn+1}, ..., \epsilon_{T+M})$ is also used).

A.3: Simulation procedure in Section 3.2.2 when $r = 0$

In this appendix, we demonstrate that in the purely noncausal case ($r = 0$) the forecasting technique derived in Section 3.2.2 reduces to that derived in Section 3.1. To simplify notation, we give details in the case $s = 1$ only.

When $s = 1$ one can readily check that (see the beginning of Section 3.2.2)

$$\boldsymbol{R} = \boldsymbol{Q}^{-1} = \begin{bmatrix} I_n & -N_1 & -N_2 & \cdots & -N_{n-1} \\ 0 & I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I_n \end{bmatrix}.$$

This implies that $\det(\boldsymbol{R}) = 1$ and, as now $R_{1,1} = I_n$ and $R_{1,k} = -N_k$ $(k = 2, ..., n)$, the density function $h_{\zeta_1,\zeta_2}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)$ employed in Section 3.2.2 takes the form

$$h_{\zeta_1,\zeta_2}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) = f\left(\zeta_{1,1} - \sum_{k=2}^{n} N_k \epsilon_{T-1+k}\right) \prod_{j=2}^{n} f\left(\epsilon_{T-s+j}\right).$$

Here we need to replace $\zeta_{1,1}$ by $\tilde{\zeta}_{1,1}(\boldsymbol{e}^+) = \tilde{v}_{1,T} - \sum_{j=n}^{M} N_j \epsilon_{T+j}$ with $\tilde{v}_{1,T} = y_T$ (see (5) and (9)). Thus, consider the expression

$$f\left(\tilde{\zeta}_{1,1}(\boldsymbol{e}^+) - \sum_{k=2}^{n} N_k \epsilon_{T-s+k}\right) = f\left(y_T - \sum_{j=1}^{M} N_j \epsilon_{T+j}\right) = f\left(\tilde{\epsilon}_T\left(\boldsymbol{\epsilon}^+\right)\right),$$

where the latter equality is obtained by specializing the definition of $\tilde{\epsilon}_T(\boldsymbol{\epsilon}^+)$ to the case $s = 1$ (see the arguments leading to (15) in Section 3.1). As now $\boldsymbol{\xi} = \boldsymbol{\epsilon}^+$, the Monte Carlo approximation (26) in Section 3.2.2 becomes

$$\hat{\mathsf{E}}_T\left(q\left(\boldsymbol{\xi}\right)\right) = \frac{\sum_{i=1}^{m} q(\boldsymbol{\epsilon}^{+(i)}) f\left(\tilde{\epsilon}_T\left(\boldsymbol{\epsilon}^{+(i)}\right)\right)}{\sum_{i=1}^{m} f\left(\tilde{\epsilon}_T\left(\boldsymbol{\epsilon}^{+(i)}\right)\right)},$$

which with $q\left(\boldsymbol{\xi}\right) = \sum_{j=0}^{M-h} N_j \epsilon_{T+h+j}^{(i)}$ equals the expression obtained for $\hat{\mathsf{E}}_T\left(y_{T+h}\right)$ in Section 3.1 in the case $s = 1$. This shows the desired result (note that now Step 1 can be skipped because the error terms $\epsilon_{T-s+1}, ..., \epsilon_T$ are not involved in the computation of forecasts).

When $s > 1$ the matrix $\boldsymbol{Q}$ is of the form

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ 0 & I_{sn(n-1)} \end{bmatrix},$$

where $\boldsymbol{Q}_{11}$ is upper triangular with unit diagonal elements. Computing the inverse of $\boldsymbol{Q}$ and using arguments similar to those above one can again show the desired result. Details are omitted.

References

Athanasopoulos, G., Vahid, F., 2008. VARMA versus VAR for macroeconomic forecasting. Journal of Business and Economic Statistics 26, 237–252.

Breidt, F.J., Davis, R.A., Lii, K.S., Rosenblatt, M., 1991. Maximum likelihood estimation for noncausal autoregressive processes. Journal of Multivariate Analysis 36, 175–198.

Breidt, F.J., Hsu, N.J., 2005. Best mean square prediction for moving averages. Statistica Sinica 15, 427–446.

Canova, F., 2007. G-7 inflation forecasts: Random walk, Phillips curve or what else? Macroeconomic Dynamics 11, 1–30.

Davis, R., Song, L., 2010. Noncausal vector AR processes with application to financial time series. Unpublished manuscript, Columbia University, New York.

Gali, J., Gertler, M., 1999. Inflation dynamics: A structural econometric approach. Journal of Monetary Economics 44, 195–222.

Geweke, J., 1996. Monte Carlo simulation and numerical integration. In: Amman, H., Kendrick, D., Rust, J. (Eds.), Handbook of Computational Economics. North-Holland, Amsterdam, pp. 731–800.

Hannan, E.J., Deistler, M., 1988. The Statistical Theory of Linear Systems. Wiley,

New York.

Lanne, M., Luoto, J., 2012. Autoregression-based estimation of the new Keynesian Phillips curve. Journal of Economic Dynamics and Control, forthcoming.

Lanne, M., Saikkonen, P., 2011. Noncausal autoregressions for economic time series. Journal of Time Series Econometrics 3, article 2.

Lanne, M., Saikkonen, P., 2012. Noncausal vector autoregression. Econometric Theory, forthcoming.

Lanne, M., Luoma, A., Luoto, J., 2012a. Bayesian model selection and forecasting in noncausal autoregressive models. Journal of Applied Econometrics 27, 812–830.

Lanne, M., Luoto, J., Saikkonen, P., 2012b. Optimal forecasting of noncausal autoregressive time series. International Journal of Forecasting 28, 623–631.

Nason, J.M., Smith, G.W., 2008. Identifying the new Keynesian Phillips curve. Journal of Applied Econometrics 23, 525–551.

Rosenblatt, M., 2000. Gaussian and Non-Gaussian Linear Time Series and Random Fields. Springer-Verlag, New York.

Rubaszek, M., Skrzypczynski, P., 2008. On the forecasting performance of a small-scale DSGE model. International Journal of Forecasting 24, 498–512.

Tables

Table 1: Estimation results of the VAR(0,2) and VAR(1,1) models for the U.S. inflation and real marginal cost.

Panel A: VAR(0,2) model								
	0.618	0		0.271	0		1.260	0.152
Φ_1	(0.094)	(-)	Φ_2	(0.090)	(-)	Σ	(0.209)	(0.091)
	0.064	0.999		-0.142	-0.065		0.152	0.609
	(0.063)	(0.088)		(0.061)	(0.086)		(0.091)	(0.101)
λ	5.801		logL	-371.741				
	(1.743)							
Panel B: VAR(1,1) model								
	-0.347	0		0.915	0		1.178	0.581
Π_1	(0.088)	(-)	Φ_1	(0.032)	(-)	Σ	(0.202)	(0.311)
	-0.257	0.929		0.562	0.041		0.581	0.868
	(0.119)	(0.033)		(0.253)	(0.089)		(0.311)	(0.317)
λ	5.305		logL	-371.222				
	(1.619)							

Notes: The numbers in the parentheses are standard errors based on the Hessian of the log-likelihood function. In the table, λ is the degrees-of-freedom parameter of the multivariate t-distribution and logL is the value of the maximized log-likelihood function.

Table 2: Mean-squared forecast errors (MSFEs) of the VAR(0,2) model described in Table 1.

Horizon	1	2	3	4	5	6	7	8
m				MSFE, y_{1t}				
10 000	1.727	2.447	3.176	3.638	4.188	4.513	4.881	5.042
100 000	1.719	2.437	3.154	3.611	4.168	4.498	4.860	5.016
200 000	1.720	2.439	3.148	3.610	4.166	4.494	4.862	5.021
500 000	1.720	2.437	3.152	3.611	4.165	4.488	4.856	5.010
m				MSFE, y_{2t}				
10 000	1.036	2.124	3.166	4.097	4.855	5.547	6.166	6.626
100 000	1.028	2.113	3.163	4.093	4.851	5.537	6.151	6.604
200 000	1.027	2.116	3.160	4.095	4.857	5.537	6.153	6.611
500 000	1.026	2.109	3.156	4.093	4.849	5.537	6.152	6.606
m				Det				
10 000	1.774	4.974	9.724	14.397	19.816	24.650	29.779	33.178
100 000	1.751	4.927	9.643	14.276	19.700	24.512	29.576	32.884
200 000	1.751	4.936	9.623	14.269	19.712	24.487	29.592	32.959
500 000	1.749	4.921	9.626	14.273	19.681	24.453	29.555	32.860

Notes: The entries are based on 10 000 replications. The sample size is 300 ($T = 300$) and m is the number of simulated realizations (see Section 3). The truncation parameter M is set at 50 (see, e.g., (11)). The MSFEs are reported separately for the components of $y_t = (y_{1t}, y_{2t})'$. Det denotes the determinant of the mean-squared forecast error matrix. The correct noncausal VAR(0,2) model is estimated without taking the restrictions in the DGP (see Table 1) into account.

Table 3: Mean-squared forecast errors (MSFEs) of the VAR(1,1) model described in Table 1. Forecasts based on importance sampling.

Horizon	1	2	3	4	5	6	7	8
m				MSFE, y_{1t}				
10 000	1.884	2.617	3.345	4.043	4.476	4.954	5.422	5.749
100 000	1.830	2.594	3.310	4.014	4.451	4.920	5.385	5.705
200 000	1.842	2.589	3.319	4.014	4.453	4.918	5.376	5.699
500 000	1.837	2.587	3.305	4.014	4.448	4.898	5.366	5.691
m				MSFE, y_{2t}				
10 000	0.986	2.045	2.967	3.901	4.696	5.393	6.153	6.669
100 000	0.979	2.035	2.940	3.871	4.675	5.358	6.116	6.615
200 000	0.980	2.039	2.945	3.878	4.668	5.345	6.091	6.606
500 000	0.980	2.033	2.941	3.870	4.660	5.340	6.097	6.597
m				Det				
10 000	1.809	4.994	9.005	14.083	18.445	23.296	28.711	32.970
100 000	1.741	4.915	8.835	13.833	18.199	22.881	28.253	32.433
200 000	1.753	4.922	8.878	13.872	18.222	22.864	28.129	32.389
500 000	1.748	4.903	8.826	13.846	18.150	22.741	28.079	32.271

Notes: See the notes to Table 2. The auxiliary density function $\varphi(\boldsymbol{\zeta}_2)$ is chosen as discussed in Section 4.2.

Table 4: Mean-squared forecast errors (MSFEs) of the VAR(1,1) model described in Table 1. Forecasts based on assuming nonsingularity of the matrix N_0.

Horizon	1	2	3	4	5	6	7	8
m				MSFE, y_{1t}				
10 000	1.813	2.587	3.304	4.011	4.449	4.901	5.358	5.681
100 000	1.812	2.580	3.300	4.000	4.433	4.901	5.361	5.683
200 000	1.812	2.578	3.300	4.001	4.440	4.900	5.364	5.688
500 000	1.811	2.579	3.300	4.003	4.435	4.898	5.365	5.686
m				MSFE, y_{2t}				
10 000	0.985	2.050	2.964	3.893	4.691	5.366	6.114	6.613
100 000	0.976	2.029	2.937	3.869	4.655	5.335	6.095	6.599
200 000	0.978	2.030	2.938	3.866	4.652	5.332	6.088	6.593
500 000	0.976	2.029	2.937	3.866	4.653	5.331	6.088	6.594
m				Det				
10 000	1.734	4.939	8.879	13.941	18.271	22.870	28.103	32.263
100 000	1.716	4.882	8.802	13.801	18.075	22.729	28.057	32.224
200 000	1.719	4.881	8.802	13.784	18.082	22.700	28.025	32.219
500 000	1.716	4.881	8.801	13.795	18.066	22.701	28.045	32.224

Notes: See the notes to Table 3.

Table 5: Relative mean-squared forecast errors (MSFEs) of the VAR(1,1) and VAR(0,2) models described in Table 1 relative to the Gaussian causal VAR(2) model.

Model	Horizon							
	1	2	3	4	5	6	7	8
				MSFE, y_{1t}				
VAR(1,1), IS	0.987	0.990	0.992	0.986	0.985	0.989	0.989	0.988
VAR(1,1)	0.971	0.986	0.987	0.983	0.982	0.985	0.987	0.986
VAR(0,2)	0.964	0.985	0.987	0.991	0.997	0.997	0.995	0.999
				MSFE, y_{2t}				
VAR(1,1), IS	1.002	1.005	1.001	1.003	1.002	0.999	0.996	0.997
VAR(1,1)	0.999	1.000	0.999	1.000	0.999	0.997	0.996	0.995
VAR(0,2)	1.000	1.003	1.004	1.004	1.002	0.996	0.996	0.997
				Det				
VAR(1,1), IS	0.990	0.994	0.995	0.989	0.987	0.987	0.985	0.986
VAR(1,1)	0.970	0.986	0.986	0.983	0.979	0.980	0.982	0.981
VAR(0,2)	0.966	0.989	0.993	0.996	1.001	0.995	0.992	0.997

Notes: See the notes to Tables 2–4. Entries below unity indicate the superiority of the noncausal models. IS refers to importance-sampling-based forecasts. The number of simulated realizations is m=200 000.

Table 6: Relative mean-squared forecast errors (MSFEs) of the VAR(1,1) model obtained with importance sampling and incorrectly assuming the nonsingularity of the matrix N_0.

	Horizon							
	1	2	3	4	5	6	7	8
MSFE, y_{1t}	0.996	0.993	0.995	0.996	0.996	0.995	0.997	0.999
MSFE, y_{2t}	1.003	1.003	0.998	0.996	0.997	0.997	0.995	0.997
Det	0.999	0.998	0.995	0.994	0.995	0.994	0.995	0.996

Notes: The values of the autoregressive coefficients are given in Footnote 1. The error term has a t-distribution with covariance matrix $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ and the value of the degree-of-freedom parameter 5.00. The entries above unity indicate larger MSFEs for importance-sampling-based forecasts. The number of simulated realizations is m=200 000.

Table 7: Mean-squared forecast errors (MSFEs) of the second-order causal and noncausal VAR(r, s) models for the U.S. inflation and marginal cost data.

Model	Forecast horizon (quarters)							
	1	2	3	4	5	6	7	8
				MSFE, inflation				
VAR(2)-N	1.073	1.426	1.694	2.075	2.769	3.379	3.969	4.387
VAR(2)-t	1.080	1.455	1.756	2.168	2.908	3.554	4.209	4.655
VAR(1,1), IS	1.068	1.373	1.499	1.777	2.365	2.800	3.188	3.436
VAR(1,1)	1.066	1.371	1.518	1.789	2.371	2.817	3.216	3.464
VAR(0,2)	1.077	1.368	1.675	2.123	2.806	3.372	3.882	4.290
				MSFE, marginal cost				
VAR(2)-N	0.838	1.346	2.210	3.053	4.414	5.921	7.463	9.286
VAR(2)-t	0.849	1.351	2.234	3.106	4.518	6.103	7.744	9.698
VAR(1,1), IS	0.849	1.384	2.319	3.223	4.622	6.173	7.675	9.482
VAR(1,1)	0.844	1.383	2.316	3.218	4.631	6.170	7.686	9.491
VAR(0,2)	0.831	1.397	2.335	3.259	4.675	6.248	7.804	9.607
				Det				
VAR(2)-N	0.887	1.779	2.984	4.731	8.709	13.358	18.990	24.595
VAR(2)-t	0.904	1.817	3.088	4.916	9.113	13.960	20.026	25.856
VAR(1,1), IS	0.902	1.779	2.829	4.302	7.877	11.650	16.062	20.435
VAR(1,1)	0.896	1.771	2.868	4.290	7.869	11.682	16.190	20.542
VAR(0,2)	0.883	1.765	3.066	5.008	9.039	13.580	19.075	24.634

Notes: The entries are the MSFEs and determinants of the MSFE matrices of causal VAR(2) and noncausal VAR(1,1) and VAR(0,2) models. N and t denote Gaussian and t-distributed errors, respectively. IS refers to importance-sampling-based forecasts. The number of simulated realizations is m=200 000.

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