BANK OF FINLAND DISCUSSION PAPERS

8 • 2001

Tuomas Välimäki
Research Department

23.5.2001

Fixed rate tenders and the overnight money market equilibrium

Suomen Pankin keskustelualoitteita
Finlands Banks diskussionsunderlag

Suomen Pankki
Bank of Finland
P.O.Box 160, FIN-00101 HELSINKI, Finland
☎ + 358 9 1831

BANK OF FINLAND
DISCUSSION PAPERS

8 • 2001

Tuomas Välimäki
Research Department
23.5.2001

Fixed rate tenders and the overnight money market equilibrium

I am grateful to Jouko Vilmunen for encouraging and helpful comments. Also the comments from David Mayes, Juha Tarkka, Tuomas Saarenheimo, Matti Virén, and Jarmo Kontulainen are gratefully acknowledged.

Suomen Pankin keskustelualoitteita
Finlands Banks diskussionsunderlag

ISBN 951-686-717-0
ISSN 0785-3572
(print)

ISBN 951-686-718-9
ISSN 1456-6184
(online)

Suomen Pankin monistuskeskus
Helsinki 2001

Fixed rate tenders and the overnight money market equilibrium

Bank of Finland Discussion Papers 8/2001

Tuomas Välimäki
Research Department

Abstract

This paper presents a general equilibrium model of the determination of equilibrium in the interbank market for overnight liquidity when the central bank uses fixed rate tenders in its liquidity provision. We consider three alternative liquidity policy rules. First, the central bank may provide the bid amounts in full. Alternatively, the central bank can scale back the bid amounts pro rata with the individual bids. For the latter case, we consider two target options for the central bank: liquidity or an interest rate. We show that the expected overnight rate remains more tightly in the hands of the central bank if the full allotment procedure or a pure interest rate targeting rule is used than if liquidity targeting is used. We will also demonstrate how optimal bidding in tender operations varies considerably according to which procedure is chosen by the central bank.

Key words: money market tenders, overnight rate of interest, averaging, central bank operational framework

Kiinteäkorkoiset huutokaupat ja yön yli -rahamarkkinoiden tasapaino

Suomen Pankin keskustelualoitteita 8/2001

Tuomas Välimäki
Tutkimusosasto

Tiivistelmä

Tutkimuksessa mallinnetaan pankkien välisten yön yli -likviditeettimarkkinoiden tasapainon määräytyminen, kun keskuspankki käyttää kiinteäkorkoisia huutokauppoja jakaessaan likviditeettiä näille markkinoille. Markkinoiden toimintaa tarkastellaan kolmen vaihtoehtoisen politiikkasäännön vallitessa. Ensimmäisessä tapauksessa keskuspankki hyväksyy kaikki pankkien tekemät tarjoukset täysimääräisesti. Muissa kahdessa tapauksessa keskuspankki rajoittaa likviditeetin tarjontaa tähdäten joko likviditeetti- tai korkotavoitteeseen, mutta jakaa likviditeettiä pankeille suhteessa niiltä saamiensa tarjousten kokoon. Tutkimus osoittaa, että koron ohjaus on tehokkainta, jos huutokauppamenettely perustuu tarjousten täysimääräiseen hyväksymiseen tai jos sovelletaan korkotavoitetta. Likviditeettisääntöä sovellettaessa koron kontrolli on tätä löyhempi. Lisäksi osoitetaan, miten pankkien tarjouskäyttäytyminen riippuu merkittävästi keskuspankin käyttämästä politiikkasäännöstä.

Asiasanat: rahamarkkinahuutokaupat, korot, vähimmäisvarantojen keskiarvoistaminen, keskuspankin toimintakehikko

Contents

1 Introduction

The overnight rate of interest is probably not the most important rate in monetary policy transmission. However, the importance of the interbank overnight market should not be understated, as it is where the central bank operates in order to implement monetary policy, and also because overnight is normally the shortest maturity for which there are well organized markets. (Hence, the yield curve can be seen reflecting the expected future values of the overnight rate.) Therefore, understanding the functioning of the monetary policy operational framework (ie the instruments and procedures of the central bank) is essential in order to be able to evaluate the monetary policy stance, and to interpret the reasons for and consequences of variations in the conditions on the overnight market. For example, in one framework a change in the overnight rate of interest can be seen indicating a change in the tightness of monetary policy, whereas in another framework changes in the overnight rate may always originate from stochastic liquidity shocks, thus bearing no information value at all.

The literature on the overnight markets is heavily concentrated on describing and explaining the stylized facts of the Fed Funds market, ie the market for interbank overnight reserves in United States. For example, according to Hamilton (1996) the observed cyclical behaviour of the Fed Funds rate may result from line limits, transaction costs and weekend accounting conventions. Also Furfine (1998) shows that the intra-maintenance period variations in Fed Funds rate are a consequence of volatility in daily interbank payment volumes. However, an exception from this (US-style) line is Perez-Quiros and Rodriguez (2000), who model the behaviour of overnight funds in the Euroarea, by using a framework, that includes standing facilities.[1] They claim that the introduction of a deposit facility stabilizes the overnight rate of interest. That paper, like most papers analyzing the Fed Funds market, abstracts from monetary policy. This means, that in the standard literature the analysis is only partial; liquidity is exogenous and there is no role for active liquidity management by the central bank. A notable exception is Bartolini, Bertola & Prati (1999), who model the interbank money markets by giving explicit role for central bank intervention. Their model is, however, not very suitable for studying the behaviour of overnight markets in Europe, where the operational framework in most countries during resent years has included standing facilities. Especially as the open market operations have been conducted in the form of *fixed rate tenders*. When using fixed rate tenders, the central bank not only provides the

[1] While finalizing this discussion paper, the author became aware of other studies on the operating framework of the European Central Bank and the euro money market, including Ayuso & Repullo (2000) and Ehrhart (2000). The paper by Ehrhart presents an experimental investigation into the bidding behaviour of the banks under a fixed rate tender procedure and exogenous (tight) supply of liquidity. Ehrhart claims that fixed rate tenders lead to continuously increasing overbidding in the operations. The model used in Ehrhart's paper, however, abstarcts from the interbank overnight market as the place where the value of liquidity is determined. The paper by Repullo & Ayuso shows that if a central bank has an asymmetric loss function that depends on the quadratic difference between the interbank rate and a target rate of the central bank, fixed rate tenders have a unique equilibrium characterized by extreme overbidding.

markets with liquidity, but it also signals the stance of monetary policy with the interest rate set in the operations. Thus, it is important to understand how the allotment procedure used by the central bank affects both the demand for liquidity in tenders and the amount of liquidity prevailing in the overnight markets.

In this paper we present a general equilibrium model of the behaviour of overnight markets, where the central bank manages liquidity by fixed rate tenders. First, we model the determination of the overnight rate of interest as a function of the money market liquidity. Then, we will analyze how the money market liquidity itself is determined under various allotment rules used by the central bank. The central weight in the analysis is put on comparing the money market equilibrium when the central bank accepts all the bids it receives in the tender in full (*full allotment procedure*) with the equilibrium when the central bank scales back the bids it receives (*proportional allotment procedure*). The operational framework according to which the central bank is here assumed to implement its monetary policy largely resembles that of the one used by the ECB between January 1999 and June 2000. However, for simplicity we have assumed here that the central bank conducts one operation each day whose maturity is overnight. The consequences of these simplifications are discussed briefly in the conclusions of the paper.

The paper is organized as follows. In section 2 we describe the functioning of the money markets, and the role of the central bank in liquidity management. Section 3 analyses the determination of the overnight rate, when the operational framework does not include averaged reserve requirement scheme. In section 4 we will introduce the dynamics that comes with averaging provision. Finally, section 5 concludes and gives a summary on the main findings of the paper.

2 Money markets and the central bank

By money markets we refer to the market where institutions enter into transactions with each other by trading unsecured debt, negotiable debt instruments or collateralized loans. For simplicity, we abstract from the fact that the interest rates of different instruments carry varying premia over the risk free yield curve. Thus, when referring to a market rate of interest for a specific maturity, we assume that this kind of unique risk free interest rate for all relevant maturities exist.

The notions of money market liquidity and bank reserves are used interchangeably throughout this paper. They both refer to the balances banks have on their settlement accounts with the central bank. By interbank trading we refer to money market deals between credit institutions that have access to the central bank operations. It is worth emphasizing that even though interbank trades redistribute the money market liquidity between the banks, they do not affect the total amount of liquidity prevailing in the market. Only transactions, that also involve the central bank, change the (aggregate) money market liquidity.

It is assumed here that one day (overnight) is the shortest maturity in the organized interbank market.[2] Thus, it is also the starting point of the yield curve. In overnight trading the value date of the transaction is the trading day (same day settlement) and the maturity date is the following banking day (ie normally the maturity of an overnight loan is three days on Fridays). We assume that normal interbank trading with instruments of longer maturities are settled with a lag of at least one banking day. Hence, the only way a bank can offset the liquidity shocks it faces is through trading in the overnight market. These shocks may stem from unexpected deposit withdrawals, new deposits or from any other unanticipated transaction with same day settlement.

Central bank objectives and its operational framework

We ignore whether the central bank uses monetary targeting, direct inflation targeting or any other procedure as an indicator or intermediate target in achieving price stability (or any other primary goal it might have). We merely assume that the central bank uses a short-term interest rate as a policy rate or as an operating target.[3] However, the maturity of this rate does not need to be overnight.

In this study, we want to model the determination of the overnight rate and especially how the determination of its expected value is affected by the operational target of the central bank. The formation of the expected overnight rate is of special interest, as it is the *expected values* of the rate that are the basis for the determination of the yield curve, and it is the rates considerably longer than overnight, that normally are assumed to be important for the transmission of monetary policy. Therefore, as we expect the operational framework to affect the expected values of the overnight rate, we also expect these operational issues to affect the transmission mechanism of the monetary policy. For example, it is obvious that transmission of the overnight rate's volatility along the yield curve to longer maturities depends crucially on whether the variations in the overnight rate affect the expected values of future overnight rates.[4]

We assume here, that the operational framework of the central bank contains three different (ECB style[5]) instruments, that can be used to meet the operational target. These are: *i) open market operations* by which the central bank actively manages the money market liquidity, *ii)* the interest rate corridor

[2]There may also be interbank markets for intraday liquidity. However, we are not interested on such markets for the purposes of this study, as intraday trades do not affect the end-of-day balances on the banks' settlement accounts.

[3]This is currently the case at least in the Euroarea, USA and Japan. See for example European Central Bank (2000), Federal Reserve Bank of new York (2000) and Bank of Japan (1999) respectively.

[4]See for example Ayuso, Haldane and Restoy (1994) who claim that the differences in the transmission of volatility from the overnight market to longer money markets in Spain, France, UK and Germany resulted primarily from differences in the operational frameworks of these countries.

[5]The operational framework of the ECB is described in detail in ECB publication "The Single Monetary Policy in Stage Three; General Documentation on ESCB Monetary Policy Instruments and Procedures", September 1998. From now on we refer to this document simply as GD.

set by the *standing facilities (ie the marginal lending facility and the deposit facility)*. The use of the standing facilities can be initiated by banks. Furthermore, the central bank can affect the demand for money market liquidity by *iii)* imposing *reserve requirements*.

Throughout this paper, we assume that the active liquidity management (open market operations) is conducted solely by fixed rate money market tenders. The ECB conducted its open market operations in this style between January 1999 and June 2000. One purpose of these tender operations is to provide the banks with refinancing. However, an at least equally important function of these operations is their role in signalling the monetary policy stance of the central bank.[6]

Besides the open market operations, the liquidity conditions in the interbank trading are affected by the standing facilities. The marginal lending rate sets an upper limit (ie the ceiling) for the secured interbank overnight rate. The central bank is always willing to provide additional liquidity at this pre-specified interest rate against eligible collateral. Thus, no bank is willing to pay more than the marginal lending rate for reserves on the interbank market. The lower limit (ie the floor) for the overnight rate is set by the rate of the deposit facility. The banks are allowed to place overnight deposits with the central bank at this pre-specified interest rate. Hence, the interest rates of the standing facilities effectively create a corridor in which the interbank overnight rate of interest may fluctuate (*the interest rate corridor*). The central bank can affect the volatility of the overnight rate eg by the width of the corridor. The central bank may also use the rates of the standing facilities in signalling the expected future stance of the monetary policy.

When the open market operations are conducted in such a fashion that they do not affect the trading day's interbank overnight liquidity (eg if transactions are settled at $T{+}1$), the supply of overnight liquidity on a specific day is fixed as long as the overnight rate stays within the corridor (ie as long as the price of borrowing liquidity from the market is cheaper than the marginal lending rate and the revenue from an interbank loan is above the deposit rate).[7] However, the supply of liquidity is affected by stochastic shocks that can be anticipated by neither the banks nor the central bank. The size of the liquidity shock for the central bank may be different from the sum of the net shocks the banks face. The shock for an individual bank is the net sum of unexpected flows into and out off its reserve account (or actually the difference between this amount and its forecast value). The liquidity shock from the central bank's viewpoint is the deviation of the net changes in the autonomous liquidity factors from their expected value. This shock might include (depending on the institutional set up of the currency area of the central bank) eg unexpected variations in the government balances with the central bank, changes in the amount of currency in circulation etc.

The third instrument at the disposal of the central bank is the reserve requirement (in case of ECB the so called minimum reserve requirement). The central bank can require the credit institutions to hold a share of their liabilities

[6] See eg the GD page 4 or European Central Bank (2000) page 49.

[7] A procedure of same day settlement could be allowed in our model without qualitatively changing the results.

as minimum reserves with the central bank. These reserves are assumed to be held in the settlement accounts of the banks. Averaging provisions may be allowed in the maintenance of the minimum reserves. If averaging is used, the compliance with the reserve requirement is judged by the average of the end-of-day balances an institution has on its reserve (settlement) account during the maintenance period.

In addition to these three instruments, the operational framework contains a crucial additional feature; overdrafts are forbidden. This means that, if a bank would otherwise be ending the day with a debit balance on its settlement account, it must cover the negative balance by borrowing from the marginal lending facility. Thus, both the aggregate end-of-day liquidity of the banking sector as a whole and the end-of-day liquidity of a single bank must always be at least zero.

The evolution of money market liquidity during a banking day

Trading in the overnight market is conducted throughout the day. Thus, in the real world there is no single overnight rate for any specific day. However, we assume here, that the overnight trading is conducted by a Walrasian auctioneer at a certain point of time during the trading day. The clearing rate used by the auctioneer is assumed to equal the weighted (by volume) average of interest rates used in the interbank overnight trading during that day. Normally such a rate is calculated by the central bank or some other institution as a reference rate for the markets.[8]

Furthermore, we will assume that there will be two distinct and independently distributed liquidity shocks for the banks during a banking day.[9] The first shock (μ) is realized before the overnight markets are cleared, and the second one (ε) realizes after the settlement. The expected value for both of these shocks is zero ($E[\mu]=E[\varepsilon]=0$). The construction of two separate shocks follows from the fact, that the aggregate net shock a bank faces consists of a continuum of small independent shocks occurring throughout the day. Now, when we model the overnight market as being cleared at a single point in the day, μ is the net effect of all shocks before that moment, and ε is the net effect of the shocks occurring after the clearance of the markets. If the overnight markets were modelled as being settled at the end of the day, ie there would be only one shock per day, there would not be any uncertainty about the end-of-day reserve balances of the banks in the overnight trading. Thus, the overnight rate would equal either the marginal lending rate or the deposit rate (depending on whether there is a shortage or a surplus in the overnight market) on the last day of the reserves maintenance period (on every day, if an averaging scheme was not applied). By the construction with two separate shocks, we will ensure that the uncertainty about each bank's reserve position will make the bank's demand schedule for reserves smoothly downward sloping.

[8] In case of the the Eurosystem, ECB calculates EONIA, which is a volume weighted average interest rate of interbank overnight deposits reported by certain panel banks. In the USA the Fed Funds Rate is the counterpart of EONIA in Europe.

[9] A similar division of the liquidity shock into two parts can also be found in Bartolini et al (1998).

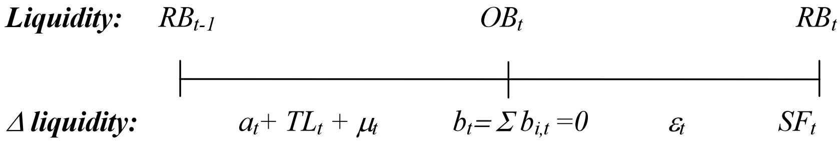

Figure 1: The evolution of money market liquidity during the day

At the beginning of a banking day t the money market liquidity equals the previous banking day's aggregate reserve balances ($RB_{t-1} = \sum RB_{i,t-1}$). Each bank knows its own balance, and the total amount is known to the central bank. The new and maturing monetary policy transactions, whose value day is t, are also known with certainty by the central bank.

Figure 1 shows the timing of the evolution of the money market liquidity on a single banking day. a_t is the sum of expected net changes in autonomous liquidity factors (including maturing central bank operations), TL_t is the amount of liquidity provided to the markets in the open market operation (tendered liquidity), and μ_t is the first liquidity shock of the day. OB_t is the amount of liquidity on the markets when the overnight markets clear (OB_t=$RB_{t-1} + a_t + TL_t + \mu_t$). We will denote its expected value by eOB_t (ie $eOB_t = RB_{t-1} + a_t + TL_t$). $b_{i,t}$ is the net amount bank i borrows from interbank markets. Lending reserves to the markets is treated as negative borrowing. The banks' aggregate net borrowing from the markets must equal zero, as in every deal there is a borrower and a lender for the same amount of reserves (ie $b_t = \sum b_{i,t} = 0$). ε_t is the second liquidity shock of the day.

Let RR_i denote the bank i's reserve requirement per day, and RR its aggregate counterpart. Now, we can define the minimum required daily balances for the remaining period (RDB):

$$RDB_{i,t} = \frac{T * RR_i - \sum_{k=1}^{t-1} RB_{i,k}}{T - (t-1)} \tag{1}$$

Where T is the number of days in the maintenance period in question. Equation 1 gives us the average amount of reserves bank i should have daily (from day t to the end of the on-going maintenance period) on its reserve balances in order to hit the reserve requirement exactly. We will denote the banking sector wide counterpart of $RDB_{i,t}$ by $RDB_t(= \sum RDB_{i,t})$. In a system without reserve averaging scheme $RDB_{i,t}$ and RDB_t will always equal RR_i and RR respectively.

Now, SF_t denotes the banks' net use of the standing facilities (ie liquidity credits - use of the deposit facility). Apart from the fact that a bank can use the standing facilities whenever it wants to, a bank should always acquire liquidity credits (LC) from the marginal lending facility, if its end-of-day reserve balances would otherwise be negative or if at the last day of the maintenance period its reserve balances would not be large enough to meet the reserve requirement (ie $LC_{i,t} = \max\ (0, -(OB_{i,t} + \varepsilon_{i,t}))$ for $t < T$ and $LC_{i,T} = \max\ (0, -(OB_{i,T} + \varepsilon_{i,T}), -(OB_{i,T} + \varepsilon_{i,T} - RDB_{i,T})))$.[10] Similarly, a rational bank will always deposit all reserves exceeding the reserve requirement into the deposit facility (ie $DF_{i,t} = \max(0, OB_{i,t} + \varepsilon_{i,t} - (T-(t-1)) * RDB_{i,t}))$ as otherwise these reserves will earn zero interest.

Finally, the reserve balances (ie the aggregate end-of-day balances) are denoted by RB_t, which is the sum of all factors included in OB_t, the second liquidity shock and the net use of the standing facilities ($RB_t = OB_t + \varepsilon_t + LC_{i,T} - DF_{i,T}$). Now, based on the optimal use of the standing facilities, we know that $RB_t \in [0, (T-(t-1)\,RDB_t)]$, and consequently the required daily balances for the remaining period can never be negative (ie $RDB_t \geq 0$).[11]

Let us add one more definition into the liquidity terminology. The excess reserves of bank i ($ER_{i,t}$) is the amount of reserves it has over the required daily balances *at the moment of overnight market clearance* ($ER_{i,t} = OB_{i,t} - RDB_{i,t}$), and its expected value will be denoted by eER ($eER_{i,t} = eOB_{i,t} - RDB_{i,t}$).

There are two different factors behind the demand for bank reserves. First, as long as overdrafts are forbidden, the banks can not have debit balances on their settlement accounts with the central bank at the end of the day. The uncertainty introduced by the assumption of a liquidity shock occurring after the overnight markets have cleared will ensure that the demand for reserves depends negatively on the interest rate (otherwise the demand for reserves could be a step function, in which case the overnight rate would equal either the marginal lending rate or the deposit rate depending on whether the liquidity is below or above zero or the reserve requirement).

Another factor behind the demand for reserves is the minimum reserve requirement (if imposed by the central bank). If the requirement is not averaged, the only thing it will add is that the cost minimizing end-of-day liquidity for a bank is the required amount instead of zero. However, if the requirement is averaged the story changes completely. In this case the demand for reserves will be similar to the case without reserve requirements only on the last day of the reserves maintenance period. Before the last banking day, an optimizing bank can have reserves in excess to the requirement or less than the requirement is. Only the average amount of reserves held with the central bank counts. Using averaging provision a bank can optimize on the cost of holding reserves by maintaining them whenever it assumes the cost of it to be the lowest during

[10] If a bank fails to get its reserve account balances to zero (or on the last day of the reserves maintenance period to the required level), the bank will face penalties that are considerably heavier than the cost of using the marginal lending facility. Thus, a rational bank will always acquire liquidity credits when facing either of these situations.

[11] $RB_t = OB_t + \varepsilon_t + LC_T - DF_T = OB_t + \varepsilon_t + \max\ (0, -(OB_t + \varepsilon_t)) - \max(0, OB_t + \varepsilon_t - (T-(t-1)) * RDB_t)$. Thus, if $OB_t + \varepsilon_t < 0$, then $RB_t = 0$, and if $OB_t + \varepsilon_t > 0$, the maximum value for RB_t is $(T-(t-1)) * RDB_t$. Now, RDB_{t+1} is minimized with the maximum value for RB_t. Therefore, $RDB_{t+1}^{\min} = \frac{T*RR - \sum_{k=1}^{t-1} RB_k - (T-(t-1))RDB_t}{T-(t-1)-1} = \frac{T*RR - \sum_{k=1}^{t-1} RB_k - \left(T*RR - \sum_{k=1}^{t-1} RB_k\right)}{T-t} = 0$. So, $RDB_{t+1}$is always at least zero, and we know that $RDB_1 = RR \geq 0$, hence, RDB_t can not be negative.

that averaging period.[12] A bank will demand less (more) reserves on a single day, if the overnight rate on that day is high (low) relative to the rate it assumes to prevail on the following days during the same period.[13] Thus, an averaging provision will enhance the interest rate elasticity of the demand for bank reserves, or to put it the other way around, the changes in the interest rate due to temporary liquidity shocks are smaller because of the averaging, *ceteris paribus*.

Next we set out a simple model on the determination of overnight rate of interest as a function of money market liquidity. After that we model the supply of liquidity as a function of banks' interest rate expectations and the central bank's operational target. To simplify the calculations that follow, we assume the banks to be homogeneous and their mass to sum up to unity. The central bank is assumed to operate only by fixed rate liquidity tenders. This is how many European central banks used to operate in the 1990's, and also how the ECB conducted its main refinancing operations during the first 18 months.

3 No (averaged) reserve requirements

3.1 Overnight rate as a function of liquidity

Let us first consider the demand for bank reserves in a system without reserve requirements. This is also the starting point for analysis of the demand for money market liquidity in an operational framework that includes an averaged reserve requirement. When modelling the demand for reserves in the overnight market, we follow the classical model introduced by Poole (1968), and frequently used by others (eg Bartolini, Bertola & Prati (1999)). The main difference of our model from that of Poole's is the introduction of the rates of the standing facilities.

Bank i's demand for overnight reserves in the interbank market can be achieved as the first order condition of the bank's profit maximizing problem. The cost of borrowing reserves (the income from lending) that bank i faces, is simply the overnight rate of interest at day T (r_T^{on}). The income from the borrowing (the cost of lending) is the interest rate of the two standing facilities (r_T^m and r_T^d) weighted by their usage.[14] The maximization problem becomes:

[12] This is some times referred to as *intertemporal arbitrage*. However, the word arbitrage may be slightly misleading, as the gain from this kind of behaviour is uncertain.

[13] If a bank holds more reserves at the beginning of a maintenance period than at the end of it, it is said to be *frontloading* reserves. In the opposite case the bank is *backloading* reserves.

[14] For the rest of this section we will drop the time subscripts (T) from the interest rates.

$$\begin{aligned}\max_{b_{i,T}} \mathrm{E}(\Pi) \quad = \quad & r^m \left[\int\limits_{-\infty}^{-ER_{i,T}-b_{i,T}} (ER_{i,T} + b_{i,T} + \varepsilon_T) f(\varepsilon_T) d\varepsilon_T \right] \\ & + r^d \left[\int\limits_{-ER_{i,T}-b_{i,T}}^{\infty} (ER_{i,T} + b_{i,T} + \varepsilon_T) f(\varepsilon_T) d\varepsilon_T \right] \\ & - r^{on} b_{i,T} \end{aligned} \tag{2}$$

Where $f(\varepsilon_T)$ is the distribution function of the stochastic error term, whose cumulative counterpart we denote by $F(\varepsilon_T)$. The first order condition with respect to interbank borrowing can be derived by using Leibniz's formula:

$$(r^m - r^d) F\left(-ER_{i,T} - b^*_{i,T}\right) + (r^d - r^{on}) = 0 \tag{3}$$

or

$$F\left(-ER_{i,T} - b^*_{i,T}\right) = \frac{r^{on} - r^d}{r^m - r^d}, \tag{4}$$

where $F\left(-ER_{i,T} - b^*_{i,T}\right)$ represents the probability of bank i being overdrawn with the optimal borrowing. We will see immediately that the optimizing bank chooses its net borrowing in order to balance the probability of ending the day with a debit balance with the relative cost of using the standing facilities.

The right hand side of equation (4) shows us the location of the overnight market rate within the interest rate corridor set by the standing facilities. The lower the market rate is within the corridor, the larger is the equilibrium borrowing for bank i. Intuitively this can be interpreted so that when the relative cost of acquiring liquidity credits $(r^m - r^{on})$ decreases compared with the (opportunity) cost from using the deposit facility $(r^{on} - r^d)$, the optimal policy for the bank is to increase its probability of being overdrawn (simultaneously the bank decreases its probability of having to rely on the deposit facility).

If the cumulative distribution function has an inverse function $(F^{-1}(\cdot))$, we can derive explicit form for bank i's borrowing function:

$$b^*_{i,T}\left(-ER_{i,T}, r^{on}\right) = -ER_{i,T} - F^{-1}\left(\frac{r^{on} - r^d}{r^m - r^d}\right) \tag{5}$$

Equation (5) shows us clearly that the optimal net borrowing equals the excess reserves (ie the gap between existing reserves and required reserves; $-ER_{i,T}$) and the inverse of the probability of a liquidity shock leaving the bank with negative end-of-day balances, given the location of the market rate within the official corridor. The optimal borrowing naturally decreases with both the excess reserves prevailing before the clearing of overnight markets and the interbank overnight rate (the rates of the two standing facilities are taken as given).

Bank i can act as a borrower or lender in the market. However, as long as the overnight market rate stays strictly inside the corridor, the money market liquidity is constant, as there will not be any transactions with the central bank. As a consequence the aggregate borrowing must be zero. We can get the market-clearing overnight rate of interest from equation (4) by aggregating over the unitary mass of banks ($b^*_{i,T} = b^*_T = 0$ and $-ER_{i,T} = -ER_T$):

$$r^{on} = r^m F(-ER_T) + r^d(1 - F(-ER_T)). \tag{6}$$

Thus, the equilibrium interbank overnight rate of interest is simply the average of the two rates of the standing facilities weighted by the probabilities of the money markets being short of or in excess liquidity. There are three factors determining the overnight rate of interest: i) the interest rate corridor, ie the rates of the standing facilities set by the central bank, ii) the distribution of liquidity shocks after the last open market operation affecting day T's liquidity, and iii) the supply of liquidity with respect to the liquidity need. In our model the rates of the standing facilities are given prior to the overnight trading.[15] If we assume the distribution of the liquidity shocks to be stable, the only varying parameter determining the overnight rate in our model is the supply of (excess) liquidity. Hence, the key questions facing the central bank are, how accurately the daily supply of liquidity is in its control, and what the effects of volatility are in money market liquidity. The answer to the first question depends i) on the central banks ability to forecast both the developments in the autonomous liquidity factors and the banks' aggregate demand for liquidity, and ii) on the central bank's ability to provide the markets with the estimated liquidity need. The effect the overnight volatility has in general depends crucially on how the counterparties interpret movements in the overnight rate as reflecting the monetary policy stance. This depends largely on the procedure the central bank uses in choosing the supplied amount.

To address the question of how the supply of money market liquidity is determined we will model the demand for bank reserves in money market tenders under different set of liquidity policy rules used by the central bank. The bidding behaviour of the banks varies with the approach the central bank has in liquidity allotment.

3.2 Determination of money market liquidity

In case of a framework without averaged reserve requirements, we assume that the central bank conducts one liquidity operation for each day. We will also assume that the structural deficit in the money market is large enough, so that

[15] In case of ECB's framework, a change in the rates of the standing facilities can be effective on the following banking day at the earliest.

the tender operations will always be liquidity providing.[16] In these liquidity increasing fixed rate tenders the central bank announces the rate of interest at which it stands ready to provide the counterparties with liquidity. After an announcement each bank may submit a bid to the central bank specifying the amount of liquidity the bank is willing to borrow at the announced rate. The central bank can accept all the bids it receives in full (*full allotment*) or it can scale the bids back proportionally to the amount bid (*proportional allotment*). Here we assume that the counterparties know in advance whether the central bank is using a full or proportional allotment strategy. If the aggregate bids the central bank receives do not exceed the amount of liquidity the central bank targets to lend, it will provide the markets with all the liquidity bid for even under the proportional allotment procedure; ie 100% acceptance of bids is not always an indication of the full allotment approach. Let us next consider these two methods separately.

3.2.1 Full allotment

Let us start by defining some terminology for the bidding strategies of the banks. First, we define the *private value* of a certain amount of expected excess reserves for a bank as a weighted average of the rates of the standing facilities, where weights are determined by the bank's probability of having to use the standing facilities with this amount of expected excess reserves. A bank has *neutral liquidity*, if its probability weighted cost of relying on the standing facilities equals the tender rate (ie the private value of neutral liquidity equals the tender rate). In *neutral bidding* a bank bids for such an amount that (when accepted in full) will leave the bank with neutral liquidity. The size of a neutral bid ($TL_{i,T}^{neutral}$) is implicitly given by:

$$\begin{aligned} & r^m * G(RDB_{i,T} - RB_{i,T-1} - a_{i,T} - TL_{i,T}^{neutral}) \\ & + r^d(1 - G(RDB_{i,T} - RB_{i,T-1} - a_{i,T} - TL_{i,T}^{neutral})) = r^T, \end{aligned} \quad (7)$$

where $G(\cdot)$ is the cumulative distribution function of the sum of the two stochastic error terms μ_T and ε_T, ie $G(RDB_{i,T} - RB_{i,T-1} - a_{i,T} - TL_{i,T}^{neutral})$ is the probability of bank i being overdrawn after acquiring $TL_{i,T}^{neutral}$ from the liquidity tender. The expected excess reserves before the realization of μ is $eER_T^{neutral} = RDB_{i,T} - RB_{i,T-1} - a_{i,T} - TL_{i,T}^{neutral}$. Now, we can write equation (7) as:

$$r^m G(-eER_{i,T}^{neutral}) + r^d(1 - G(eER_{i,T}^{neutral})) = r^T. \quad (8)$$

Equation (8) just states the fact that the private value of neutral expected excess reserves equals the tender rate. We also know that this is the exact

[16] By large enough we mean here, that the probability of the whole banking sector ending the day with debit balances would be close to one, if no liquidity were provided through tender operation. If this weren't the case, the central bank might sometimes have to use liquidity draining operations instead of liquidity providing ones. This assumption does not otherwise limit the analysis.

amount a bank would bid for in a fixed rate tender with full allotment liquidity provision, if there were no secondary market for liquidity.[17]

Strategic overbidding occurs, if a bank bids for more than the neutral bidding strategy would suggest, in order to profit from the (positive) difference between the tender rate and the banks estimate of the market overnight rate ($r^T <$E$[r^{on}]$). Accordingly, *strategic underbidding* occurs when a bank bids for less liquidity than the neutral strategy would require, to profit from the bank's estimation of a negative difference between the tender rate and market overnight rate ($r^T >$E$[r^{on}]$).

By equation (6) we know that the overnight rate of interest is a function of the (excess) money market liquidity. Thus, in addition to the central bank's allotment tactics, the bidding behaviour of a single bank depends on the bidding strategies of other banks. In equilibrium the bidding strategy of the representative bank must be such that with the equilibrium liquidity the expected overnight rate of interest (ie the price of liquidity at the clearance of the market) will equal the price of liquidity at the tender. This is derived from the fact that, if E$[r^{on}] > r^T$, every (atomistic) bank would make maximum profits by increasing its bid up to the maximum level (or placing an infinitely large bid if there were no maximum bid), and selling the extra liquidity in the overnight market. However, in such a case the total liquidity would be infinitely large or at least large enough to bring the overnight rate down to its minimum value (ie E$[r^{on}] = r^d$), which would contradict the assumption of the expected overnight rate exceeding the tender rate. Also, if E$[r^{on}] < r^T$, every bank would maximize their profits by placing a zero bid (ie not participating the tender at all), and buying the missing liquidity from the overnight market. However, in such a case the total liquidity on the interbank market would be sufficiently low, that the expected value for the overnight rate would rise to the ceiling (ie E$[r^{on}] = r^m > r^T$). *Therefore, the only possible sustainable equilibrium is such that the difference between the expected overnight rate and the tender rate is zero,* $E[r^{on}] = r^T$. In such a case, no bank can make positive expected profits by changing its bid. We also know by equation (6) that the overnight rate is a decreasing function of the money market liquidity, which includes the tendered reserves. Thus, *there can be only one level of expected liquidity that can be sustained as an equilibrium.*

When all banks are bidding according to the neutral strategy, the overnight rate will become:

$$\begin{aligned} r^{on} &= r^m F(RDB_T - RB_{T-1} - a_T - TL_T^{neutral} - \mu_T) \\ &\quad + r^d(1 - F(RDB_T - RB_{T-1} - a_T - TL_T^{neutral} - \mu_T)). \end{aligned} \tag{10}$$

[17] If there were no interbank market for central bank reserves, the profit maximization problem of bank i at the liqudity auction would be very similar to that of decribed in equation (2). In this case the maximization would be taken w.r.t $TL_{i,T}$ instead of $b_{i,t}$, $(ER_{i,T} + \varepsilon_T)$ should be replaced by $(eER_{i,T} + \nu_T)$ and $f(\varepsilon_T)d\varepsilon_T$ by $g(\nu_T)d\nu_T$. Thus, the FOC would become:

$$r^m G\left(-eER_{i,T}^*\right) + r^d\left(1 - G\left(-eER_{i,T}^*\right)\right) = r^T. \tag{9}$$

Now, the optimal expected excess reserves, $-eER_{i,T}^*$, (defined implicitly in equation (9)) just equals the neutral expected excess reserves in equation (8).

Now, it can be shown, that (at the time of the tender operation) the expectation of the cumulative distribution function of the second shock (expectation taken over the distribution of μ) will equal the cumulative distribution function of the sum of the two independent shocks (ie $\mathrm{E}_{f\mu}[F(\cdot)] = \mathrm{G}(\cdot)$; in the following we will denote $\mathrm{E}_{f\mu}[F(\cdot)]$ simply by $\mathrm{E}[F(\cdot)]$ to shorten the notation).[18] Thus, with neutral bidding, the expected value of the overnight rate is given by:

$$\begin{aligned} & \mathrm{E}[r^{on}] = r^m \mathrm{E}\left[F(RDB_T - RB_{T-1} - a_T - TL_T^{neutral} - \mu_T)\right] \\ & +r^d \left\{1 - \mathrm{E}\left[F(RDB_T - RB_{T-1} - a_T - TL_T^{neutral} - \mu_T)\right]\right\} \\ = \; & r^m \mathrm{E}\left[F(-ER_T^{neutral})\right] + r^d \left\{1 - \mathrm{E}\left[F(-ER_T^{neutral})\right]\right\} \\ = \; & r^m G(-eER_T^{neutral}) + r^d \left[1 - G(-eER_T^{neutral})\right] = r_T^T. \end{aligned} \tag{11}$$

Equation (11) tells us, that the expected value of the overnight rate, when the banks use neutral bidding, equals the tender rate. Thus, *in the only sustainable equilibrium the total bids must equal the neutral bidding strategy for the banks* ($TL_T^{neutral} = TL_T^*$).[19]

Furthermore, from equation (11) we have:

$$\mathrm{E}\left[F(-ER_T^*)\right] = G(-eER_T^*) = \frac{r_T^T - r_T^d}{r_T^m - r_T^d}, \tag{12}$$

which defines the equilibrium borrowing implicitly as a function of the interest rates used by the central bank. We can see that equilibrium bidding will leave the money market with liquidity that equates the probability of it being overdrawn with the relative location of *the tender rate* within the interest rate corridor.

Now, if the cumulative distribution function $G(\cdot)$ has the inverse function $G^{-1}(\cdot)$, we can derive the explicit form for the equilibrium bidding:

$$eER_T^* = -G^{-1}\left(\frac{r_T^T - r_T^d}{r_T^m - r_T^d}\right) \Leftrightarrow TL_T^* = LG_T - G^{-1}\left(\frac{r_T^T - r_T^d}{r_T^m - r_T^d}\right), \tag{13}$$

[18] Let $\nu = \mu + \varepsilon$. It can be shown, that $G(\nu) = \int F_\varepsilon(\nu - \mu) f_\mu(\mu) d\mu$. Where f_ε, f_μ, F_ε, and F_μ refer to distributions and cumulative distributions of the error terms ε and μ respectively. By the definition of expectation, we have $G(\nu) = \mathrm{E}_{f_\mu}[F_\varepsilon(\nu - \mu)]$, where the expectation is taken over the distribution of μ. The proof of this is given in technical appendix A.

[19] Note, that the unique equilibrium we have derived here does not contain any information on how the liquidity is distributed among the banks in the tender. From the point of view of a single *atomistic* bank (which takes the total money market liquidity as given), every bid size will lead to zero expected profits, as long as the expected overnight rate equals the tender rate. If one would like to restrict the number of possible distributions of the tendered liquidity, one possibility would be to impose an extra assumption, according to which there is positive probability (that could be infinitesimal) that a bank can't enter the interbank market on that day. In such a case there would be a unique equilibrium for each individual bank, in which each bank will bid its neutral liquidity. The reason is that as long as the bank can enter the interbank market any bid is equally good for the bank, but in the infinitesimally probable case that it is not able to enter into transactions with other banks, it is optimal to bid according to the neutral strategy (as with neutral bidding the private value for the liquidity the bank bid for, equals the tender rate). Similar result (the uniqueness in a single bank's bidding) could also be derived by introducing a fixed cost that banks will face when they enter the interbank market.

where LG_T is the (estimated) liquidity gap between required daily balances and the sum of morning balances and the autonomous liquidity factors ($LG_T = RDB_T - RB_{T-1} - a_T$). In this model the amount of bank reserves demanded at the tender will depend on the liquidity gap, expected overnight rate (ie tender rate), the rates of the two standing facilities and the distribution of liquidity shocks. The central bank sets the expectation over the market overnight rate of interest indirectly by the tender rate, and the rates of the standing facilities are also announced directly by the central bank. Thus, if the shock distribution is taken as given, the central bank is able to determine the demand for reserves (thus the expected money market liquidity) by choosing the location of the tender rate within the interest rate corridor.

Note that, if the shock distribution is symmetric and the central bank applies a symmetric interest rate corridor (ie $\frac{r_T^T - r_T^d}{r_T^m - r_T^d} = \frac{1}{2}$), the equilibrium expected amount of excess reserves will be zero (as $G(0) = \frac{1}{2}$ for symmetric shock distribution). If the shock distribution is skewed to the left (*right*), the banks will (on average) have positive (*negative*) excess reserves when the interest rate corridor is symmetric.

The actual overnight rate on a particular day will deviate from its expected value (the tender rate) because of the (net) liquidity shocks occurring between the allotment of the tender operation and the clearing of the interbank overnight markets (μ). However, *the variation in the actual overnight rate does not contain any information on the stance of the monetary policy.* It is merely a consequence of the sum of net errors made by the banks in estimating their need for liquidity in tender operations. Hence, the volatility should not be transmitted to longer-term interest rates (interest rates that are more relevant in the monetary policy transmission).

We further clarify the determination of the overnight rate and the relevance of the two liquidity shocks by figure 2. In drawing this figure, we have assumed for clarity that both shocks are normally distributed, and the interest rate corridor is symmetric around the tender rate.[20]

In figure 2, S^T is the perfectly elastic supply of tender reserves, and D^T denotes the demand for reserves during the operation (given by equation (13), or if $G^{-1}(\cdot)$ does not exist, implicitly given by equation (12)). The equilibrium amount of reserves expected to prevail at the clearance of the overnight market (eOB_T) is given by the equality of the demand and supply (point a). The expected (equilibrium) value of the overnight rate is r^T, and in case of symmetric corridor, the equilibrium (ie the expected level of) excess reserves is 0.

The equilibrium liquidity gives us the location of the inelastic part of $\mathrm{E}\left[S^{o/n}\right]$, the expected supply of liquidity at the overnight market.[21] The true supply of liquidity after two alternative realizations of the first shock (μ^+, and μ^-) is given by the two dashed lines. The demand for liquidity at the clearance of the overnight market is denoted by $D^{o/n}$. With liquidity close to the ex-

[20] Demand functions in all figures in this paper are based on the assumption of normally distributed liquidity shocks.

[21] The perfectly elastic parts of overnight supplies are naturally at the level of the rates of the standing facilities, as the banks can get all the liquidity they want at the marginal lending rate, and they can deposit liquidity in the central bank at the deposit rate.

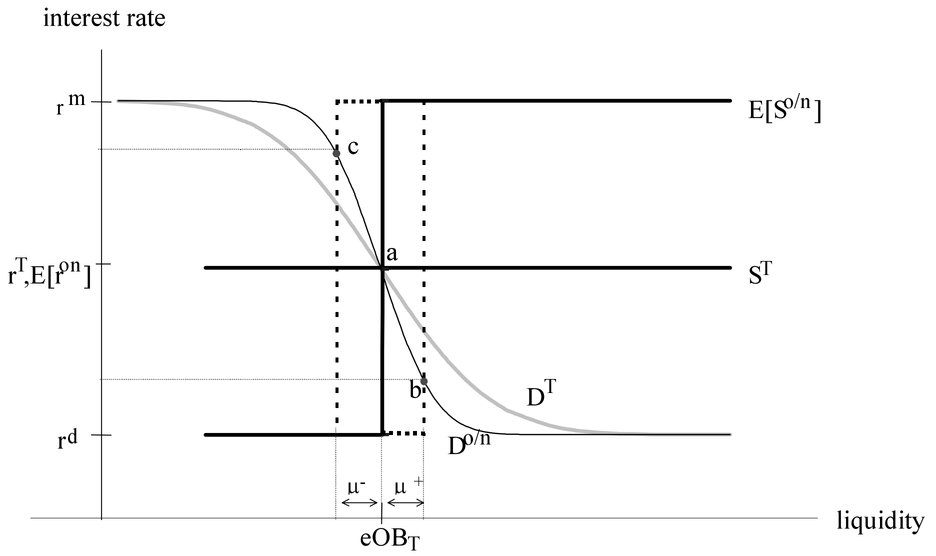

Figure 2: Determination of overnight rate; symmetric corridor

pected its value, the interest rate elasticity of $D^{o/n}$ (based on $F(\cdot)$) is smaller than that of D^{T} (based on $G(\cdot)$), as the variance of the remaining shock (ε) is smaller than the variance of the total shock ($\mu+\varepsilon$) (the stochastic error terms μ and ε are independently distributed).[22] This means, that the variations in the overnight rate of interest due to a shock of given size is larger after some of the liquidity uncertainty has disappeared. If the first shock of the day is positive (μ^{+}), the realized value of the overnight rate will be lower than its expected value (point b). Similarly, the overnight rate increases up to point c due to a negative liquidity shock.

The volatility of the overnight rate in this setting depends on the timing of the clearance of the interbank market, as μ and ε reflect the share of the aggregate shock occurring before and after the clearance of the market respectively.[23] The earlier the markets clear, the smaller the share of the flow of the daily shocks occurring before the clearance (smaller σ_{μ} and larger σ_{ε}), and the closer $D^{o/n}$ is to D^{T}. Intuitively, early clearing of the interbank market increases the uncertainty of a bank's end-of-day balance at the clearance of the market, which increases the interest rate elasticity of the demand for reserves. Thus, *the volatility of the overnight interest rate is lower on markets that are active already in the mornings, compared with the markets that use interbank trading merely to settle the foreseen liquidity needs of the banks.*

Finally, let us consider the case in which the interest rate corridor is asymmetric. Figure 3 shows the determination of the overnight rate when the tender rate is in the lower part of the corridor.

Here, again the equilibrium at the tender operation gives the expected value for both the liquidity and the overnight rate (point a). The expected value of

[22] Note that we can get distribution $g_{\mu+\varepsilon} \sim N(0, \sigma_{\mu+\varepsilon})$ from distribution $f_{\varepsilon} \sim N(0, \sigma_{\varepsilon})$ through a mean preserving spread.

[23] The aggregate shock consists of a continuum of small independent shocks.

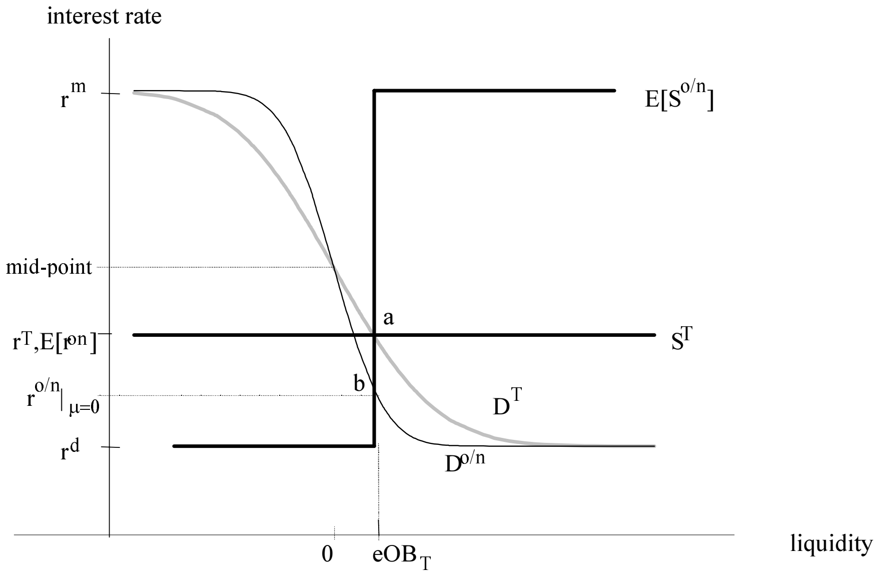

Figure 3: Determination of overnight rate; asymmetric corridor

the overnight rate still equals the tender rate. However, the expected liquidity is now larger than zero, as the relative cost of having to use the deposit facility is lower than the cost of acquiring credit from the marginal lending facility, and thus, the banks are willing to increase the probability of using the deposit facility.

Note, that even though the expected value of overnight rate interest equals the tender rate ($\mathrm{E}\left[r^{o/n}\right] = r^T$), the overnight rate at the expected liquidity is lower than the tender rate ($r^{o/n} \mid_{\mu=0} < r^T$, see point b in figure 3). This result is obvious, as we know that, having assumed the normally distributed error terms, the demand for reserves is convex (concave) at liquidity levels above (below) zero, and that the relative curvature of $D^{o/n}$ is higher than that of D^T. This means, that we expect to see the overnight rate realized below the tender rate more frequently than above it, if the tender rate is in the lower part of the interest rate corridor. Also the interest rate variations due to liquidity shocks are not symmetric around the expected value. This again results from the convexity of demand at rates below the middle of the corridor.

Similar, but opposite effects can be shown for the case where the tender rate is in the upper part of the corridor.

The central bank can affect the amount of excess reserves demanded, and the volatility of the interbank overnight rate by choosing both the width of the interest rate corridor and the location of the tender rate within the corridor. These effects should be taken into account, if the central bank wants to use the rates of the standing facilities as an independent signalling device.

3.2.2 Proportional allotment

In the case of proportional allotment, the banks know, that the central bank has a target for liquidity, and that it will try to allot liquidity according to this target regardless of the total amount bid by the banks. Let us define the targeted amount as TL^s. Now, the actual tendered liquidity (TL) will not (always) be the total amount of bids (TL^d). The amount of liquidity the central bank actually provides to the markets is either the amount targeted by it or the aggregate amount of bids, whichever is the smaller ($TL = \min(TL^d, TL^s)$). Thus, the banks must take into account the behaviour of the central bank as well as the behaviour of the other banks when preparing their bids.

When can the central bank control also the supply of liquidity provided to the market in fixed rate tenders? Let us first assume, that the banks expect the central bank to have a target for liquidity, such that it will (on average) leave the markets with neutral liquidity (ie liquidity by which the expected overnight rate of interest would equal the tender rate, $TL^s = TL^{neutral}$).[24] We also assume, that the central bank's estimate of the banks' demand for reserves in order to have neutral liquidity is unbiased. If the banks now used the neutral bidding strategy (like they did under full allotment), the liquidity supplied to the markets would be the smaller of the two variables that have the same mean: i) the central bank's estimate of tendered reserves needed for neutral liquidity (which is based on the central bank's forecast of the autonomous liquidity factors a^{CB}), and ii) the sum of banks' estimates on their reserve needs for neutral liquidity (ie the aggregate bid (TL^*), which is based on the banks' forecast of the autonomous liquidity factors $a = \sum_i a_i$). The banks' aggregate estimate of the autonomous liquidity factors does not need to (and normally does not) coincide daily with that of the central bank, even though they both are unbiased estimators of the same stochastic variable. Therefore, the overnight rate with liquidity based on the neutral demand would normally differ from that of based on the neutral supply, even though they share the same expected value (ie $p(r^{on}|TL^* \neq r^{on}|TL^{CB}) > 0$, even though $\mathrm{E}[r^{on}|TL^*] = \mathrm{E}[r^{on}|TL^{CB}] = r^T$). Consequently, the expected value of the overnight rate with the liquidity actually tendered ($TL = \min\left(TL^*, TL^{CB}\right)$) would be above its expected value with either of these single liquidity variables ($\mathrm{E}[r^{on}|TL] = \mathrm{E}[r^{on}|\min(TL^*, TL^{CB})] > \mathrm{E}[r^{on}|TL^*] = \mathrm{E}[r^{on}|TL^{CB}] = r^T$), as the overnight rate of interest is a decreasing function of liquidity. In such a case, the representative bank is evidently able to make profitable deviation from the neutral bidding strategy. By increasing its bid, a bank is going to have excess liquidity at the tender rate, and the income from selling this extra liquidity in the money market is expected to be higher than the tender rate. Similarly, underbidding is ruled out as a sustainable equilibrium strategy in

[24] In section 4.2 (where the reserve requirement is based on averaging), we divide the proportional allotment procedure into liquidity targeting and interest rate targeting. In liquidity targeting the central bank has set liquidity directly as the target, whereas in interest rate targeting the amount the central bank is willing to lend will be derived indirectly from the banks demand function. Here, both of these two approaches would produce similar results, as the reserve holding is not based on averaging. Thus, the neutral liquidity target we have here can be thought of as a direct liquidity target or to be derived from a neutral interest rate target (where the target rate equals the tender rate).

this setting, as the expected liquidity would then be smaller than with neutral bidding, and so the incentive to deviate from the overbidding strategy would be even stronger than from neutral bidding. Therefore, all sustainable equilibria with this kind of proportional allotment procedure must result in overbidding. If the aggregate amount of bids exceeds the estimated neutral level sufficiently ($TL^d >> TL^*$), the tendered liquidity will always equal the central bank's target amount ($TL = \min(TL^d, TL^{CB}) = TL^{CB}$).[25] Hence, the supply of daily liquidity would be determined solely by the target of the central bank. Consequently, the expected value for interbank overnight rate would equal the tender rate.

The amount of overbidding can't always be determined uniquely in this setting. Eg if $2TL^* \geq TL^{CB}|_{a^{CB}=a^{\max}}$, the total bids amounting to twice the real liquidity need would lead to the same result as total bids amounting to three times the real need. Therefore, any such amount of total bids that is large enough to maintain $TL^d \geq TL^{CB}|_{a^{CB}=a^{\max}}$ would be an equilibrium. Now, from a single bank's point of view any bid would lead to zero profit as long as it can be sure that the central bank can control the liquidity according to its target (ie as long as $p(TL^d \geq TL^{CB}|_{a^{CB}=a^{\max}}) = 1$). However, if there is even the slightest probability that the aggregate bids might be lower than the central bank's target amount, it would be optimal to bid the maximum value that one can bid for.[26]

What if the central bank has some other policy than to provide the markets with neutral liquidity? Let us first assume, that the central bank wants to squeeze the markets (ie provide the markets with liquidity below its neutral level) in order to keep the overnight rate of interest (or its expected value to be exact) above the tender rate. This approach, when expected by the counterparties, raises the incentive for overbidding, as the income from selling the extra liquidity is increased compared with the neutral situation. Thus, the

[25] How much the aggregate bids need to exceed the expected neutral level depends on the maximum size the central bank's estimate of the autonomous liquidity factors can be ($a^{\max}$). If $TL^d \geq TL^{CB}|_{a^{CB}=a^{\max}}$, the aggregate bids will allways exceed the amount the central bank is willing to provide the markets with, and consequently there will never be full allotment.

[26] If the central bank did not limit the bid size in any way, the optimal bid would in principle be infinite, however, in practice the size of a bid would still be limited at least to be a numerical value. Furthermore, the bid size could also be limited by the central bank (eg the ECB requires the banks to be able to cover the amount of reserves they are allotted for by adequate collateral) or by market imperfections (eg the banks usually have limited credit lines that are needed to distribute the liquidity in the overnight market). In case there were such collateral requirements or credit lines, the maximum bid would be limited by the probability of being allotted for more reserves than would be optimal taking these limitations into account. Thus, the optimal bid of a single bank would depend (partly) on its expectation of the allotment ratio (ie allotted liquidity/aggregate bids) in the tender - the lower the expected allotment ratio is, the lower is the probability of reaching these limits, and consequently, the more one may bid in the tender. As in this set up there is no natural focal point for expectations on the allotment ratio, the bank could use the allotment ratio of the previous tender (or average of such ratios in the past few tenders) as such a point while preparing its bid. If this was the case, the bank's optimal bid would increase from tender to tender (as the focal point diminishes continuously), until the allotment ratio would reach such a low level that the bank could be sure of the central bank having the control of the allotted liquidity (ie $p\,(\text{aggregate bids} \geq \text{central bank's target liquidity}) = 1$).

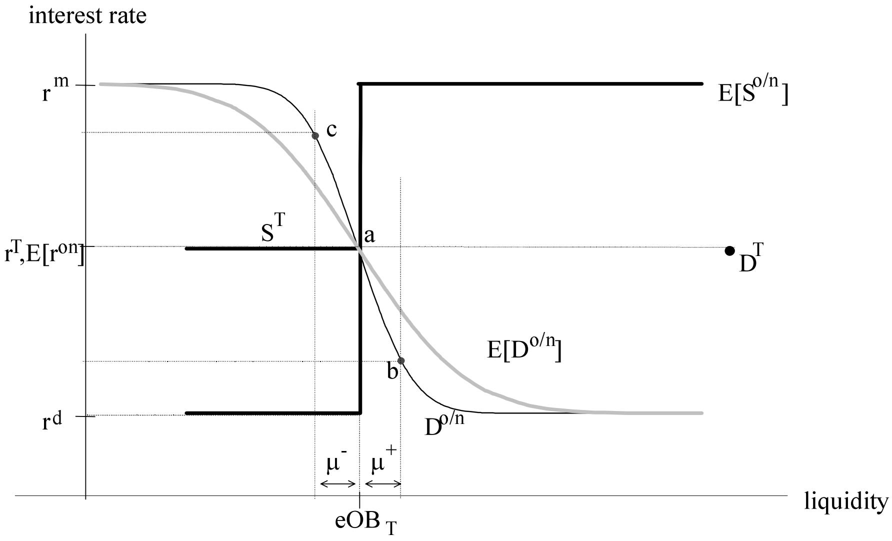

Figure 4: Determination of overnight rate; symmetric corridor

central bank would be able to steer both the overnight rate and the money market liquidity with this kind of policy. However, the rationale for using fixed rate tenders in this vein could very well be questioned. If the central bank behaves like this, it actually uses the supply of liquidity as its policy variable instead of the tender rate. Then, the whole process of implementing monetary policy would be more transparent to the public, if variable rate tenders were chosen. Furthermore, it is hard to find a rationale for the transfer of profit to the successful bidders caused by the use of fixed rate tenders with liquidity supply rationed below the neutral demand. This procedure would benefit those who can make the largest bids (above their neutral liquidity demand). Hence, this procedure would eventually lead to infinitely large bids, if the bid size is not somehow rationed.

In the opposite case, the central bank's strategy would be to maintain the overnight rate below the tender rate. This could be achieved by flooding the market with excess reserves. However, the central bank would not be able to do this, as the banks would not be willing to provide it with large enough bids (ie $TL^d < TL^s$), if the price of liquidity is expected to be lower in the markets than in the tender operation. In this case, the central bank would not be able to control the quantity of money market liquidity, and this strategy would produce outcome similar to the full allotment case.

Figures 4 and 5 clarify the determination of overnight rate under proportional allotment procedure with neutral liquidity target, both for symmetric interest rate corridor and asymmetric corridor.[27] The only differences between these two figures and those of under the full allotment procedure (figures 2 and 3) are in the demand for and supply of liquidity at the tender operation. Now, D^T is an arbitrary point at the level of the tender rate and at huge liquidity (relative to the real need). The supply is again perfectly elastic, but only up

[27]In figure 4, like in the rest of the figures in the paper, we assume the liquidity shocks t be distributed normally.

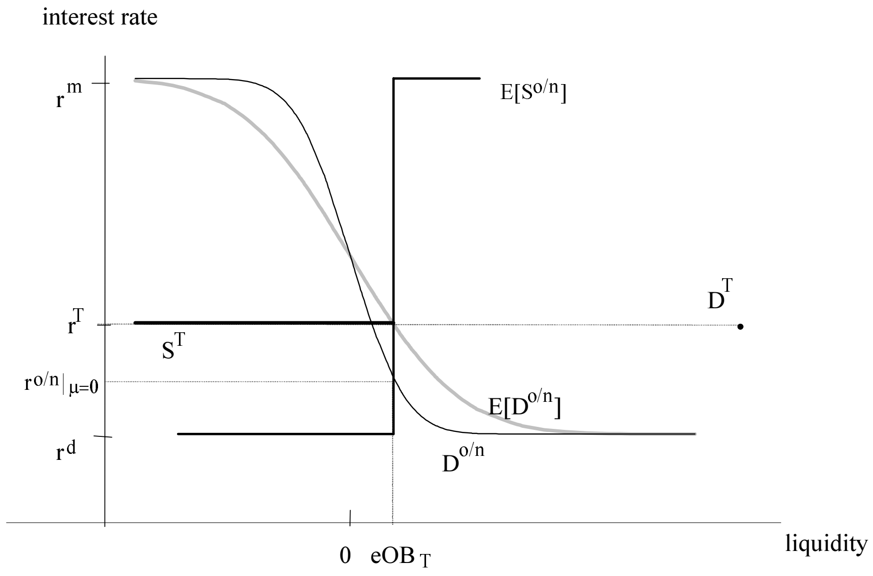

Figure 5: Determination of overnight rate; asymmetric corridor

to the amount targeted by the central bank. This target amount is given by the central bank's expectation of the demand for liquidity at the tender rate (ie $\mathrm{E}\left[S^{o/n}\right]$ at r^T). Thus, $\mathrm{E}\left[D^{o/n}\right]$ is similar to D^T. The difference between them is that D^T is based on $G(\cdot)$, which is the banks' expectations over $F(\cdot)$, and $\mathrm{E}\left[D^{o/n}\right]$ is based on the central banks expectation of $F(\cdot)$.

3.3 Comparing the two approaches

We next summarize the findings from the two previous sections, and try to answer the question: how do these two equilibria with different allotment mechanism used (full allotment and proportional allotment with neutral liquidity target) differ from each other?

- *The demand for and supply of liquidity*

  The demand for liquidity at the clearance of the overnight market does not depend on the allotment procedure, as it is merely a function of the prevailing money market liquidity, the interest rates of the standing facilities and the distribution of liquidity shocks. Also the shape of liquidity supply at the overnight market is independent of the approach used in allotting the liquidity. The supply is perfectly inelastic at the level of overnight balances from deposit rate to the marginal lending rate, and the supply is perfectly elastic at the rates of the standing facilities.

  However, the demand and supply at the tender operation differ according to the allotment procedure. If the proportional allotment method is used, the supply is perfectly elastic only up to the liquidity target of the central bank, while it is perfectly elastic without limits in case of full allotment. The demand for liquidity is arbitrarily large relative to the real need under proportional allotment with neutral liquidity target. In case of full allotment (or proportional allotment with the tender rate in the upper part of the corridor) the *equilibrium demand* is determined by the probability-weighted cost of using the standing facilities.

Due to the differences in the demand for and supply of liquidity at the tender, the location of the inelastic part of the supply of overnight liquidity at the clearance of the market may differ according to the allotment procedure.

- *The level and volatility of the overnight rate*

  With both procedures the expected value of the overnight rate will equal the tender rate. If the interest rate corridor is symmetric, the relative volatility of these approaches depends on the size of the first liquidity shock of the day. If the central bank can estimate the evolution of the total liquidity better than the corresponding aggregated estimates of the banks are (ie if $\mu^{CB} < \sum \mu_i$), the volatility of the overnight rate of interest is smaller under proportional allotment method. However, this is not necessarily the case, particularly, if the central bank publishes its estimate before the tender operation.

  In case of an asymmetric interest rate corridor, the volatility depends again on the relative accuracy of the liquidity estimates. However, now it depends also on the relative accuracy of the estimates of the cumulative distributions. It is not however obvious, that $\mathrm{E}^{\mathrm{CB}}[F(\cdot)]$ will be a more accurate estimate of $F(\cdot)$, than $G(\cdot)$ is. Hence, without further assumptions we can't say whether the volatility of the overnight rate is larger under full allotment or not.

- *Signalling monetary policy and transmission of volatility*

  In the case of full allotment, the expected value of the overnight rate for a specific day is always the value expected to be used in the tender operation affecting that day's liquidity. Thus, the yield curve (that is based on future values of the overnight rate) should reflect only the expectations on the evolution of the tender rate. These expectations should not have anything to do with the overnight rate realized on a specific day, as its deviation from the tender rate is merely produced by forecast errors by the banks. Thus, the signals given by the tender operations are unambiguous, and the volatility of the overnight rate should not be transmitted to longer periods.

  The same reasoning applies also for the case with proportional allotment, as long as the strategy used in choosing the level targeted by the central bank is known to the public (or at least to the counterparties). If the target must be read from the past behaviour of the central bank (ie past allotment decisions), variations in realizations overnight might be interpreted as changes in the monetary policy stance. Thus, in such a case it would not be certain that the volatility is not transmitted to longer maturities. This, harmful, transmission could be avoided *either* explicitly by the central bank announcing the allotment policy *or* by it making the liquidity policy implicitly public through publishing the liquidity forecast, on which it bases its liquidity allotment decision.

- *Using interest rate corridor as an independent signalling device*

  From the previous analysis, it is clear, that a symmetric interest rate corridor is simplest for the central bank to operate with as long as we can expect the liquidity shocks to be symmetrically distributed. This results from the fact that if the corridor is symmetric, the demand for liquidity at the tender will equal the demand for liquidity at the clearance of the market. Also the variation of the overnight rate round the tender rate is symmetric with a symmetric corridor. However, the central bank might like to give monetary policy signals independent from the tender rate via the rates of the standing facilities. Eg having a tender rate in the lower part of the corridor could indicate, that the central bank sees the probability of its next change in the tender rate to be upwards.

  Using the corridor independently is rather complicated with the proportional allotment method. If the tender rate lies in the upper part of the corridor, the central is not able to meet its target liquidity, and consequently the procedure will in fact be similar to full allotment. Also, if the tender rate is in the lower part of the corridor, the central bank must adjust its liquidity target up from 0 (or the level of the reserve requirement), to keep the target amount neutral.[28] Estimating the new target liquidity (after a change in the tender rate's location within the corridor) can be difficult especially if the shock distributions are not constant over time.

  The use of the rates of the standing facilities as independent policy instruments is perhaps not so difficult with the full allotment procedure. However, in this case the central bank must remember that the asymmetry of the corridor affects the demand for excess reserves, and consequently it also affects the cost of the framework to the banks.

[28] The target amount of the central bank will differ from zero, if the shock distribution is asymmetric. The amount would be positive (negative), if the distribution were skewed to the left (right).

4 The model with averaged reserve requirements

If compliance with the reserve requirement is judged by the average value of reserve holdings during a reserves maintenance period instead of a daily requirement, the demand for daily reserves changes dramatically. As in the previous chapter, we assume here that the central bank conducts liquidity operations daily. We also assume, that a liquidity operation will mature on the day when the following operation is settled. Thus, the maturity of the liquidity provided is overnight unless we relax the assumption of the frequency of the operations. Furthermore, we will continue to assume that the central bank tenders are liquidity providing and that the structural liquidity deficit of the money market enlarged by the reserve requirement is "large enough", if the banks do not bid for liquidity in a tender operation.[29]

The determination of the overnight rate on the last day of the maintenance period is similar to the case described in the previous chapter. Thus, the $\mathrm{E}[r_T^{on}]$ will equal the probability weighted average of the expected value of the marginal lending rate and the deposit rate at the end of the period.

$$\mathrm{E}\left[r_T^{on}\right] = \mathrm{E}\left[r_T^{m}\right] G(-eER_T) + \mathrm{E}\left[r_T^{d}\right] (1 - G(-eER_T)), \tag{14}$$

Where the expected excess reserves (eER_T) equals the money market liquidity at the time of overnight markets clearing subtracted with the minimum required daily balances ($eOB_T - RDB_T$). As in the case without averaged reserve requirements, we divide the following analysis of the demand for liquidity according to which approach the central bank uses in its allotment decisions.

4.1 Full allotment

4.1.1 Penultimate day (T-1)

As we saw in the previous chapter, banks are willing to bid for neutral liquidity in the last operation of the maintenance period (affecting day T's reserves), when a full allotment procedure is used by the central bank. Thus, the expected value (at the last tender) of the last day's overnight rate will equal the tender rate ($\mathrm{E}_T\left[r_T^{on}\right] = r_T^T$). At the interbank market clearance on T-1, the banks know that their liquidity holdings on that day do not affect the last days overnight rate, as the situation in the overnight market will be neutralized in the last tender operation. Thus, the expected value of the last day's overnight rate equals the expected value of the last tender rate, if the expectations are taken at T-1 (or before) ($\mathrm{E}_{T-1}\left[r_T^{on}\right] = \mathrm{E}_{T-1}\left[r_T^T\right]$). Consequently, *in the case of pure averaging*[30], the cost of borrowing (income from lending) an extra unit of liquidity from the interbank markets on day T-1 would be r_{T-1}^{on}, and the

[29] "Large enough" here means, that the probability of ending the day with debit balances is close to one, if no liquidity is provided through tender operation. If this were not the case, the central bank should use liquidty draining operations instead of liquidity providing ones under some circumstances. This assumption does not otherwise limit the analysis.

[30] By pure averaging we refer to a system, where the end-of-day balances of a credit institution is not limited by any regulations, other than the reserve requirement.

expected income from (cost of) that extra liquidity would derive from being able to avoid borrowing next day at $\mathrm{E}\left[r_T^T\right]$. So, for every level of reserves the market-clearing overnight rate on T-1 would be the interest rate expected to be used in the last operation affecting this maintenance period's liquidity (ie $r_{T-1}^{on} = \mathrm{E}_{T-1}\left[r_T^T\right]$). Hence, no bank would be willing to borrow (lend) at rates above (below) the tender rate. *However, central banks do not normally allow for pure averaging.* Averaging provisions do not normally allow overdrafts.

Here, we will maintain the assumption, that overdrafts are forbidden. Ie if a bank would otherwise be ending a day with a debit balance, it has to cover the deficit by liquidity credit taken from the marginal lending facility. In this case, the cost of borrowing liquidity from interbank markets at T-1 is still r_{T-1}^{on} times the borrowed amount. However, the expected income from it consists of three parts: *i)* the amount of marginal credits expected to be avoided through the interbank borrowing times the marginal lending rate[31], *ii)* the amount of liquidity expected to be deposited into the deposit facility (ie the amount of reserves exceeding the requirement for the whole maintenance period) times the deposit rate, and *iii)* the avoidance of having to borrow tomorrow at $\mathrm{E}\left[r_T^T\right]$ in order to fulfil the reserve requirement times the amount of reserve deposits the bank is expected to have by the end of the day.

The profit maximization problem of a bank operating in the overnight market at the penultimate day of the maintenance period is given in appendix B, which also shows how equation (15), that describes the determination of the overnight rate, is derived from the first order condition (w.r.t. the interbank borrowing) of the profit maximization problem:

$$\begin{aligned} r_{T-1}^{on} &= \mathrm{E}_{T-1}\left[r_T^T\right]\{1 - F(-OB_{T-1}) - [1 - F(2RDB_{T-1} - OB_{T-1})]\} \\ &+ r_{T-1}^m F(-OB_{T-1}) + r_{T-1}^d\left[1 - F(2RDB_{T-1} - OB_{T-1})\right]. \end{aligned} \tag{15}$$

We can also write equation (15) as the expected change in the overnight rate of interest (ie the difference between today's overnight rate and the expected rate to be used in the final tender operation):

$$\begin{aligned} r_{T-1}^{on} - \mathrm{E}_{T-1}\left[r_T^T\right] &= \left[\left(r_{T-1}^m - \mathrm{E}_{T-1}\left[r_T^T\right]\right)F(-OB_{T-1})\right] \\ &+ \left(r_{T-1}^d - \mathrm{E}_{T-1}\left[r_T^T\right]\right)\left[1 - F(2RDB_{T-1} - OB_{T-1})\right] \end{aligned} \tag{16}$$

Equation (16) says that in equilibrium the expected change in the overnight rate between the two final days of the maintenance period, equals the probability weighted spreads between the expected tender rate and the current rates of the standing facilities.

If the interest rate corridor is symmetric[32] and the banks do not anticipate a change in the tender rate, the overnight rate at T-1 will be below that expected for the last day, only if the probability of being overdrawn at T-1 is smaller than the probability of fulfilling the reserve requirement for the whole period

[31] $F(-OB_{T-1})$ is the probability of the liquidity shock ε_{T-1} being less than $-OB_{T-1}$.

[32] By symmetric interest rate corridor we refer to the situation, where the tender rate is the mid-point of the interest rate corridor (i.e. $r_t^T = \frac{r_t^m + r_t^d}{2}$).

at T-1.[33] If we assume the liquidity shocks to be distributed symmetrically, we will see that the overnight rate is expected to decrease between T-1 and T, as long as the amount of liquidity traded at the overnight market at T-1 is less than the minimum required daily balances to be held at T-1 and T (if $OB_{T-1} < RDB_{T-1}$, then $r^{on}_{T-1} > \mathrm{E}_{T-1}\left[r^T_T\right] = \mathrm{E}_{T-1}\left[r^{on}_T\right]$). Similarly, with a symmetric corridor and symmetric shock distributions, the overnight rate is expected to increase during the two last days, if the liquidity at the overnight clearance at T-1 is over the minimum required daily balances.

From equation (16) we know, that the overnight rate of interest on the penultimate day of the maintenance period is an increasing function in all central bank rates (expected value of the last tender rate, current deposit rate and current marginal lending rate ;$\frac{\partial r^{on}_{T-1}}{\partial \mathrm{E}_{T-1}\left[r^T_T\right]}, \frac{\partial r^{on}_{T-1}}{\partial r^m_{T-1}}, \frac{\partial r^{on}_{T-1}}{\partial r^d_{T-1}} > 0$), and a decreasing function in both current reserve holdings (money market liquidity at the time of clearing) and the minimum required daily balances ($\frac{\partial r^{on}_{T-1}}{\partial OB_{T-1}}, \frac{\partial r^{on}_{T-1}}{\partial RDB_{T-1}} < 0$). The RDB itself is increasing in the reserve requirement and decreasing in the past reserve holdings ($\frac{\partial RDB_{T-1}}{\partial RR} > 0, \frac{\partial RDB_{T-1}}{\partial \Sigma_{j=1}^{T-2} RB_j} < 0$).

The bidding behaviour and the determination of equilibrium liquidity

The demand for liquidity in the penultimate tender (affecting the liquidity on T-1) will depend on the expected value for the overnight rate for that day. Now, we know that the price of overnight liquidity at T-1 is a decreasing function of the overnight balances on that day. Thus, with reasoning similar to the case with no averaging (see section 3.2.1), the banks will in equilibrium be bidding for liquidity until the expected value of today's overnight rate equals today's tender rate ($\mathrm{E}_{T-1}\left[r^{on}_{T-1}\right] = r^T_{T-1}$).[34] Let us denote the expected change in the tender rate between the last two days of the period by Δr^T (ie $\Delta r^T = E_{T-1}\left[r^T_T\right] - r^T_{T-1}$). Based on equation (16), and the facts that in equilibrium $\mathrm{E}_{T-1}\left[r^{on}_{T-1}\right] = r^T_{T-1}$ and $\mathrm{E}\left[F\left(-OB^*_{T-1}\right)\right] = G\left(-eOB^*_{T-1}\right)$ we will have:

$$\begin{aligned} \mathrm{E}_{T-1} \quad & \left[r^{on}_{T-1}\right] - \mathrm{E}_{T-1}\left[r^T_T\right] = r^T_{T-1} - \mathrm{E}_{T-1}\left[r^T_T\right] = -\Delta r^T = \\ & \left(r^m_{T-1} - \mathrm{E}_{T-1}\left[r^T_T\right]\right) G(-eOB^*_{T-1}) \\ & + \left(r^d_{T-1} - \mathrm{E}_{T-1}\left[r^T_T\right]\right)\left[1 - G(2RDB_{T-1} - eOB^*_{T-1})\right], \end{aligned} \tag{17}$$

where eOB^*_{T-1}denotes the expected overnight balances at the clearance of the overnight market with equilibrium bidding (ie $eOB^*_{T-1} = RB_{T-2} + a_{t-1} + TL^*_{T-1}$). Equation (17) implicitly defines the optimal bidding behaviour of the banks; equilibrium bidding is such that it balances the expected change in the price of liquidity with the probability weighted difference between the current rates of the standing facilities and the expected tender rate for tomorrow.

[33] With a symmetric corridor and constant tender rate $\left|r^m_{T-1} - \mathrm{E}_{T-1}\left[r^T_T\right]\right| = \left|r^d_{T-1} - \mathrm{E}_{T-1}\left[r^T_T\right]\right|$. Thus, we must have $F(-OB_{T-1}) < 1 - F(2 * RDB_{T-1} - OB_{T-1})$, as otherwise the RHS of the equation 16 would not be negative.

[34] If this were not the case, the banks could be making profit from increasing their bids, if $\mathrm{E}_{T-1}\left[r^{on}_{T-1}\right] > r^T$, or by lowering their bids, if $\mathrm{E}_{T-1}\left[r^{on}_{T-1}\right] < r^T$. Hence, in equilibrium the overnight rate is expected to remain constant at the level of the tender rate during the two last days of the reserves maintenance periods.

Let us divide the analysis of the equilibrium liquidity according to the interest rate expectations of the banks:

Neutral interest rate expectations

By neutral interest rate expectations we refer to the situation, where the banks do not anticipate a change in the tender rate, ie $E_{T-1}\left[r_T^T\right] = r_{T-1}^T$. Let us denote this rate simply by r^T. Equation 17 can, under neutral interest rate expectations, be reduced into:

$$\begin{aligned}(r_{T-1}^m - \ r^T) \ \ & G(-eOB_{T-1}^*) = \\ & \left(r^T - r_{T-1}^d\right)\left[1 - G(2RDB_{T-1} - eOB_{T-1}^*)\right].\end{aligned} \tag{18}$$

Equation (18) implicitly defines the optimal bidding to be such that it balances the probability weighted difference between the rates of the standing facilities and the tender rate. The LHS of equation (18) is positive, and monotonically decreasing in liquidity. Also the RHS of it is positive, however, it monotonically increases with liquidity. Thus, there always exists a level of liquidity, that satisfies the equilibrium condition. We can conclude, that *at T-1 the optimal bidding is a function of the tender rate's location within the interest rate corridor, and the probability of having to rely on the standing facilities.*

Let us assume for a moment, that the interest rate corridor is symmetric, and the liquidity shocks are distributed symmetrically. In this case, we know by equation (11) that the equilibrium liquidity of tomorrow (at T) is RDB_T.[35] By the symmetry assumptions, equation (18) reduces further to $G(-eOB_{T-1}^*) = G(-2RDB_{T-1} + eOB_{T-1}^*)$,[36] from which we easily see, that $eOB_{T-1}^* = RDB_{T-1}$. Ie the banks are expected to hold reserves according to their minimum required daily balances at T-1. Thus, with $RB_{T-1} = eOB_{T-1}^*$ the RDB_T would equal RDB_{T-1}, however, the expected value for RDB_T will be higher than RDB_{T-1}, at least if eOB_{T-1}^* is very low (ie if $F\left(-OB_{T-1}\right)$ is significantly above zero).[37] This means that the mean value of the money market liquidity is slightly larger on the last day of the maintenance period than the equilibrium liquidity for the previous day is, if the required daily balances at T-1 is relatively low ($eOB_{T-1} = RDB_{T-1} \leq E[RDB_T] = E[OB_T]$).

If the tender rate was in the upper part of the corridor $(\left(r^m - r^T\right) < \left(r^T - r^d\right))$, the equilibrium liquidity must leave the probability of being overdrawn larger than the probability of being forced to use the deposit facility (ie $G(-eOB_{T-1}^*) > 1 - G(2RDB_{T-1} - eOB_{T-1}^*)$). In this case, the overnight liquidity is expected to increase more during the last two days than with

[35] Under these symmetry assumptions equation 11 reduces to $(r^m - r_T^T)\left[G(-eER_T^*) - 1 + G(-eER_T^*)\right] = 0 \Rightarrow G(-eER_T^*) = \frac{1}{2} \Rightarrow eER_T^* = 0 \Rightarrow eOB_T^* = RDB_T$.

[36] Remembering, that for symmetric shock distribution $G\left(-x\right) = 1 - G\left(x\right)$.

[37] This holds, as for the last day:
$RDB_T = \begin{cases} 2RDB_{T-1} - OB_{T-1} - \varepsilon_{T-1}, \text{ if } \varepsilon_{T-1} > -OB_{T-1} \\ 2RDB_{T-1}, \text{ if } \varepsilon_{T-1} < -OB_{T-1} \end{cases}$, thus $E[RDB_T] = (2RDB_{T-1} - eOB_{T-1})\left[1 - F\left(eOB_{T-1}\right)\right] + 2RDB_{T-1} * F\left(-eOB_{T-1}\right)$, which reduces in this symmetric case to $E[RDB_T] = RDB_{T-1}\left[1 + F\left(-RDB_{T-1}\right)\right]$. More generally
$E[RDB_t] = \frac{(T-t+1)RDB_{t-1} - OB_{t-1}}{T-t}\left[1 - F\left(-OB_t\right)\right] + \frac{(T-t+1)RDB_{t-1}}{T-t}F\left(-OB_t\right)$.

a symmetric corridor, as $G(-eOB^*_{T-1}) > 1 - G(2RDB_{T-1} - eOB^*_{T-1}) \Rightarrow G(eOB^*_{T-1}) < G(2RDB_{T-1} - eOB^*_{T-1}) \Rightarrow eOB^*_{T-1} < RDB_{T-1}$, and thus, $\mathrm{E}[OB^*_T] = \mathrm{E}[RDB_T] > RDB_{T-1} > eOB^*_{T-1}$.

Similarly, if $\left(r^m - r^T\right) > \left(r^T - r^d\right)$, then $(G(-eOB^*_{T-1}) < (1 - G(2RDB_{T-1} - eOB^*_{T-1}))$, and the direction of the evolution of liquidity on the last two days depends on the magnitude of the asymmetry, as well as on the size of RDB_{T-1}.

Expectations of increased interest rates

If the banks anticipate an increase in the tender rate during the remainder of the period ($r^T_{T-1} < \mathrm{E}_{T-1}\left[r^T_T\right]$ ie $\Delta r^T > 0$) the demand for liquidity in the penultimate tender will increase considerably, as the banks perceive the price of today's central bank liquidity cheap compared with that of tomorrow's. To get an equilibrium in the tender at T-1, the banks place bids again in order to equate the expected value of today's overnight rate with today's tender rate ($\mathrm{E}_{T-1}\left[r^{on}_{T-1}\right] = r^T_{T-1}$). However, at the moment of the overnight trading r^{on}_{T-1} is a function of r^T_T instead of r^T_{T-1} (see equation (16)). Thus, the expected overnight liquidity must now be larger than in case of neutral expectations, as the RHS of (17) has to be negative.

Equation (17) tells us, that the banks will bid for liquidity until the difference between the rates of the two standing facilities and the expected tender rate weighted by the probabilities with which they are expected to be used, equals the negative of the expected difference in the rates of the two remaining tenders. With expectations of increased interest rates the expected difference between the two tender rates is positive ($\Delta r^T = E_{T-1}\left[r^T_T\right] - r^T_{T-1} > 0$). Thus, in order to get a negative value on the RHS of equation (17) the probability of using the marginal lending facility must be lower than in case of neutral interest rate expectations (ie $G(-eOB^{*,\text{increasing exp.}}_{T-1}) < G(eOB^{*,\text{neutral exp.}}_{T-1})$). This means, that the equilibrium liquidity at T-1 will be larger with expectations of increases, than in the case with neutral expectations ($eOB^{*,\text{increasing exp.}}_{T-1} > eOB^{*,\text{neutral exp.}}_{T-1}$). As the liquidity at T-1 is larger with expectations of increases, the $RDB^{\text{increasing exp.}}_T < RDB^{\text{neutral exp.}}_T$, and consequently $eOB^{*,\text{increasing exp.}}_T < eOB^{*,\text{neutral exp.}}_T$. So, the expectation of an increase in the tender rate between the two last days of the maintenance period does not carry over to the market overnight interest rate, but it is transmitted to the equilibrium overnight liquidity.

Expectations of decreased interest rates

Following the approach above, with expectations of decreases in interest rates during the remainder of the maintenance period ($r^T_{T-1} > \mathrm{E}_{T-1}\left[r^T_T\right]$) the RHS of equation (17) must be positive in equilibrium. To have this, the banks should bid for less liquidity than in the case with neutral expectations ($eOB^{*,\text{decreasing exp.}}_{T-1} < eOB^{*,\text{neutral exp.}}_{T-1}$). As in the case with expectations of increases, the overnight rate does not react to the expected fall in the tender rate. The expectations are reflected merely in the amount of overnight liquidity in the money market.

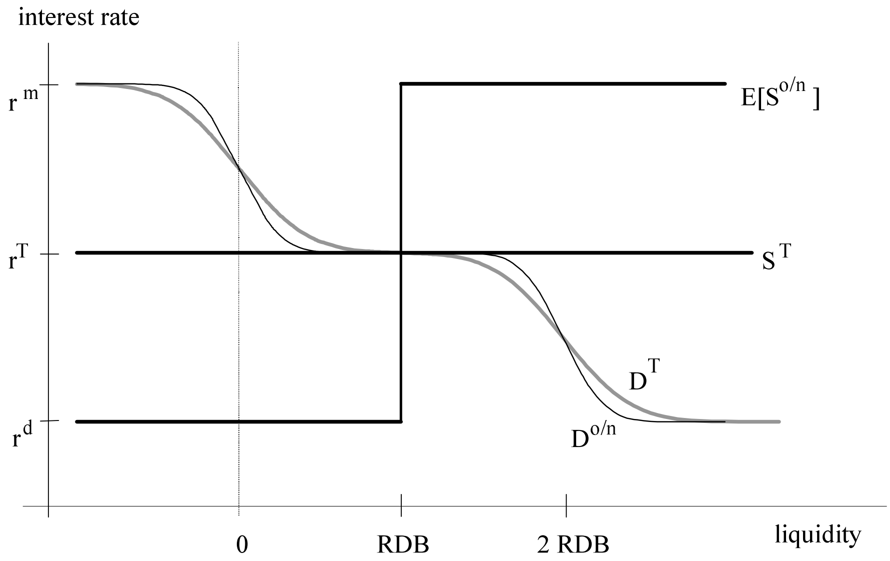

Figure 6: Determination of overnight rate at T-1; symmetric corridor and neutral interest rate expectations

To sum up, if the central bank uses full allotment, the overnight rate at T-1 will equal the tender rate affecting the liquidity at T-1, whatever expectations the banks have on the tender rate for the last day. However, the equilibrium liquidity depends on the interest rate expectations, ie $eOB_{T-1}^{*,\text{increasing exp.}} > eOB_{T-1}^{*,\text{neutral exp.}} > eOB_{T-1}^{*,\text{decreasing exp.}}$.

Figure 6 shows us the determination of the overnight rate on the penultimate day of the maintenance period, and the effect the averaging provision has. Once again S^T and D^T denote the demand and supply in the tender. The vertical part of the expected overnight supply is at RDB_{T-1}, which is the level of liquidity demanded at r^T. The demand for reserves at the clearance of the overnight market is now very elastic at liquidity levels close to the equilibrium. Thus, stochastic liquidity shocks do not affect the overnight rate of interest as much as in the case without the averaging provision.

However, even though the interest rate elasticity of the demand for liquidity increases with the averaging provision, we are not able to state, that the volatility of the overnight rate of interest decreases with it. Figure 7 shows us the case where banks are expecting a rise in the tender rate ($r_{T-1}^T < r_T^T$). The part of demand curve $D^{o/n}$ that seems to be most elastic is still around the minimum required daily balances for the rest of the period (RDB_{T-1}). However, the equilibrium liquidity provided to the market is now well above this level. Thus, we are not able to say unambiguously, whether the demand for liquidity is now more or less elastic than under the case without averaging provision. We may conclude, that *depending on the expectations of the banks, the averaging provision may lower the interest rate variability. However, the interest rate expectations the banks have under the averaging provision will lead to variations in the equilibrium liquidity, and consequently also to variations in the volatility of the overnight rate.*

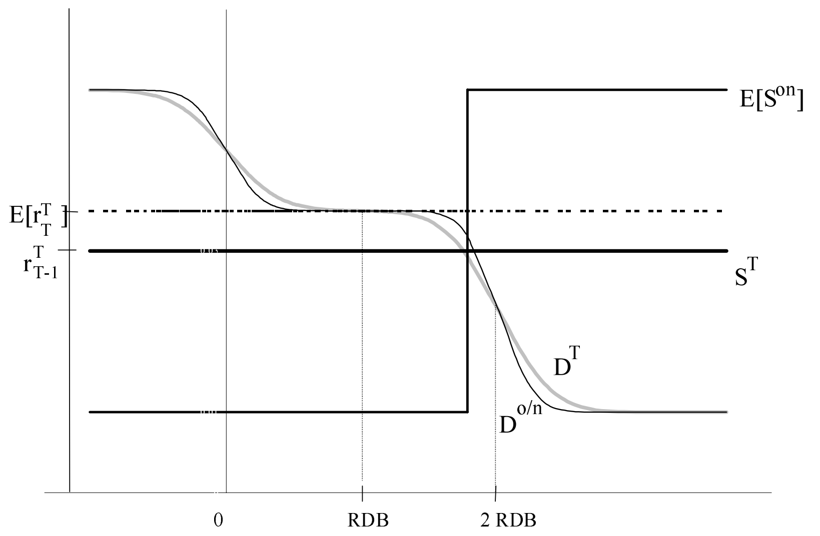

Figure 7: Determination of overnight rate at T-1; increasing interest rate expectations

4.1.2 Earlier days (1,2,...,T-3,T-2)

Let us now move on to analyzing the situation on the days prior to the last two days of the reserves maintenance period. In the penultimate day, the banks already had the luxury of averaging; as long as RDB_{T-1} was positive, the probability of having to rely on either of the standing facilities was below one. The amount of liquidity the banks held on that day did not affect the liquidity conditions on the following day, as the situation was neutralized in the last tender operation. The analysis of the situation prior to the last two days becomes a bit more complicated, as the liquidity held at t (t=1,2,...,T-2) affects the cost of holding reserves (ie the probability of having to use the standing facilities) and consequently the demand for liquidity in the tenders held at t+1,...,T-1. The channel of this effect is through RDBs of the following days.

The cost of borrowing (income from lending) reserves from overnight markets at t ($t = 1, 2, ..., T-2$) is $r_t^{on} * b_{i,t}$. The income from the liquidity bought (cost of liquidity sold) is again a mixture of several components: *i)* the marginal lending rate times the expected avoidance of being overdrawn today , *ii)* the deposit rate times the expected amount to be placed into the deposit facility today, and *iii)* the expected avoidance of having to borrow either from the central bank or from the markets later during the same maintenance period. With full allotment the banks know, that the equilibrium *ex ante* price of overnight liquidity at t,...T equals the tender rate for that day.[38] For simplicity we assume here that the central bank will not change the tender rate more that once during the remainder of the maintenance period. This assumption should not be too restrictive as eg in case of the ECB the maximum number of main

[38] Otherwise, a bank could be making profit by changing its bid in the tender as we have seen before.

refinancing operations during one maintenance period is five (so, at the first operation there are only three or four operations where the tender rate could be changed). We also assume, that the banks are unaware of the timing of the possible change. Thus, in case a rate change is expected the banks expect it to be effective already in the next operation.[39] The expected value for the future tender rate will be denoted by $\mathrm{E}_t\left[r_f^T\right]$.[40]

Besides these three factors that were used also in the determination of overnight rate at $T-1$, we have now a fourth component affecting the overnight rate at $1, 2, ..., T-3, T-2$. *iv)* An increase in reserve balances held at t lowers the minimum required daily balances for the remaining period of the following days ($\frac{\partial RDB_j}{\partial RB_t} < 0; j = t+1, ..., T-1$). The cost of liquidity uncertainty the banks face during the rest of the period is a decreasing function of RDB_j, as the probability of having to rely on the standing facilities on a particular day is a decreasing function of the required daily balances for that day, and a liquidity shock can be neutralized in the following tender operation only if it does not force the banks to use standing facilities on that day ($\frac{\partial \text{ cost of uncertainty at } j}{\partial RDB_j} = \frac{\partial \text{ cost of uncertainty}}{\partial \text{ prob. of using s.f.}} \frac{\partial \text{ prob. of using s.f.}}{\partial RDB_j} < 0$).[41] The cost of uncertainty on the last day of the maintenance period depends, as we saw earlier, on the rates of the standing facilities and on the distribution of liquidity shocks. However, on day j the same is true only, if $eRDB_j = 0$ (ie if the reserve requirement has already been fulfilled for the whole period, the banks do not have the averaging possibility anymore). Otherwise, we have to take into account that borrowing reserves has an extra effect on the maximization problem by affecting the probability of being forced to use the standing facilities (through the RDB_j's). From now on we will call this fourth determinant in the profit maximization problem *the dynamic cost factor (dcf)*.

The profit maximization problem of a bank operating at the interbank overnight market is explicitly given in appendix C. The first order condition for the profit maximization problem with respect to b_t gives us (after aggregation over the unitary mass of banks) the overnight rate of interest at t as a function of liquidity:

[39] If the banks could be certain, that the expected change will not occur in the next operation, but it could be effective in the following one, the front- or backloading of reserves (resulting from the expectations) that this model suggest would be divided between this operation in hand and the next one.

[40] $\mathrm{E}\left[r_f^T\right]$ does not need to be a single value that is expected to occur with probability one. It can be derived as a probability weighted average of the whole range of possible realizations for r_f^T, i.e. $\mathrm{E}\left[r_f^T\right] = \int_{r_f^{T,\min}}^{r_f^{T,\max}} r_f^T h\left(r_f^T\right) dr_f^T$, where $h\left(r_f^T\right)$ is the probability density function of the expected future tender rate, $r_f^{T,\min}$ and $r_f^{T,\max}$ are the lowest and highest expected realizations for the rate.

[41] The probability of using the deposit facility on day t is $\left(1 - F\left[(T-j+1)RDB_j - OB_j^*\right]\right)$. Thus, the probability decreases when RDB_j increases, *ceteris paribus*. If the probability of using the deposit facility decreased, and the probability of using the marginal lending facility were unchanged, the overnight rate for that day would decrease. Then, the banks would be lowering their demand for liquidity to restore the equilibrium between the overnight rate and the tender rate. Thus, OB_j^* would increase, which lowers the probability of being overdrawn and increases the probability of having to use the deposit facility. When the markets are in balance again, the probability of using either of the standing facilities is smaller than it was before the increase in RDB_j.

$$
\begin{aligned}
r_t^{on} =& \mathrm{E}_t\left[r_f^T\right] + \left(r_t^m - \mathrm{E}_t\left[r_f^T\right]\right) F(-OB_t) \\
&+ \left(r_t^d - \mathrm{E}_t\left[r_f^T\right]\right)\{1 - F\left[(T-t+1)RDB_t - OB_t\right]\} \\
&+ \sum_{j=t+1}^{T-1}\Bigg(\mathrm{E}\left[r_j^m - r_f^T\right]\frac{\partial eOB_j^*}{\partial b_t} G(-eOB_j^*) \\
&+ \mathrm{E}\left[r_j^d - r_f^T\right]\left[-(T-j+1)\frac{\partial eRDB_j}{\partial b_t} + \frac{\partial eOB_j^*}{\partial b_t}\right] \\
&\times \left\{1 - G\left[(T-j+1)eRDB_j - eOB_j^*\right]\right\}\Bigg),
\end{aligned} \tag{19}
$$

where $G(-eOB_j^*)$ and $(1-G((T-j+1)RDB_j - eOB_j^*))$ are the probabilities of having to use the two standing facilities at j. These probabilities are affected by interbank lending today, as lending today lowers both $eRDB_j$ and eOB_j^*.

Noting, that $\frac{\partial eOB_j^*}{\partial b_t} = \frac{\partial eOB_j^*}{\partial eRDB_j}\frac{\partial eRDB_j}{\partial b_t}$, and taking the partial derivative $\frac{\partial eRDB_j}{\partial b_t}$, we can open equation (19) a little bit further to see explicitly the dcf as a function of the probability of not having to rely on the standing facilities today:[42]

$$
\begin{aligned}
r_t^{on} =& \mathrm{E}_t\left[r_f^T\right] + \left(r_t^m - \mathrm{E}_t\left[r_f^T\right]\right) F(-OB_t) \\
&+ \left(r_t^d - \mathrm{E}_t\left[r_f^T\right]\right)\{1 - F\left[(T-t+1)RDB_t - OB_t\right]\} \\
&+ \{F\left[(T-t+1)RDB_t - OB_t\right] - F(-OB_t)\} \\
&\times \sum_{j=t+1}^{T-1}\Bigg(\mathrm{E}\left[r_j^m - r_f^T\right]\frac{\partial eOB_j^*}{\partial eRDB_j} G(-eOB_j^*)\left(\frac{-1}{T-j+1}\right) \\
&+ \mathrm{E}\left[r_j^d - r_f^T\right]\left[1 - \left(\frac{1}{T-j+1}\right)\frac{\partial eOB_j^*}{\partial eRDB_j}\right] \\
&\times \left\{1 - G\left[(T-j+1)eRDB_j - eOB_j^*\right]\right\}\Bigg)\Lambda_j
\end{aligned} \tag{20}
$$

Equations (19) and (20) tell us, that the overnight rate is the tender rate expected to prevail over the rest of the maintenance period, increased by the probability weighted cost of having to rely on marginal lending today, and decreased both by the probability weighted cost of cost of having to rely on deposit facility today (at t) and by the growth of the cost of future uncertainty that comes with the extra borrowing.

We know, that in equilibrium (under full allotment) the banks are bidding for liquidity until $\mathrm{E}_t\left[r_t^{on}\right] = r_t^T$.[43] Thus, the equilibrium condition for the money market at t $(t = 1, 2, ..., T-1)$ is:

[42] In equation (20) we will use definition $\frac{\partial eRBD_{i,j}}{\partial b_t} = \frac{-[F(IB_t)-F(-OB_t)]}{T-j+1}\Lambda_j$, where $\Lambda_j = 1 + \sum_{k=t+1}^{j-1}\frac{\partial eOB_k^*}{\partial eRDB_k}\frac{-1}{T-k+1}\left(1 + \sum_{l=t+1}^{k-1}\frac{\partial eOB_l^*}{\partial eRDB_l}\frac{-1}{T-l+1}\right.$
$\left.\times ... \times \left\{1 + \frac{\partial eOB_{t+2}^*}{\partial eRDB_{t+2}}\frac{-1}{T-t-1}\left[1 + \frac{\partial eOB_{t+1}^*}{\partial eRDB_{t+1}}\left(\frac{-1}{T-t}\right)\right]\right\}\right)$.
See appendix C for its calculation.

[43] Again, if this were not the case, banks could make positive profit by changing their bidding behaviour.

$$
\begin{aligned}
\mathrm{E}_t\left[r_t^{on}\right]-\mathrm{E}_t\left[r_f^T\right] &= r_t^T-\mathrm{E}_t\left[r_f^T\right]=\left(r_t^m-\mathrm{E}_t\left[r_f^T\right]\right)G(-eOB_t^*) \\
&+\left(r_t^d-\mathrm{E}_t\left[r_f^T\right]\right)\left\{1-G\left[(T-t+1)RDB_t-eOB_t^*\right]\right\} \\
&+\sum_{j=t+1}^{T-1}\left(\mathrm{E}_t\left[r_j^m-r_f^T\right]\frac{\partial eOB_j^*}{\partial eRDB_j}\frac{\partial eRDB_j}{\partial b_t}G(-eOB_j^*)\right. \\
&+\mathrm{E}_t\left[r_j^d-r_f^T\right]\left(\frac{\partial eOB_j^*}{\partial eRDB_j}-(T-j+1)\right)\frac{\partial eRDB_j}{\partial b_t} \\
&\left.\times\left\{1-G\left[(T-j+1)eRDB_j-eOB_j^*\right]\right\}\right)
\end{aligned}
\tag{21}
$$

or

$$
\begin{aligned}
\mathrm{E}\left[r_t^{on}\right]-\mathrm{E}_t\left[r_f^T\right] &= r_t^T-\mathrm{E}_t\left[r_f^T\right]=\left(r_t^m-\mathrm{E}_t\left[r_f^T\right]\right)G(-eOB_t^*) \\
&+\left(r_t^d-\mathrm{E}_t\left[r_f^T\right]\right)\left\{1-G\left[(T-t+1)RDB_t-eOB_t^*\right]\right\} \\
&+\left\{G\left[(T-t+1)RDB_t-eOB_t^*\right]-G(-OB_t^*)\right\} \\
&\times\sum_{j=t+1}^{T-1}\left(\mathrm{E}\left[r_j^m-r_f^T\right]\frac{\partial eOB_j^*}{\partial eRDB_j}G(-eOB_j^*)\left(\frac{-1}{T-j+1}\right)\right. \\
&+\mathrm{E}\left[r_j^d-r_f^T\right]\left(1-\left(\frac{1}{T-j+1}\right)\frac{\partial eOB_j^*}{\partial eRDB_j}\right) \\
&\left.\times\left\{1-G\left[(T-j+1)eRDB_j-eOB_j^*\right]\right\}\right)\Lambda_j;
\end{aligned}
\tag{22}
$$

which both implicitly define the optimal bidding of the banks at t (given $TL_t^* = eOB_t^* - RB_{t-1} - a_t$). From equations (21) and (22) we see, that in equilibrium the banks bid for liquidity that equates the expected change in the tender rate with the probability weighted costs of using the standing facilities today and the dynamic cost factor. As the optimal bidding at t is a function of future optimal bidding (implicitly given by eOB_j^*), the equilibrium liquidity eOB_t^* must be calculated recursively using backward induction. This means, that we must first solve OB_T^* as a function of RDB_T (which is known at T), and use this to solve for eOB_{T-1}^* as a function of RDB_{T-1} and so on. Thus, while deciding on its bid at t, a bank must calculate the optimal path of reserve holdings for all days remaining in the current maintenance period.

If the banks have fulfilled their reserve requirement for the whole maintenance period already before t ($RDB_t = RDB_{t+1} = ... = RDB_T = 0$), the extra borrowing does not affect the future uncertainty anymore, as the dynamic cost factor becomes zero. Thus, the rest of the period will be similar to the case without averaging, and the equilibrium bidding is defined simply by:

$r_t^T - \mathrm{E}_t\left[r_f^T\right] = \left(r_t^m - \mathrm{E}_t\left[r_f^T\right]\right)G(-eOB_t^*) + \left(r_t^d - \mathrm{E}_t\left[r_f^T\right]\right)\left[1 - G(-eOB_t^*)\right]$ or $G(-eOB_t^*) = \frac{r_t^T - r_t^d}{r_t^m - r_t^d}$.

To see the effect the averaging provision has, we are interested in cases where RDB_t is strictly positive. If $RDB_t > 0$, the dynamic cost factor is negative (we know, that $\frac{\partial RDB_j}{\partial RB_t} < 0$ and $\frac{\partial eOB_j^*}{\partial RDB_j} > 0$). Therefore, the dynamic cost factor always encourages the banks to postpone the holding of reserves.

3-day maintenance period as an example

To get some intuition of the optimal borrowing determined by (21), let us consider the very simplest case in which the *dcf* is present. Assume T=3 (or equivalently $t = T - 2$) and the liquidity shocks are normally distributed ($\mu_t \sim$N$(0, \sigma_\mu^2)$and $\varepsilon_t \sim$N$(0, \sigma_\varepsilon^2) \Rightarrow \nu_t \sim$N$(0, \sigma_\nu^2)$). Now, the equilibrium equation at t=1 becomes:

$$\begin{aligned} & r_1^T - \mathrm{E}_1\left[r_2^T\right] = (r_1^m - \mathrm{E}\left[r_f^T\right])N(-eOB_1^*) \\ & + \left(r_1^d - \mathrm{E}\left[r_f^T\right]\right)\left[1 - N(3RDB_1 - eOB_1^*)\right] \\ & + \left[N(3RDB_1 - eOB_1^*) - N(-eOB_1^*)\right] \\ & \times \left\{ \mathrm{E}\left[r_2^m - r_f^T\right]\left(-\frac{1}{2}\right)\frac{\partial eOB_2^*}{\partial eRDB_2}N(-eOB_2^*) \right. \\ & \left. + \mathrm{E}\left[\left(r_2^d - r_f^T\right)\right]\left(1 - \frac{1}{2}\frac{\partial eOB_2^*}{\partial eRDB_2}\right)\left[1 - N(2eRDB_2 - eOB_2^*)\right] \right\} \end{aligned} \tag{23}$$

Where N$(\cdot)$ is the cumulative distribution function of the normally distributed aggregate shock. The dynamic cost factor (ie the third term on the RHS) is always negative. Under neutral or decreasing interest rate expectations the LHS of the equation is non-negative (ie $r_1^T - \mathrm{E}_1\left[r_2^T\right] \geq 0$). Hence, with such expectations the banks should aim at liquidity that will leave the probability weighted cost of using the marginal lending facility lower than the probability weighted cost of using the deposit facility today.

In case of neutral interest rate expectations ($\mathrm{E}\left[r_f^T\right] = r_1^T$, which we denote as r^T), and symmetric interest rate corridor, we know, that $eOB_2^* = eRDB_2$ (thus $\frac{\partial eOB_2^*}{\partial eRDB_2} = 1$). We also know, that $G(-OB) = 1 - G(OB)$ for symmetric shock distributions. Thus, $1 - N(2eRDB_2 - eOB_2^*) = N(-eOB_2^*)$, and we can write equation (23) as:

$$\begin{aligned} 0 &= (r_1^m - r^T)\mathrm{N}(-eOB_1^*) + \left(r_1^d - r^T\right)\left(1 - \mathrm{N}(3RDB_1 - eOB_1^*)\right) \\ & + \left[\mathrm{N}(3RDB_1 - eOB_1^*) - \mathrm{N}(-eOB_1^*)\right](-\frac{1}{2})\mathrm{E}\left(r_2^m - r_2^d\right)\mathrm{N}(-eRDB_2) \end{aligned} \tag{24}$$

This can be further reduced under the symmetric interest rate corridor (ie $r^m - r^T = -(r^d - r^T) = 0.5(r^m - r^d)$) to:

$$\begin{aligned} & \mathrm{N}(-eOB_1^*) - \mathrm{N}(-3RDB_1 + eOB_1^*) = \\ & \mathrm{N}(-\frac{3RDB_1 - eOB_1^*}{2})\left[\mathrm{N}(3RDB_1 - eOB_1^*) - \mathrm{N}(-eOB_1^*)\right]. \end{aligned} \tag{25}$$

Equation (25) says, that with equilibrium bidding the difference between the probabilities of overdrawing and being forced to use the deposit facility today, will equal the probability of overdrawing tomorrow after not being forced to use the standing facilities today. We could easily solve equation (25) for the equilibrium liquidity (hence also for the equilibrium bidding), if we knew the variances of the shock distributions. In table 1, we have calculated the equilibrium overnight balances for different variances in the shock distribution as well as for three different interest rate expectations. Here we have assumed that the reserve requirement is 100 units. We also assume that when the banks expect the central bank to change its tender rate, they expect it to do so by 0.25 %-points between the first and the second tenders ($3\% \rightarrow 3,25\%$ or $3\% \rightarrow 2,75\%$). Furthermore,we assume, that the corridor is expected to be symmetric during the remaining period (the assumed width 4%):

Table 1. **Equilibrium liquidity at different levels of uncertainty**

$\sigma_{\mu+\varepsilon} \setminus eOB_1^*$	neutral expect.	incr. expect.	decr. expect.
10	94	276	12
20	100	252	24
50	101	181	60

Table 1 illustrates us the fact, that the equilibrium liquidity is a function of both interest rate expectations and the distribution of the liquidity shocks (when the standard deviation of the shocks is normally distributed). If banks are expecting the central bank rates to be constant, the equilibrium bidding will leave the market with less reserves the smaller is their volatility. Intuitively this means that the more certain they can be on their end of day balances, the more they can postpone adjusting the reserves maintenance, and thus lower the cost of future uncertainty (ie probability of having to rely on the standing facilities in the future).

The equilibrium liquidity is, however, much more affected by interest rate expectations than by the volatility. If the banks expect a rate rise (rate cut) they will try to front- (back-) load the reserves. *The lower is the volatility of the liquidity, the more front- or backloading will occur.* This again is natural, as the more certain you the evolution of reserves, the greater the incentive to take advantage on the expected difference between today's and expected future values of the overnight rate.

Figures 8, 9 and 10 demonstrate the determination of the overnight rate on the first day of a three-day maintenance period. When drawing these figures, we have assumed, that the reserve requirement is 100/day (ie also $RDB_1 = 100$), and that the two daily liquidity shocks are both normally distributed with zero mean, the standard deviation of the first shock is 10 and that of the second is 20. In figures 9 and 10 the tender rate is expected to be changed by 25 basis points (bps) from the starting value of 3%. The thicker and lighter curves illustrate the demand for liquidity at the tenders, whereas the demand during the overnight market clearance is given by the thinner darker curves.

From figure 8 we see, that the first shock of the day must be very large compared with the total liquidity to make the overnight rate deviate significantly from the tender rate. The equilibrium liquidity is at the level of the required daily balances for the remaining maintenance period. The interest rate elasticity of the demand for liquidity seems to be large at liquidity levels from around 0.5RDB up to 2RDB.

Figure 9 shows us the case, where the banks expect the central bank rates to be decreased by 25 bps. The expectations will affect the demand for reserves

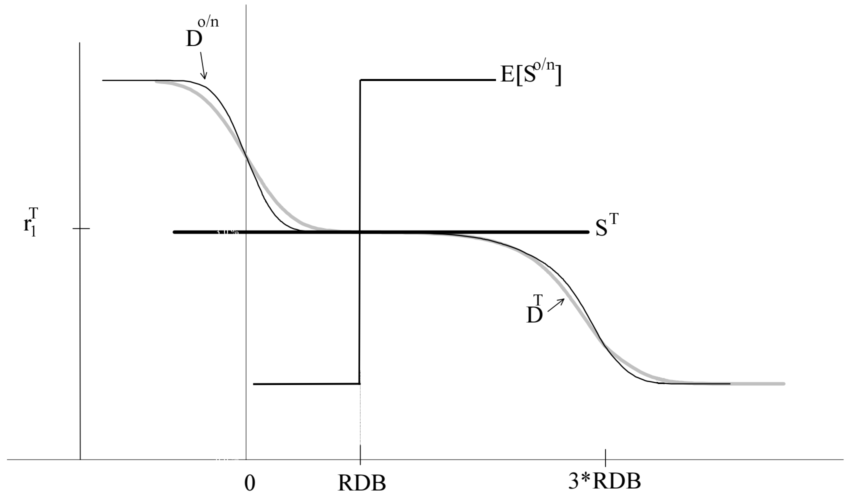

Figure 8: Determination of overnight rate on the first day of a 3-day maintenance period; neutral interest rate expectations

heavily. The banks will try to postpone the reserve holding to the second day, when it will be cheaper to hold them. The interest rate elasticity of the demand is much less at the equilibrium liquidity than it was in case of neutral expectations. Now, the value of the overnight rate expected at the tender (ie the tender rate) is higher than the overnight rate would be, if the first shock equals its expected value (ie if $\mu = 0$). This again results from the combination of the convexity of the demand at these low levels of liquidity, and the fact that some of the uncertainty has faded away between the tender operation and clearance of the overnight market.

Figure 10 illustrates the opposite case, where the banks expect the central bank to increase its rates. In this case the banks will frontload liquidity, as its price on tomorrow is expected to be higher. We will see from the figure, that the difference between the expected overnight rate and the overnight rate at expected liquidity is smaller in this case than if a rate cut were expected. This result comes from the fact, that the dynamic cost factor is larger at high levels of liquidity. Hence, the demand functions are not symmetric around their inflection points. Due to the *dcf* the demand function is more convex at low liquidity than it is concave at high levels of liquidity.

The effect the liquidity volatility has on the equilibrium liquidity is illustrated by figure 11. It shows us how the equilibrium liquidity decreases from $eOB' = 266$ to $eOB = 232$, as the standard deviation of the liquidity shock is doubled from 20 to 40 (the darker demand curve is based on the higher standard deviation).[44] Thus, the magnitude of frontloading (with increasing expectations) clearly depends on the liquidity volatility. Similar effects could be illustrated for neutral and decreasing interest rate expectations.

The reserve requirement defines directly the minimum daily balances for the remaining period at the first day of the reserves maintenance period (ie

[44] The demand curves here are based on normally distributed liquidity shocks, the amount of reserve requirement is 100, and the tender rate is expected to be raised from 3% to 3.5%.

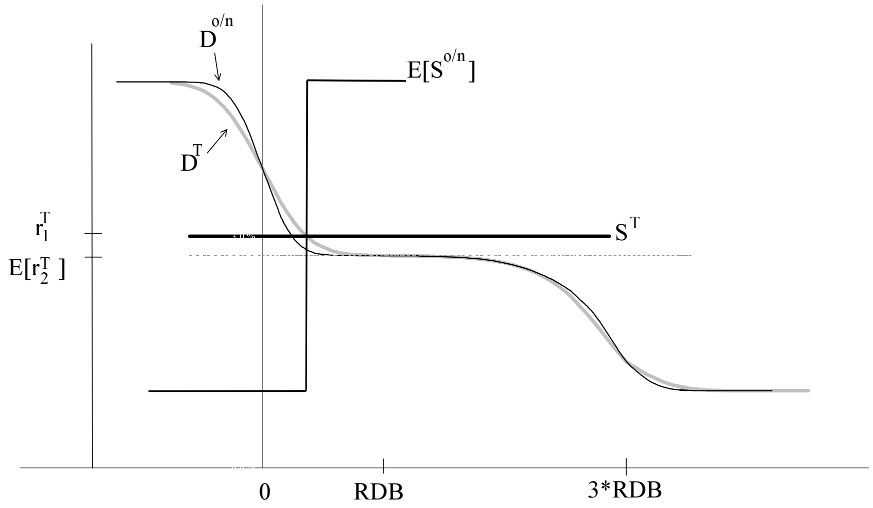

Figure 9: Determination of overnight rate on the first day of a 3-day maintenance period; decreasing interest rate expectations

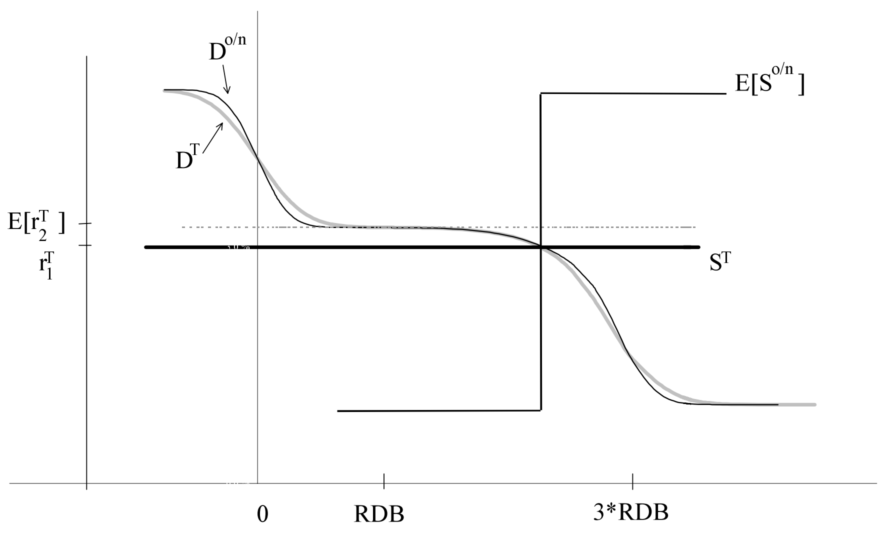

Figure 10: Determination of overnight rate on the first day of a 3-day maintenance period; increasing interest rate expectations

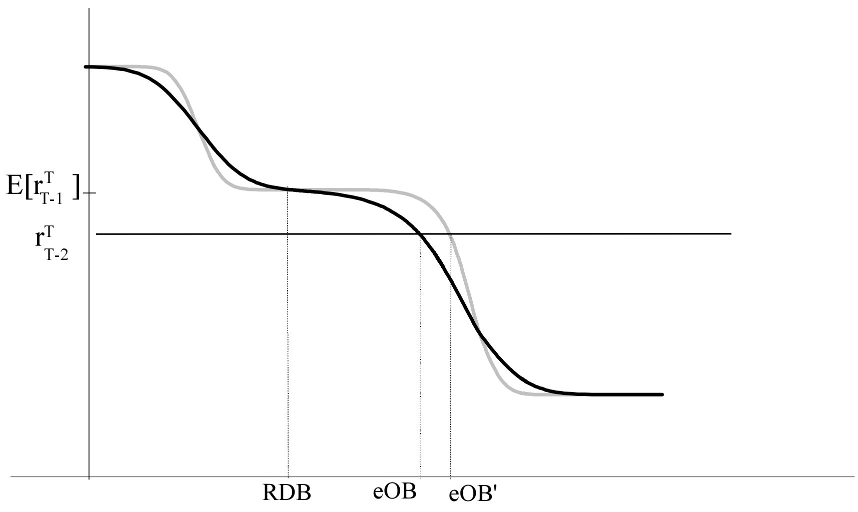

Figure 11: The effect of an increase in the volatility of liquidity shocks on the equilibrium liquidity

$RBD_1 = RR$). As the equilibrium liquidity eOB_1^* is a function of the liquidity uncertainty, we notice that eOB_1^*/RBD_1 is decreasing in RR (at least as long as the shock distribution is independent of the reserve requirement). Thus, whether the banks with neutral interest rate expectations will be front- or backloading reserves at the beginning of the maintenance period depends on the size of the reserve requirement compared to the liquidity volatility. Now, if the equilibrium liquidity at $t = 1$ is larger than the reserve requirement ($eOB_1^*/RBD_1 > 1$, ie the banks are frontloading reserves at the beginning of the period), we expect RDB_2 to be smaller than RDB_1. Therefore, $eOB_2^*/eRBD_2$ will be larger than eOB_2^*/RR. Thus, if $eOB_1^*/RBD_1 > 1$ we expect the overnight balances to decrease on the following days (as $\frac{\partial eOB_j^*}{\partial eRBD_j} > 0$), however, we do not expect the frontloading of the reserves to disappear (as $\frac{\partial\left(eOB_j^*/eRBD_j\right)}{\partial eRBD_j} < 0$). Similarly, if the initial reserve requirement is low (s.t. $eOB_1^*/RBD_1 < 1$), we expect the equilibrium liquidity to increase as time passes, but we do not expect the backloading of reserves to disappear (if the interest rate expectations do not change).

Figure 12 illustrates the effect the size of the reserve requirement has on the equilibrium liquidity. The darker demand curve is based on reserve requirements of 100, whereas the lighter is based on that of 200. Both demand curves assume the standard deviation of liquidity shocks to be 25, and the tender rate to be raised from 3% to 3.5%. Here, the $eOB = 2.6RDB$ while $eOB' = 2.8RDB'$, which illustrates us the fact that eOB_t/RDB_t is increasing in RR.

To sum up the findings of this section we may conclude that, if the monetary policy framework includes averaging provision in the reserve holding, and if the central bank uses a full allotment procedure in liquidity provision, the following will hold:

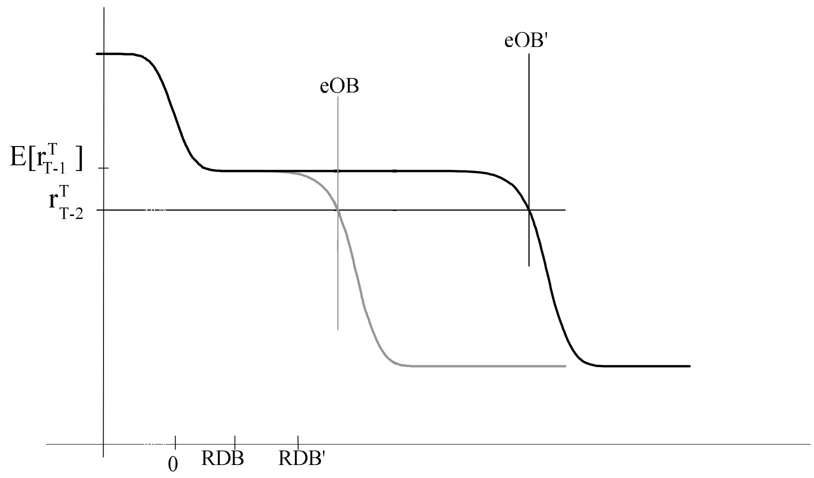

Figure 12: The effect of the size of reserve requirement on the equilibrium liquidity

1. The expected value of the overnight rate of interest will equal the tender rate expected for that day.

2. The timing of the reserve holdings within the maintenance period depends firstly, on interest rate expectations of the banks, but also on the distribution of liquidity shocks and on the size of the reserve requirement. The central bank can affect the timing of the reserve holding and the banks' possibility of doing intraperiod arbitrage by choosing the width of the interest rate corridor and the position of the tender rate within the corridor.

3. The volatility of the overnight rate depends on interest rate expectations. If the banks have neutral expectations the interest rate elasticity of the demand for liquidity is expected to be very large, ie the stochastic liquidity shocks are not expected to swing the overnight rate far from the tender rate. However, the demand will become less elastic as the equilibrium liquidity moves (due to expected changes in central bank rates) towards zero or fulfillment of the whole requirement.

4. The value of the overnight rate at the expected overnight liquidity might be geared towards the expected new tender rate, as some of the liquidity uncertainty vanishes between the tender operation and the clearance of the overnight market. The size of this effect depends on the amount of uncertainty resolved before the clearance (ie how large the difference between demand for reserves at the tender and at the interbank market clearance is). The earlier the market clears the smaller this effect will be. Thus, if the interbank market is active throughout the day, the volatility of the overnight rate is expected to be smaller. Also this effect will be more evident, if a rate cut is expected.

5. The variations in the demand for liquidity and in the volatility of the interbank overnight rate of interest are largely the result of changes in the demand for reserves due to expected movements in the central bank rates. This volatility could be avoided by timing the changes in official interest rates. If the central bank chose to adjust its rates only at the first tender operation of each maintenance period, the speculative demand for reserves would vanish, and the overnight rate would be very stable around the tender rate (at least before the last day of the maintenance period).

Some qualifications on the model Expectations of changes in the central bank rates will produce heavy variations in the equilibrium liquidity in the model presented above. The effect of the rate changes is likely to be much more moderate, if we introduce market imperfection into the model. Eg if the banks faced collateral requirements in the central bank lending and line limits in interbank dealing, the banks' incentive to deviate from a path of steady reserve holdings could be diminished substantially. A similar effect would rise, if the banks were risk averse in the sense that they are not interested only in maximizing the expected profits, and the volatility of profits also entered into the utility function. This kind of risk aversion could lower the banks' willingness to speculate on the (uncertain) future rate changes by front- or backloading reserves.

4.2 Proportional allotment

Besides the full allotment procedure, there are several alternative rules, that the central bank can use for liquidity allotment in fixed rate tenders. Here, we will concentrate on two simple policy rules, to keep the analysis of this section manageable. According to the first rule, the central bank tries to minimize the variations in the money market liquidity (*liquidity targeting*). In this approach the central bank could provide the markets at t with liquidity, that would *either* bring the expected required daily balances for the remaining period to the reserve requirement (ie with the targeted liquidity (eOB_t^{CB}) we would have $RDB_{t+1} = RR$) *or* it could provide the markets simply with RDB_t (with targeted liquidity we would have $RDB_{t+1} = RDB_t$). Ie the amount of liquidity allotted by the central bank aims at minimizing the variations in liquidity either for the whole period or for the rest of the period. The difference between these two policies is very small. Here, we will assume, that *in liquidity targeting the central bank aims at always providing the markets with liquidity s.t.* $eOB_t^{CB} = RDB_t$.[45]

The alternative policy rule studied here is that the central bank tries to provide the market with liquidity that will keep the (expected) overnight rate as close to a target value (set by the central bank itself) as possible (*interest*

[45] Thus, we assume, that the central bank minimizes the liquidity variations for the rest of the period (by this procedure $eOB_t = eOB_{t+1} = ... = eOB_T$), and does not try to counter the effect of previous liquidity shocks in new operations.

rate targeting). This target value (r_t^{targeted}) may or may not equal the tender rate.

The demand for overnight balances at the clearance of the market is a function of the money market liquidity, shock distribution and both current and expected future central bank rates. Hence, it does not depend on the approach that was used in the allotment of liquidity in tender operation. Thus, if we substitute the expected value of the future overnight rate for the expected value of the future tender rate in equations (6), (15) and (20), we will have the equations determining the overnight rate at T, $T-2$ and $T-j$ ($j = 2, ..., T-1$) respectively. The substitution is needed, as the expected overnight rate of a given date does not need to equal the expected tender rate with the proportional allotment procedure. We saw already in chapter 3, proportional allotment reduces to full allotment, if the demand for liquidity at the (fixed rate) tender does not exceed the amount the central bank is willing to provide the markets with. Thus, we are now mainly interested in cases where the expected value for the overnight rate equals or exceeds the tender rate. In such a case the banks will be increasing their bids from the optimal level under full allotment (ie they will be overbidding) in order to profit from the expected difference in the price of liquidity in the tender operation and in the interbank market.

4.2.1 Liquidity targeting

The expected value of the overnight rate of interest on the last day of the maintenance period is given by $\mathrm{E}[r_T^{on}] = r_T^m G(-eER_T) + r_T^d [1 - G(-eER_T)]$. With *liquidity targeting* we know (by definition), that $eER_T = eOB_T^{CB} - RDB_T = 0$ as long as the central bank is able to allot liquidity according to its target. To limit the number of cases we have to study, we assume from now on that the shock distributions are symmetric (unless otherwise mentioned). Hence, $G(0) = 0.5$ and the expected value for the last day's overnight rate will equal the mid-point of the interest rate corridor ($\frac{r_T^m + r_T^d}{2}$), if the banks place enough bids in the tender.

In section 3.2.2 we saw that the banks are overbidding[46], if the expected value of the last day's overnight rate is not below the tender rate. Thus, the central bank will receive enough bids and consequently is able to control the supply of the daily liquidity, if the last day's tender rate is not in the upper part of the corridor (ie $r_T^T \leq \frac{r_T^m + r_T^d}{2}$). If the rate was in the upper part (ie $r_T^T > \frac{r_T^m + r_T^d}{2}$), the banks could be able to make positive profit by lowering their bids below the liquidity targeted by the central bank. In this case, the central bank would not receive enough bids relative to its target, and the equilibrium would be determined as in the case of full allotment. Therefore, the expected

[46] Ie the actual bid amount is greater than the optimal bid under full allotment procedure. The optimal bid under the full allotment procedure is referred to also as the real liquidity demand of the bank.

value for the last day's overnight rate is the higher of the tender rate and the mid-point of the corridor:

$$E\left[r_T^{on}\right] = \max\left(\frac{r_T^m + r_T^d}{2}, r^T\right). \tag{26}$$

From now on, we assume , that the central bank uses a symmetric interest rate corridor (ie $r_t^T = \frac{r_t^m + r_t^d}{2}$) while operating with liquidity targeting policy. Hence, the expected overnight rate on the last day of the maintenance period will naturally equal the tender rate.[47] The reason for assuming a symmetric corridor being used with liquidity targeting is based on the following facts. First, the central bank would not be able to meet its target, if the tender rate were in the upper part of the corridor (as we saw above). Secondly, if the tender rate were in the lower part of the corridor, i) the stance of current monetary policy would be determined by the mid-point of the corridor instead of the fixed tender rate, and ii) as we will later see, this kind of situation would lead to infinite bidding by the banks, and into windfall gains to successful bidders.

Penultimate day (T-1)

With liquidity targeting, the expected amount of overnight balances of the banks at T-1 is RDB_{T-1}, as long as the central bank has control over daily money market liquidity (ie as long as the demand for liquidity in the tender exceeds the amount of reserves the central bank is willing to provide the markets with). Otherwise, the expected overnight balances will equal the optimal balances under the full allotment procedure (eOB_{T-1}^*). Let us define $z_t = \min(eOB_t^*, RDB_t)$, where z_t is the expected overnight balances at the clearance of the market under proportional allotment with liquidity targeting. Now, the following will hold for the overnight rate at the penultimate day of the maintenance period:[48]

$$\begin{aligned} r_{T-1}^{on} &= \mathrm{E}_{T-1}\left[r_T^T\right]\left[F(2RDB_{T-1} - z_{T-1} + \mu_{T-1}) - F(-z_{T-1} + \mu_{T-1})\right] \quad (27) \\ &\quad + r_{T-1}^m F(-z_{T-1} + \mu_{T-1}) + r_{T-1}^d\left[1 - F(2RDB_{T-1} - z_{T-1} + \mu_{T-1})\right]. \end{aligned}$$

Note, that we can use $\mathrm{E}_{T-1}\left[r_T^T\right]$ as the price of borrowing tomorrow, as $\mathrm{E}_{T-1}\left[r_T^{on}\right] = \mathrm{E}_{T-1}\left[r_T^T\right]$ both with liquidity targeting and in case of full allotment. The overnight rate on the penultimate day will again equal the probability weighted average of the central bank rates (expected tender rate, and current rates of the standing facilities). The expected value for the overnight rate at $T-1$ is now given by:

$$\begin{aligned} \mathrm{E}\left[r_{T-1}^{on}\right] &= \mathrm{E}\left[r_T^T\right] + \left(r_{T-1}^m - \mathrm{E}\left[r_T^T\right]\right) G(-z_{T-1}) \\ &\quad + \left(r_{T-1}^d - \mathrm{E}\left[r_T^T\right]\right)\left[1 - G(2RDB_{T-1} - z_{T-1})\right] \end{aligned} \tag{28}$$

By subsection 4.1 we know, that the relation between eOB_{T-1}^* and RDB_{T-1} depends on the interest rate expectations of the banks and the symmetry of

[47] If the shock distribution was not symmetric, the interest rate corridor should also be asymmetric for $r_T^m G(0) + r_T^d [1 - G(0)] = r^T$ to hold.

[48] For the determination of equation (27), see from appendix B how equation (15) is derived from the profit maximization problem of a single bank.

the corridor around the tender rate. Here, we have assumed the interest rate corridor to be symmetric. Thus, according to subsection 4.1 we expect to see: i) $eOB^*_{T-1} > RDB_{T-1}$, if the overnight rate is expected to increase during the last two days of the maintenance period, ii) $eOB^*_{T-1} = RDB_{T-1}$, if the rate is expected to remain constant, and iii) $eOB^*_{T-1} < RDB_{T-1}$, if the banks expect the overnight rate to decrease.[49] So, we expect the central bank to be in a position to allot the targeted amount as long as the banks do not expect the overnight rate to decrease between T-1 and T.

Let us assume for a moment, that the central bank does have the control over the daily liquidity supply (ie $eOB^*_{T-1} \geq RDB_{T-1} \Rightarrow z_{T-1} = RDB_{T-1}$). The expected overnight rate at T-1 will be given by:

$$\begin{aligned} \mathrm{E}\left[r^{on}_{T-1}\right] &= \mathrm{E}\left[r^T_T\right] + \left(r^m_{T-1} - \mathrm{E}\left[r^T_T\right]\right) G(-RDB_{T-1}) \\ &\quad + \left(r^d_{T-1} - \mathrm{E}\left[r^T_T\right]\right)\left[1 - G(RDB_{T-1})\right], \end{aligned} \tag{29}$$

which can also be written as the difference of the expected overnight rate for today and that of the following banking day:

$$\mathrm{E}\left[r^{on}_{T-1}\right] - \mathrm{E}\left[r^T_T\right] = \left(\frac{r^m_{T-1} + r^d_{T-1}}{2} - \mathrm{E}\left[r^T_T\right]\right) 2G(-RDB_{T-1}). \tag{30}$$

The expected value for the overnight rate at $T-1$ will equal the tender rate expected to be used in the last tender (which we have assumed to be in the middle of the interest rate corridor for the last day) only, if the banks have neutral expectations on the central bank rates (in that case $r^T_{T-1} = \frac{r^m_{T-1}+r^d_{T-1}}{2} = \frac{r^m_T+r^d_T}{2} = \mathrm{E}\left[r^T_T\right]$). This results from the fact that, if the banks are expecting the central bank to increase its rates, the term $\frac{r^m_{T-1}+r^d_{T-1}}{2} - \mathrm{E}\left[r^T_T\right]$ will become negative (RHS<0), and consequently the expected overnight rate has to be smaller than the expected tender rate to be used in the last operation.[50] Similarly, if the central bank is expected to cut its rates, the overnight rate is expected to decrease during the last two days of the period. This means, that our assumption of the central bank being in control of the daily liquidity would be correct, and that the overnight rate indeed given by equation (29), if the central bank rates are expected to remain constant or to be raised.[51] However, if a rate cut is expected, the liquidity allotment is not determined by the central bank's target. In such a case, the

[49] If the interest rate corridor were not symmetric, with neutral interest rate expectations we would expect $eOB^*_{T-1} > RDB_{T-1}$, if the tender rate were in the upper part of the corridor, and $eOB^*_{T-1} < RDB_{T-1}$, if it were in the lower part of the corridor.

[50] The expected value for overnight rate would equal today's tender rate, if $RDB_{T-1} = 0$. However, it is extremely unlikely, that the liquidity shocks could bring RDB_{T-1} down to zero, if the central bank uses liquidity targeting.

[51] We have just shown, that $\mathrm{E}_{T-1}\left[r^{on}_{T-1}\right] < \mathrm{E}_{T-1}\left[r^{on}_T\right]$, if the banks expect the central bank either to keep its rates constant or to increase them. With reasoning similar to section 3.2.2, we know, that in this case the banks will be overbidding (i.e. $eOB^*_{T-1} < eOB^{actual}_{T-1}$). This will further enhance the fact that the central bank has control over the expected liquidity, if the expectations are either neutral or increasing.

equilibrium liquidity and the expected overnight rate would be (similarly to the case with full allotment) $z_{T-1} = eOB^{*}_{T-1}$ and $\mathrm{E}\big[r^{on}_{T-1}\big] = \mathrm{E}\big[r^{T}_{T-1}\big]$.[52]

If we modify equation (30) slightly, we easily see that when the central bank can control the liquidity, the expected overnight rate will be between today's tender rate and that of expected for tomorrow ($r^{T}_{T-1} \leq \mathrm{E}\big[r^{on}_{T-1}\big] < r^{T}_{T}$).[53] We also see, that this rate approaches asymptotically the expected tender rate for T, as RDB_{T-1} increases (ie as the probability of going overdrawn diminishes). We know, that under liquidity targeting, the expected value for RDB_{T-1} is the average reserve requirement (RR). If the aim of averaging the reserve requirement is to increase the interest rate elasticity of the demand for reserves, we might expect, that the central bank sets RR well above the average size of a stochastic liquidity shock. Thus, we expect most of the interest rate expectations to be absorbed by the overnight rate already at $T-1$. For example, if the standard deviation of *normally distributed* liquidity shocks is 25% of the average liquidity (ie the reserve requirement), the expected value of overnight rate at $T-1$ will absorb more than 99% of the expected change in the tender rate.

Now, we may conclude, that *with interest rate targeting the expected value for overnight rate at the second last day equals the current tender rate, if the banks have neutral or decreasing interest rate expectations, and is very close to the expected tender rate of the last operation, if a rate rise is expected.*

Figure 13 illustrates us the determination of overnight rate, when the central bank uses a proportional allotment procedure with liquidity targeting, and when the banks expect a rate rise. The figure shows us, that the expected overnight rate at $T-1$ will be very close to the expected tender rate for tomorrow. We also see, that the money market equilibrium is expected to be found from the highly elastic part of the demand curve. Thus, the variations in the overnight rate reflect changes in interest rate expectations rather than the effect of stochastic liquidity shocks.

Earlier days (1, ... ,T-3, T-2)

In case of full allotment, there was a considerable difference in the determination of money market equilibrium between the penultimate day of the maintenance period and the days before that. This difference occurs because

[52] If the interest rate corridor was asymmetric, the central bank would be able to control the supply of daily liquidity as long as i) the probability weighted average of the mid-point of the corridor is not less than the current tender rate (i.e. $\mathrm{E}\big[r^{mid}_{T}\big]\left[1-2G\left(-RDB_{T-1}\right)\right] + r^{mid}_{T-1}2G\left(-RDB_{T-1}\right) \geq r^{T}_{T-1}$) when the tender rate is kept in the lower part of the corridor, or ii) the probability weighted average of the expected future tender rate and the current mid-point is not less than the current tender rate (i.e. $\mathrm{E}\big[r^{T}_{T}\big]\left[1-2G\left(-RDB_{T-1}\right)\right] + r^{mid}_{T-1}2G\left(-RDB_{T-1}\right) \geq r^{T}_{T-1}$) when the tender rate is kept in the lower part of the corridor.

[53] By using the fact, that with symmetric interest rate corridor $r^{T}_{T-1} = \frac{r^{m}_{T-1}+r^{d}_{T-1}}{2}$, we can write equation (30) as:

$$E\left[r^{on}_{T-1}\right] - r^{T}_{T-1} = \left(E_{T-1}\left[r^{T}_{T}\right] - r^{T}_{T-1}\right)\left[1-2G(-RDB_{T-1})\right] \tag{31}$$

For expectations of increased interest rate, the RHS of equation (31) must be non-negative, as $2G\left(-RDB_{T-1}\right) \leq 1$ (assuming symmetric shock distribution we have $G(0) = 0.5$ and $RDB_{T-1} \geq 0$).

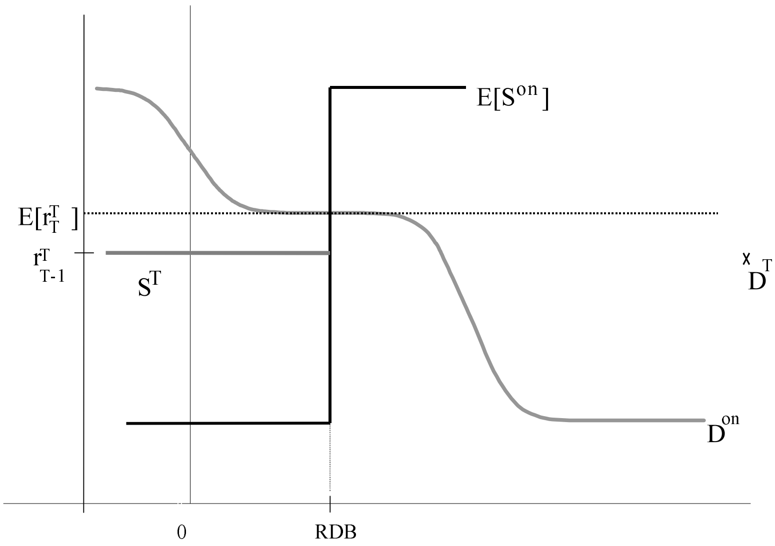

Figure 13: Determination of overnight rate at $T-1$; proportional allotment with liquidity targeting and increasing interest rate expectations

in the early part of the period, the banks have to take into account the effect their liquidity holdings have on the required daily balances for the remaining period on the following days, where as on the penultimate day this dynamic cost factor is absent. Also here the *dcf* affects the demand for reserves on the earlier days. However, this effect will affect the amount of liquidity provided to the markets only if the central bank is not able to control the (daily) amount of reserves to be allotted to the markets. As on the penultimate day of the maintenance period, the banks will be overbidding ($eOB_t^{actual} > eOB_t^*$) at t, if the expected overnight rate is not lower with the required minimum daily balances than with the equilibrium liquidity under full allotment (ie overbidding occurs if $\mathrm{E}[r_t^{on}|_{eOB_t=RDB_t}] \geq \mathrm{E}\left[r_t^{on}|_{eOB_t=eOB_t^*}\right] = r_t^T$). Thus, the control over the daily liquidity supply is in the hands of the central bank, as long as the demand for reserves in a tender under full allotment would be at least equal to the RDB_t (as the overnight rate is a monotonically increasing function of the liquidity). Now, if the control over the supply of reserves is in the hands of the central bank, the overnight rate at t (obtained once again as the first order condition of the banks' profit maximizing problem) will be:[54]

[54]For the derivation of equation 32 see from appendix C how equation 19 was derived as first order condition of the profit maximization problem of a bank. Note, that here we have to substitute eOB_j^* for $eRDB_j$ as the equilibrium liquidity is determined by the central bank target instead of the optimal demand of the banks. Thus, we also replace $\frac{\partial eOB_j^*}{\partial b_t}$ by $\frac{\partial eRDB_j}{\partial b_t}$.

$$
\begin{aligned}
r_t^{on} =& \mathrm{E}_t\left[r_f^{on}\right] + \left(r_t^m - \mathrm{E}_t\left[r_f^{on}\right]\right) F(-OB_t) \\
&+ \left(r_t^d - \mathrm{E}_t\left[r_f^{on}\right]\right) \{1 - F\left[(T-t+1)RDB_t - OB_t\right]\} \\
&+ \sum_{j=t+1}^{T-1} \left(\mathrm{E}\left[r_j^m - r_f^{on}\right] G(-eRDB_j) + \mathrm{E}\left[r_j^d - r_f^{on}\right](-T+j)\right. \\
&\left.\times \{1 - G\left[(T-j)eRDB_j\right]\}\right) \frac{\partial eRDB_j}{\partial b_t},
\end{aligned} \tag{32}
$$

where $\mathrm{E}_t\left[r_f^{on}\right]$ is the expected overnight rate for the remaining days within the same period. The expected value for the overnight rate will be given by (note that $eOB_t = RDB_t$):

$$
\begin{aligned}
\mathrm{E}_t\left[r_t^{on}\right] = & \max\left[r_t^T, \mathrm{E}_t\left[r_f^{on}\right] + \left(r_t^m - \mathrm{E}_t\left[r_f^{on}\right]\right) G(-RDB_t)\right. \\
&+ \left(r_t^d - \mathrm{E}_t\left[r_f^{on}\right]\right) \left[1 - G((T-t)RDB_t)\right] \\
&+ \sum_{j=t+1}^{T-1} \left(\mathrm{E}\left[r_j^m - r_f^{on}\right] G(-eRDB_j) + \mathrm{E}\left[r_j^d - r_f^{on}\right](-T+j)\right. \\
&\left.\left.\times \{1 - G\left[(T-j)eRDB_j\right]\}\right) \frac{\partial eRDB_j}{\partial b_t}\right]
\end{aligned} \tag{33}
$$

That is, the expected value of the overnight rate is given by taking the expected value of equation (32) as long as this produces a rate that is not lower than the tender rate. Otherwise the expected overnight will equal the tender rate as in the case of full allotment. Equation (32) shows, that if the central bank can allot liquidity according to its target, the expected overnight rate at t will equal the expected future overnight rate, current marginal lending rate and the deposit rate all weighted by the probabilities of their occurrence,[55] and the dynamic cost factor. If the central bank has the control over the expected liquidity, the liquidity targeting rule indicates, that $eRDB_{t+i} = eRDB_{t+2} = ... = eRDB_{T-1}$. Thus, $\frac{\partial eRDB_{t+1}}{\partial b_t} = -\frac{1}{T-t}\{G\left[(T-t+1)RDB_t - OB_t\right] - G(-OB_t)\}$, and equation (33) can be further modified into:

$$
\begin{aligned}
\mathrm{E}_t\left[r_t^{on}\right] = & \max\left[r_t^T, \mathrm{E}_t\left[r_f^{on}\right] + \left(r_t^m - \mathrm{E}_t\left[r_f^{on}\right]\right) G(-RDB_t)\right. \\
&+ \left(r_t^d - \mathrm{E}_t\left[r_f^{on}\right]\right) \{1 - G\left[(T-t)RDB_t\right]\} \\
&+ \{G\left[(T-t+1)RDB_t - OB_t\right] - G(-OB_t)\} \left(-\frac{1}{T-t}\right) \\
&\times \sum_{j=t+1}^{T-1} \left(\mathrm{E}\left[r_j^m - r_f^{on}\right] G(-RDB_{t+1}) + \mathrm{E}\left[r_j^d - r_f^{on}\right](-T+j)\right. \\
&\left.\left.\times \ \{1 - G\left[(T-j)RDB_{t+1}\right]\}\right)\right].
\end{aligned} \tag{34}
$$

[55] Ie the lowest expected overnight rate weighted by the probability of not using the standing facilities, marginal lending rate weighted by the probability of being overdrawn, and the deposit rate weighted by the probability of fulfilling the whole reserve requirement.

For example, at $T-2$ the expected overnight rate will be determined as:

$$\begin{aligned}\mathrm{E}\left[r^{on}_{T-2}\right] = \; & \max\left\{r^{T}_{t}, \mathrm{E}_{T-2}\left[r^{on}_{T-1}\right] + \left(r^{m}_{T-2} - \mathrm{E}_{T-2}\left[r^{on}_{T-1}\right]\right) G(-RDB_{T-2})\right. \\ & + \left(r^{d}_{T-2} - \mathrm{E}_{T-2}\left[r^{on}_{T-1}\right]\right) G(-2RDB_{T-2}) \\ & -0.5\mathrm{E}\left[r^{m}_{T-1} - r^{d}_{T-1}\right] G(-eRDB_{T-1})[G\left(2RDB_{T-2}\right) \\ & \left.-G\left(-RDB_{T-2}\right)]\right\}\end{aligned} \tag{35}$$

which defines $\mathrm{E}\left[r^{on}_{T-2}\right]$ to be the higher of the current tender rate and the probability weighted average of the rates of the standing facilities and the expected future overnight rate, decreased by the dynamic cost factor. In this case the *dcf* is half of the width of the interest rate corridor weighted by the probability of being overdrawn tomorrow, if one did not have to rely on the standing facilities today.

The key motive for having an averaged reserve requirement is probably the effect it has on the interest rate elasticity of the demand for reserves. Thus, we may assume that the size of the requirement is set to be large compared with the standard deviation of liquidity. If this were the case, the probability of having to rely on either of the two standing facilities would be relatively low with liquidity being at the level of the required daily balances for the remaining period. Also, the effect the *dcf* has would be minimal in this case. Therefore, in case of increasing interest rate expectations, the expected value of the overnight rate for today (at the liquidity targeted by the central bank) would be very close to the value of the overnight rate expected to prevail in the remaining days of the maintenance period ($\mathrm{E}\left[r^{on}_{T-2}\right] \simeq \mathrm{E}\left[r^{T}_{T-1}\right] \Rightarrow \mathrm{E}\left[r^{on}_{T-2}\right] \simeq \mathrm{E}\left[r^{on}_{T-1}\right] \simeq \mathrm{E}\left[r^{T}_{T-1}\right]$ etc.). Consequently, the expected value of today's overnight rate would exceed the current tender rate, and the central bank would indeed be able to control the supply of overnight liquidity. Similarly, if the current tender rate were higher than the expected mid-point on the last day of the period (ie a rate cut is expected), the expected overnight rate with liquidity at the level targeted by the central bank would fall below the current tender rate. Hence, the demand for reserves in the tender operation would not be high enough for the central bank to be able to allot liquidity according to its target, and again determination of the money market equilibrium would follow the case of full allotment.

However, when the banks have neutral interest rate expectations, the optimal amount of liquidity the banks will bid for under the full allotment procedure is an increasing function of the liquidity volatility as we saw in section 4.1.2. If the volatility of liquidity is high compared to the equilibrium liquidity, the central bank will get enough bids to be in control of the daily supply of reserves, and the expected overnight rate could increase slightly over the tender rate. However, if the central bank has chosen the reserve requirement to be high compared with the volatility, we might expect the banks to be willing to backload the reserve holdings, and consequently the equilibrium would be determined as in the case of full allotment. Therefore, while choosing the size of the reserve requirement the central bank must take into account that a higher requirement will increase the interest rate elasticity of the demand for reserves, but it might also lower the central bank's ability to control the daily supply of liquidity. Thus, we expect that there is an upper limit for the reserve requirement the central bank can apply with liquidity targeting. The central bank might like to increase the interest rate elasticity of the demand for liquidity by increasing its reserve requirement, but only up to the point where it will still be in control of the daily supply of liquidity under neutral interest rate expectations.

We have seen that the expected value of the overnight rate will be at level close to that of the expected mid-point of the interest rate corridor throughout the maintenance period, if the central bank can control the expected supply of money market liquidity. The central bank has control over the expected supply of money market liquidity, if the interest rates are expected to be raised or with neutral interest rate and high enough liquidity uncertainty. If a rate cut is expected the expected overnight rate will be at the level of the current tender rate. Furthermore, as long as a rate cut is not anticipated by the banks, the equilibrium overnight rate is expected to be realized near the required daily balances for the remaining period, where the demand for reserves has a relatively high interest rate elasticity. Consequently, the variations in the overnight rate will largely reflect changes in expectations of the central bank rates in the near future. However, if a rate cut is expected, the banks will be backloading their reserve holdings, and the equilibrium overnight rate is to be found on the less elastic part of the demand curve (as in the case of full allotment). Thus, with decreasing interest rate expectations the volatility of the overnight rate is reflecting the stochastic variations in the money market liquidity.

4.2.2 Interest rate targeting

The determination of the money market equilibrium under proportional allotment with interest rate targeting has a lot of similar features not only to those of under liquidity targeting, but also to those of full allotment. The main difference between interest rate targeting and liquidity targeting is, that under interest rate targeting policy the amount of liquidity the central bank aims at providing the market with (eOB_t^{CB}) is implicitly derived from the central bank's interest rate target. Ie the central bank is willing to provide the market with reserve balances that will bring the expected value of the overnight rate to the level of the target $\mathrm{E}\left[r_t^{on}|_{eOB_t=eOB_t^{CB}}\right] = r_t^{\text{targeted}}$. The central bank would not have control over the daily supply of liquidity, if the expected value of the target were smaller than the expected overnight under full allotment (ie if $\mathrm{E}\left[r_t^{\text{targeted}}\right] < r_t^T$). If this were the case, the banks could make positive expected profits by lowering their bids from eOB_t^{CB} down to eOB_t^*.[56] Thus, the expected money market liquidity will be at the level chosen by the central

[56] The gain from a lower bid would be the price of liquidity at the tender (r_t^T), and the expected cost of it would be the price of liquidity at the interbank market ($\mathrm{E}\left[r_t^{\text{targeted}}\right]$). Thus, the expected gain from making a lower bid is positive as long as the representative bank bids according to its neutral strategy as in the case with full allotment procedure.

bank (eOB_t^{CB}) only, if $r_t^{\text{targeted}} = \mathrm{E}\left[r_t^{on}|_{eOB_t=eOB_t^{CB}}\right] \geq \mathrm{E}\left[r_t^{on}|_{eOB_t=eOB_t^*}\right] = r_t^T$.[57] Otherwise, the expected equilibrium liquidity under interest rate targeting will equal the equilibrium liquidity under full allotment (eOB_t^*).

To distinguish the properties typical for proportional allotment with interest rate target, we are from now on interested in cases where the target of the central bank is set (and is expected to be set) at least to the level of the tender rate $r_t^{\text{targeted}} \geq r_t^T$. Now, in this case we know, that $eOB_t^{CB} \leq eOB_t^*$. If the target rate equalled the tender rate, and the banks bid according to the neutral strategy, the money market liquidity at the clearance of the market would be the minimum of the two variables with the same mean, ie $\min(OB_t^{CB}, OB_t^*)$[58]. Thus, the overnight rate at t would be $r_t^{on}\left(\min\left(OB_t^{CB}, OB_t^{CB}\right)\right)$, which is above $r_t^{on}\left(eOB_t^{CB}\right) = r_t^{on}\left(eOB_t^{CB}\right) = r_t^T$. Ie the expected value for the overnight rate would be above the tender rate, if the banks bid according to neutral strategy. Therefore, the banks have an incentive to bid for more liquidity than their demand for it under the full allotment would be. In fact the equilibrium bidding strategy would be such that the bid amount should exceed the maximum value the central bank with neutral strategy can be expected to provide to the markets ($TL^{\text{bid amount}} \geq TL^{CB,\max}$).[59] With this kind of overbidding the central bank will always get bids for more reserves than it is willing to provide the markets with, and consequently the expected value for the overnight rate at t will equal the expected value of the central bank's target.

On the last day of the maintenance period, the overnight rate is again given by $r_T^{on} = r_T^m F(-ER_T) + r_T^d\left[1 - F(-ER_T)\right]$. Thus, the amount of liquidity to be allotted by the central bank at T will be implicitly given by:

$$\mathrm{E}\left[r_T^{on}\right] = r_T^m H(-eOB_T^{CB} + RDB_T) + r_T^d\left[1 - H(-eOB_T^{CB} + RDB_T)\right] = r_T^{\text{targeted}},$$

where $H = \mathrm{E}_{f_\mu}^{CB}\left[F(-eER_T^{CB})\right]$.[60] If the inverse of the expected cumulative distribution function $H^{-1}(\cdot)$ exists, the money market liquidity the central bank aims at when making its decision over the allotment will be given by:

$$eOB_T^{CB} = RDB_T - H^{-1}\left(\frac{r_T^{\text{targeted}} - r_T^d}{r_T^m - r_T^d}\right).$$

[57] $r_t^{\text{targeted}} \geq r_t^T \Leftrightarrow \mathrm{E}\left[r_t^{on}|_{eOB_t=eOB_t^{CB}}\right] \geq \mathrm{E}\left[r_t^{on}|_{eOB_t=eOB_t^*}\right] \Rightarrow OB_t^{CB} \leq eOB_t^*$, as r_t^{on} is monotonically decreasing with eOB_t. Based on our earlier argument we know, that if the liquidity the central bank wants to allot to the market is not smaller than the liquidity bid for under full allotment, the banks will be strategically overbidding (i.e. $OB_t^{CB} \leq eOB_t^* \Rightarrow TL^s < TL^* < TL^{actual}$).

[58] Even though eOB_t^{CB} and eOB_t^* share the same value on average they can differ from each other on daily basis, as the expectation over the development in autonomous liquidity factors may differ between the central bank and the banks (a_t^{CB} may be different from a_t).

[59] In case of the 76 ECB main refinancing operations that were conducted as fixed rate tenders, the allotment ratios were less than 10% of the total bid amount in 62 operations and below 5% in 29 operations. The ratios for tenders conducted in 2000 shows us even more dramatic overbidding; the allotment ratio was below 10% in every single tender and below 5% in all but three out of the 24 tenders.

[60] In $\mathrm{E}_{f_\mu}^{CB}[\cdot)]$ the central bank's expectations are taken over the shock distribution f_μ.

Now, the difference between the target rate and the actually realizing overnight rate results mainly from the stochastic liquidity shock emerging between the tender operation and the clearance of the market. However, part of the difference might come from the central bank's inability to estimate the real cumulative distribution of shocks (ie $H(\cdot) \neq F(\cdot)$). Also, the symmetry of the corridor around the central bank's target affects the distribution of the overnight rate realizations (ie $H(\cdot)$ (like $G(\cdot)$) has wider distribution than $F(\cdot)$).

Penultimate day (T-1)

On the penultimate day of the maintenance period, the overnight rate expected to prevail on the last day again enters into the equation determining the current rate. Thus, the central bank's allotment decision, eOB^{CB}_{T-1}, is implicitly given by:

$$\begin{aligned} r^{\text{targeted}}_{T-1} = \; & \mathrm{E}^{CB}\left[\mathrm{E}^{banks}_{T-1}\left[r^{\text{targeted}}_{T}\right]\right]\left[H(2RDB_{T-1} - eOB^{CB}_{T-1}) - H(-eOB^{CB}_{T-1})\right] \\ & + \left(r^{m}_{T-1}\right) H(-eOB^{CB}_{T-1}) + r^{d}_{T-1}\left[1 - H(2RDB_{T-1} - eOB^{CB}_{T-1})\right], \end{aligned} \tag{36}$$

where $\mathrm{E}^{CB}\left[\mathrm{E}^{banks}\left[r^{\text{targeted}}_{T}\right]\right]$ is the central bank's expectation for the last day's target rate anticipated by the banks at T-1. Equation (36) tells us, that the central bank should provide the markets with liquidity by which the probability weighted sum of the target rate the central bank anticipates the banks to expect for the last day and the rates of the standing facilities will equal today's target rate. The effect an expected change of the interest rate target has on the amount of liquidity to be allotted can be clearly seen, if we modify equation (36) slightly:

$$\begin{aligned} & r^{\text{targeted}}_{T-1} - \mathrm{E}^{CB}\left[\mathrm{E}^{banks}\left[r^{\text{targeted}}_{T}\right]\right] = \\ & \left\{\left(r^{m}_{T-1} - \mathrm{E}^{CB}\left[\mathrm{E}^{banks}\left[r^{\text{targeted}}_{T}\right]\right]\right) H(-eOB^{CB}_{T-1})\right\} \\ & + \left(r^{d}_{T-1} - \mathrm{E}^{CB}\left[\mathrm{E}^{banks}\left[r^{\text{targeted}}_{T}\right]\right]\right)\left[1 - H(2RDB_{T-1} - eOB^{CB}_{T-1})\right] \end{aligned} \tag{37}$$

From (37) we see, that the central bank allots liquidity in order to balance the expected change in the target rate with the probability weighted cost of using the standing facilities. The planned allotment is a decreasing function of the expected target (as long as the RDB_{T-1} is strictly positive, ie the probability of using the standing facilities is not one[61]), and a decreasing function of the location of the target rate within the interest rate corridor (ie the lower the target rate is within the corridor, the higher is the relative cost of going overdrawn, which is to be compensated with lower probability of going overdrawn associated with the larger liquidity).

With neutral interest rate expectations and a symmetric interest rate corridor, equation (37) tells us, that the liquidity provided by the central bank will equal the required daily balances for the remaining period

[61] If the probability of using the standing facilities were 1, we would have $RDB_{T-1} = 0 \Rightarrow H(-eOB^{CB}_{T-1}) = H(2RDB_{T-1} - eOB^{CB}_{T-1}) \Rightarrow r^{\text{targeted}}_{T-1} = r^{m}_{T-1}H(-eOB^{CB}_{T-1}) + r^{d}_{T-1}(1 - H(-eOB^{CB}_{T-1}))$, and the expected overnight rate for today would be independent of the expected target rate for tomorrow.

($eOB^{CB}_{T-1} = RDB_{T-1}$).[62] If the target rate were in the upper part of the corridor (ie if the cost of acquiring liquidity credit were lower than the (opportunity) cost of having to use the deposit facility), the liquidity provided by the central bank should leave the markets with a larger probability of overdrawing than in overfilling the reserve requirement (ie $eOB^{CB}_{T-1} > RDB_{T-1} \Rightarrow H(-eOB^{CB}_{T-1}) < 1 - H(2RDB_{T-1} - eOB^{CB}_{T-1})$). Similarly, if the target rate were in the lower part of the corridor, the markets should be provided with liquidity that will make going overdrawn less probable than overfilling the requirement ($eOB^{CB}_{T-1} < RDB_{T-1}$).

If the banks were anticipated to expect the central bank to increase the target rate, the central bank would aim at providing enough liquidity to equate the negative of the expected rate increase with the probability weighted difference between the rates of the standing facilities and the expected target rate, ie the central bank should provide the market with extra liquidity in order to make the RHS negative enough. Similarly, under decreasing anticipated expectations the central bank would offer the banks so little liquidity that it would bring the probability of overdrawing larger than the probability of overfilling the reserve requirement.

Earlier days (1, ... ,T-3, T-2)

In the earlier days of the maintenance period, the overnight liquidity targeted by the central bank is implicitly given by:[63]

$$
\begin{aligned}
& r_t^{\text{targeted}} - \mathrm{E}^{CB}\left[\mathrm{E}_t^{banks}\left[r_f^{\text{targeted}}\right]\right] = \left(r_t^m - \mathrm{E}^{CB}\left[\mathrm{E}_t^{banks}\left[r_f^{\text{targeted}}\right]\right]\right) H(-eOB_t^{CB}) \\
& + \left(r_t^d - \mathrm{E}^{CB}\left[\mathrm{E}_t^{banks}\left[r_f^{\text{targeted}}\right]\right]\right)\left\{1 - H([T-t+1)RDB_t - eOB_t^{CB}]\right\} \\
& \sum_{j=t+1}^{T-1}\left(\mathrm{E}^{CB}\left[\mathrm{E}_t^{banks}\left[r_j^m - r_f^{\text{targeted}}\right]\right]\frac{\partial eOB_j^{CB}}{\partial eRDB_j}\frac{\partial eRDB_j}{\partial b_t}H(-eOB_j^{CB})\right. \\
& + \mathrm{E}^{CB}\left[\mathrm{E}_t^{banks}\left[r_j^d - r_f^{\text{targeted}}\right]\right]\left(-(T-j+1)\frac{\partial eRDB_j}{\partial b_t} + \frac{\partial eOB_j^{CB}}{\partial b_t}\right) \\
& \left.\times\left\{1 - H\left[(T-j+1)eRDB_j - eOB_j^{CB}\right]\right\}\right) \qquad (38)
\end{aligned}
$$

This equation determining the expected overnight rate at t under the proportional allotment procedure with interest rate targeting resembles very much the one determining the expected rate under full allotment. The more so, if the central bank always equates the target rate with the tender rate. Thus, any

[62] Under neutral interest rate expectations $r_{T-1}^{\text{targeted}} - \mathrm{E}^{CB}\left[\mathrm{E}_{T-1}^{banks}\left[r_T^{\text{targeted}}\right]\right] = 0$ and a symmetric corridor $r_{T-1}^m - \mathrm{E}^{CB}\left[\mathrm{E}^{banks}\left[r_T^{\text{targeted}}\right]\right] = -\left(r_{T-1}^d - E^{CB}\left[\mathrm{E}^{banks}\left[r_T^{\text{targeted}}\right]\right]\right)$, equation (37) can be written as $H(-eOB^{CB}_{T-1}) = 1 - H(2RDB_{T-1} - eOB^{CB}_{T-1})$. Thus, $eOB^{CB}_{T-1} = RDB_{T-1}$.

[63] For the derivation of equation (38) see appendix C. For how the overnight rate at t is determined from the bank's profit maximization problem see equation see (19). Recall above, that with interest rate targeting the central bank tries to equate the overnight rate with its target value ($\mathrm{E}\left[r_t^{on}|_{eOB_t = eOB_t^{CB}}\right] = r_t^{\text{targeted}}$).

differences between the amount of liquidity the central bank plans to provide the markets with and the equilibrium liquidity under full allotment must come from differences between *either* $\mathrm{E}^{CB}\left[\mathrm{E}_t^{banks}\left[r_f^{\text{targeted}}\right]\right]$ and $\mathrm{E}_t\left[r_f^T\right]$ *or* $H(\cdot)$ and $G(\cdot)$. Consequently, the actual overnight rate realized under proportional allotment with interest rate targeting will differ from the overnight target rate because of the stochastic liquidity shock or due to the central bank's inability to estimate the expectations of the banks or the shock distributions correctly. Also, the path of reserve holdings with this approach resembles that of under full allotment. When interest rate cut (rise) is expected, the central should backload (frontload) the reserve holding considerably to keep the expected overnight rate at its target. In addition to the interest rate expectations, the current and future probabilities of being forced to use the standing facilities will affect the planned allotment.

Now, with rational expectations and symmetric information the central bank should be able to work out the banks' expectations of the central bank interest rates (ie $\mathrm{E}^{CB}\left[\mathrm{E}_t^{banks}\left[r_f^{\text{targeted}}\right]\right]$ reduces into $\mathrm{E}\left[r_f^{\text{targeted}}\right]$). In such a case, the only difference between equations (21) and (38) is in the cumulative distribution functions $G(\cdot)$ and $F(\cdot)$ (assuming $r^{\text{targeted}} = r^T$). Still the central bank's task of determining the amount of liquidity to be allotted can be very troublesome when there are expectations of interest rate changes. The difficulties might arise from the fact that it could be very hard to estimate the tail probabilities of $F(\cdot)$ (ie $H(-RDB)$ could be a much better estimator of $F(-RDB)$ than $H(-3RDB)$ is of $F(-3RDB)$). However, the central bank's estimate of the liquidity needed in the interbank market should not be a systematically biased estimator of the true liquidity needed even at these high or low levels of reserves.

We will demonstrate the effect of the central bank having biased estimates of the banks' expectations of the interest rate development by figures 14 and 15. To avoid insisting on the central bank being able to reproduce the expectations of the private banks, one can consider these figures as examples of the effects of the problems in estimating the tail probabilities of the cumulative shock distribution. Figure 14 illustrates the determination of the expected supply of overnight liquidity when the central bank's estimate is lower than the true value of the banks' expectations of the increase in the tender rate will be. We see, that because of the erroneous estimate of the interest rate expectations, the estimated demand for liquidity will fall below the banks' true demand at the current tender rate. Consequently, the overnight rate is very likely to be realized above the tender rate. Similarly, if the central bank's estimate was overstating the true expected increase in the target rate, the expected supply would be too large relative to the demand for liquidity at the overnight market. Hence, in this case the overnight rate would most likely be realized below the tender rate. Note, that the central bank is able to allot "too much" liquidity here, as the banks are overbidding at equilibrium (ie bidding for more than their true demand is).

Figure 15 illustrates the similar effects (of a biased estimate of interest rate expectations), when a rate cut is expected. In figure 15 the central bank underestimates the rate cut expected by the banks. Thus, the supply of liquidity

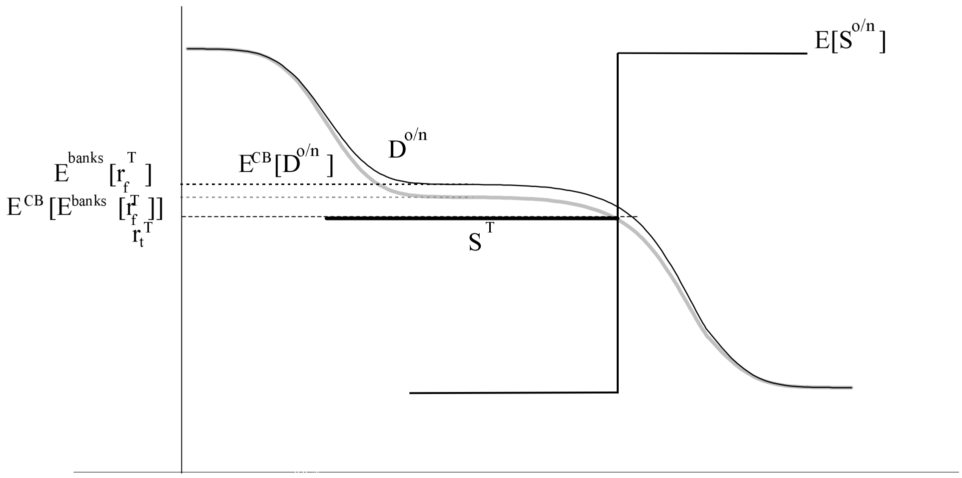

Figure 14: The effect of a too small estimate (by the central bank) over the rate rise expected by the banks'

is too high relative to the real demand. Consequently, the overnight rate is expected to fall below the tender rate due to this incorrect estimate. Similarly, an inadequately high estimate would lead to lack of overnight liquidity, and the overnight rate would most likely rise over the tender rate.

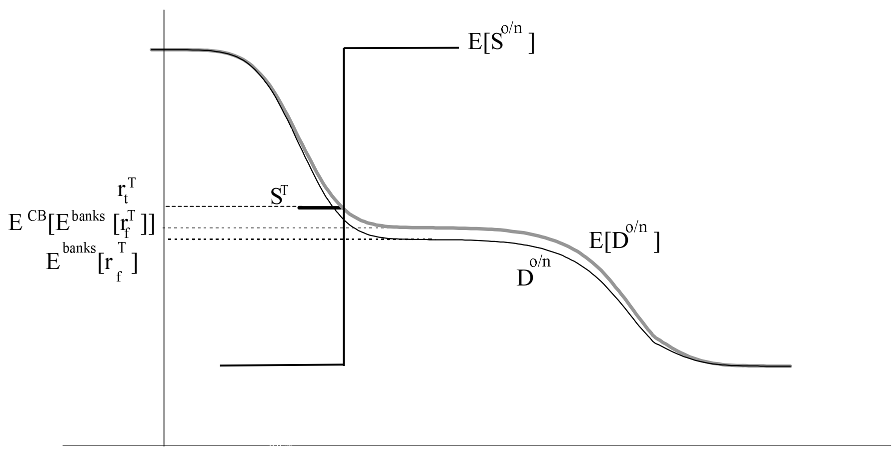

Figure 15: The effect of a too small estimate (by the central bank) over rate cut expected by the banks'

5 Summary and conclusions

In this paper we have built up a model of the determination of equilibrium in the overnight money market, when the central bank steers the market with an interest rate corridor and open market operations that are conducted in the form of fixed rate tender operations. The demand for overnight reserves at a given price is shown to be a function of expected future rate to be used in the tenders, current and expected rates of the standing facilities and the distribution of liquidity shocks. The supply of reserves is not exogenous in this model. It depends both on the liquidity policy on which the central bank bases its allotment decisions and on the banks' aggregate demand for reserves in the tender operation (with the given policy). Three alternative liquidity policies are considered in the paper. First, in a full allotment procedure, the central bank provides the markets with all the reserves the banks bid for in the tender. Secondly, in proportional allotment procedure the central bank scales the bids back in proportion to the individual bids. In case of the proportional allotment procedure, we study two different policy rules; in liquidity targeting, the central bank aims at holding the money market liquidity constant throughout the remainder of the maintenance period, whereas in interest rate targeting it intends to provide the market with liquidity that would bring the overnight rate to the targeted level.

The determination of the money market equilibrium when the central bank does not provide the banks with averaging possibility was studied in chapter 3. We saw that with both full allotment procedure and proportional allotment under interest targeting the expected value for the overnight rate will equal the tender rate.[64] The volatility of the rate was seen to depend on the distribution of shocks. Thus, the relative volatility of the overnight rate between these two approaches depends on the relative accuracy of estimates of the shock distributions made by the banking sector as a whole vs the one made by the central bank.

In case of full allotment, the variations in the overnight rate entirely reflect the stochastic and temporary liquidity shocks (ie the forecasting errors of the banks). Thus, these variations do not affect the expected values of the future overnight rates, and consequently they are not transmitted along the yield curve to longer maturities. Therefore, the signals given by the central bank rates should be unambiguous. The same is true with the proportional allotment procedure, as long as the policy is known to the public. However, if the targeted policy must be read from the past behaviour of the central bank, variations in the realized overnight rate of interest could be (mis)interpreted as changes in the monetary policy stance. In such a case, one can't be certain that the overnight volatility is not transmitted to longer maturities.

The symmetric interest rate corridor is the simplest one for the central bank to operate with (at least as long as the shock distribution can be expected to be symmetric). However, the rates of the standing facilities can be used as independent policy rates, if full allotment is used. The asymmetry of the

[64] If the averaging scheme is not used, proportional allotment with interest rate targeting is similar to the case with liquidity targeting.

corridor would certainly affect the demand for excess reserves, but it would not affect the expected tender rate. With the proportional allotment procedure, independent signalling with the corridor is very complicated, as, if the tender rate were in the upper part of the corridor, the central bank would not receive bids for as much liquidity as it wants to provide the market, and consequently, the determination of the overnight rate would be similar to the case of full allotment.[65]

The central bank can affect the volatility of the interbank overnight rate by choosing the width of the interest rate corridor. It was also shown that the stochastic overnight volatility depends on the timing of the interbank market. We saw that the volatility is lower in a market that is already active in the morning, compared to one that use interbank trading merely to settle the foreseen liquidity needs of the banks.

The analysis of the money market equilibrium under averaged reserve requirement is conducted in chapter 4. The demand for reserves is not similar on different days of the reserves maintenance period. On the last day of the period the demand is similar to that of the case without averaging, ie the demand for bank reserves at a given overnight rate depends on the current central bank rates and the distribution of liquidity shocks. However, already on the penultimate day the interest rate expectations enter the demand function. The higher the expected tender rate for the following day, the more reserves the banks will demand today at a given overnight rate. On days prior to the penultimate day, the demand for reserves is also affected by the dynamic cost factor, ie holding more reserves today will increase the future cost of liquidity uncertainty associated with the probability of having to rely on the standing facilities.

The main characteristics of the money market equilibrium can be summarized under the three different liquidity policy rules.

1. Expected level of the overnight rate

In case of full allotment the expected overnight rate equals the tender rate for that day. Thus, the expected overnight rate for a given day in the future will equal the tender rate expected to prevail on that day.

Under the proportional allotment procedure with interest rate targeting, the expected value for the overnight rate will (by definition) equal the target rate of the central bank, as long as the policy is known to the counterparties and the

[65] This effect is also part of the reason why we stuck to *neutral* interest rate targeting (ie the target rate equals the tender rate) in analyzing proportional liquidity allotment procedures when the central bank does not allow for averaging. If the target rate were below the tender rate the banks would not bid enough in the tender, and this procedure would not be proportional allotment procedure anymore. If the target rate were above the tender rate, there would be enormous excess bidding and the monetary policy would be implemented by choosing the liquidity instead of setting the interest rate (which we implicitly assume while applying fixed rate tenders). Also this procedure would lead into unjustified transfer of expected profits to successful bidders.

target is not set below the tender rate.[66] Therefore, in this case the expected overnight rate for a given future day will equal the expected target rate for that day.

The determination of the expected overnight rate under liquidity targeting policy will depend on the central bank's ability to control the daily supply of liquidity. If the banks expect the central bank to increase the tender rate during the following days, the expected overnight rate will immediately respond to these expectations, by *rising to the new higher level* expected (or to level very close to the expected one). However, if the central bank is expected to cut the tender rate within the remainder of the ongoing reserves maintenance period, the expected overnight will not react to these expectations. It will stay at the level of the current tender rate, as under full allotment procedure. In case where the banks have neutral interest rate expectations, the overnight is expected to realize at the level of the (constant) tender rate.

The fact that under liquidity targeting the expected overnight rate will differ from the tender rate, if a rate hike is expected, means that the central bank does not have total control over the (expected) price of bank reserves. Thus, it might well be the case that the monetary policy signals given through the tender rate are not as unambiguous as with either full allotment or with interest rate targeting. Also, the fact that the overnight rate will react only to expectations of a rate increase, while an expected rate cut would be reflected in the equilibrium liquidity, may well lead the counterparties and the public to (falsely) assume that the central bank has asymmetric preferences over the deviations of the overnight rate from the tender (or target) rate, even though the asymmetry lies in the structure of the operational framework rather than in the preferences.[67]

2. Expected overnight liquidity

The expected equilibrium liquidity under the full allotment procedure will depend on the required daily balances for the remaining period, the interest rate expectations of the banks and on the distribution of liquidity shocks. The higher the tender rate is expected to be during the rest of the maintenance period, the more banks are willing to hold liquidity today at the given tender rate. Hence, we expect the banks to frontload reserve holdings, if a rate hike is expected, and to backload them, if they anticipate a rate cut. The lower the liquidity volatility is, the more the banks are willing to substitute their interest rate view for steady reserve keeping. Ie the larger the probability of being forced to use the standing facilities with a given level of liquidity at

[66] If the target were below the tender rate, this approach would be similar to the case of full allotment, and consequently the expected rate would equal the tender rate instead of the target. Thus, we expect the central bank to set the target rate at least to equal the tender rate.

[67] This becomes more evident, if we think about the case where the money markets operate on liquidity surplus rather than on deficit. In such a case the tenders would be liquidity draining instead of liquidity providing operations. Therefore, the central bank would be not be in control of the daily liquidity supply, if a rate increase were expected. Thus, the expected overnight rate would equal the current tender rate with increasing interest rate expectations, but it would fall down to the expected future level, if a rate cut were expected.

the overnight interbank trading, the less the banks would seek "intertemporal arbitrage" (to front- or backload the reserve holdings).

Under liquidity targeting, the central bank is able to control the daily supply of liquidity as long as the expected future overnight rate with the target liquidity is not lower than the expected future tender rate. Thus, with increasing expectations or with neutral expectations (and high enough volatility), the expected overnight liquidity will be at the target level of the central bank, ie at the required daily balances for the remaining period. However, if a rate cut is expected (and possibly with neutral liquidity and very low stochastic liquidity shocks), the expected supply of overnight liquidity will equal that of under full allotment (ie the banks would backload reserve holdings).

With interest rate targeting the equilibrium liquidity depends on the difference between the target rate and the tender rate. In (the very unlikely) case, where the target is set below the tender, this whole approach melts down to full allotment. If the target is set at the level of the tender rate or above it, the expected liquidity will also be determined as in the case of full allotment. However, in this case the equilibrium liquidity will be a decreasing function of the *central bank's expectations over the rate the banks anticipate* prevailing during the remainder of the period, and on the central bank's expectation of the distribution of the second liquidity shock for each day. Consequently, there will be frontloading of reserve holdings, if rate increase expectations are anticipated by the central bank, and backloading will occur, if rate cut expectations are anticipated.

3. Volatility of the overnight rate

With full allotment the spread between the overnight rate and the tender rate will depend entirely on the liquidity shock that is realized between the tender operation and the clearance of the overnight market. The interest rate elasticity of the demand for overnight reserves will increase as the liquidity approaches either zero or the amount of reserves that would fulfill the requirement for the whole period. Thus, the volatility of the overnight rate grows when the banks want to front- or backload the reserves due to interest rate expectations. We also expect the variations to be asymmetric when the banks expect a change in the rate. The reason for the asymmetry lies in the shape of the demand curve at the equilibrium liquidity; it is expected to be convex, if a rate cut is expected (due to the backloading of reserves), and concave, if a rate rise is expected (as there will be frontloading of reserves).

The interest rate expectation also affects the interest rate volatility during the remainder of the maintenance period (however, not on the two last days of the maintenance period), as the reserve holdings at t affect the required daily balances for the remaining period (RDB) on the following days. For example, if the banks expect the tender rate to be increased, they will hold more reserves today, which will decrease the RDB for the following days. Thus, the probability of a bank having to use either of the standing facilities on the following days increases, which will consequently lower the interest rate elasticity of these days. Similarly, if a rate cut is expected, the volatility today will increase, but the volatility on the following days will diminish.

When the central bank has a target for liquidity and it is expected to be in control of the daily supply of the liquidity, the expected overnight liquidity at t will naturally be the target liquidity and the expected overnight rate for that day will be close to the tender rate expected to prevail in the future. In this case the interest rate elasticity of the demand for liquidity should be high, as the probability of being forced to use the standing facilities at the targeted liquidity is low. Hence, the variations in the realized overnight rate should more reflect variations in the interest rate expectations than the stochastic variations in autonomous liquidity factors. However, if the banks expect a rate cut in the near future, the daily supply of liquidity is no longer in the hands of the central bank. Thus, the overnight rate will be determined as in case of full allotment, and its volatility is related to the stochastic liquidity shocks.

In the proportional allotment procedure with interest rate target, the variations in the spread between the target rate and the overnight rate reflect the stochastic errors the central bank makes in estimating both the banks' demand for liquidity and the development in the autonomous liquidity factors. As in case of full allotment, the volatility of the overnight rate under interest rate targeting depends on the interest rate expectations, as they affect the part of the demand curve at which the equilibrium will be realized. Thus, the volatility is expected to be higher, if a rate change is expected than with neutral expectations.

We have seen, that the *stochastic volatility* of the overnight rate should be lower with liquidity targeting than with the other procedures, if an interest rate change is expected to occur in the near future. However, this need not be the case for total volatility of the rate, at least, if the rate expectations vary a lot over time. Furthermore, the stochastic volatility should not be harmful for the conduct of monetary policy, as long as it is totally understood by the counterparties and the public that these variations originate solely from prediction errors of the development of the liquidity (and also by errors in estimating the effect of interest rate expectations on the demand for liquidity in case of interest rate targeting). This should not produce any difficulties in the full allotment case, as the counterparties are always aware that the liquidity errors are due to errors made by themselves. However, the central bank must pay great attention to making its goals understood and believed by the counterparties, if it uses interest rate targeting. One possible method to do this could be to announce the target of the central bank explicitly.

The relative size of volatility under full allotment and under the interest rate targeting depends on the relative size of the aggregate estimate error (in forecasting the autonomous liquidity factors) made by the banks and the central banks error both in estimating the development of liquidity and in anticipating the effect interest rate expectations have on the demand for liquidity. The central bank might have superior knowledge of the development of the autonomous liquidity factors compared to the banks. However, this need not to be the case, at least, if the central bank publishes its liquidity forecast prior to each tender operation. Furthermore, the estimation of the effect of interest rate expectation on the demand for reserves will be a tough task for the central bank.

Under the full allotment procedure, the banking sector as a whole will bid according to the neutral strategy, ie overbidding can never be sustained as an equilibrium strategy in full allotment. Assuming *either* an infinitesimal probability for a bank not being able to participate in the interbank market on any particular day *or* by introducing a fixed cost of entering the market this result can be extended to the level of single banks. In such a case every bank will bid according to its real liquidity need, and the allocation of liquidity in the tender will follow the expected true liquidity demand by the banks.

The bidding behaviour under liquidity targeting will depend on the interest rate expectations of the banks. If the banks expect the tender rate to be increased, the banks will be overbidding to profit from the expected difference between the tender rate and the overnight rate. In such a case the profits of a bank will depend directly on the amount of liquidity it is allotted. Thus, a bank's optimal bid would eventually be infinitely large, if the size of a bid is not somehow limited. The case where the banks have neutral expectations (and the true demand for liquidity (ie demand under full allotment) is not less than the RDB) will also produce overbidding equilibria. If the banks bid according to a neutral strategy, the expected value of the overnight rate would in this case rise above the tender rate. Thus, by overbidding the banks ensure, that the supply of liquidity will be determined by the central bank. However, when the real demand is close to the RDB, the overbidding will not lead to positive expected profits, it will merely close the opportunity for such profits. If a rate cut is expected (or with neutral expectations and equilibrium bidding below the RDB), the banks will be bidding according to their real demand, as in the case of full allotment.

Under interest rate targeting, the banks will place bids in excess of their real demand for liquidity either to profit from the expected difference between the overnight rate and the target (if the target is above the tender rate) or to close the window for such an opportunity to exist (if the target rate equals the tender rate).

The bidding behaviour of the banks is very different under each of these procedures, as we have seen. There is no overbidding under full allotment (at least at the aggregate level), ie all banks will be bidding according to their real demand for reserves. In interest rate targeting, multiple overbidding equilibria exist. These equilibria are characterized only by the fact that the aggregate bids should amount to more than the central bank can be expected to be willing to provide. The problem here is that the allocation of liquidity is somehow arbitrary, unless the number of equilibria can be reduced.[68] One way for the central bank to limit this number could be to signal the most probable allotment ratio to the banks (ie the expected percentage of the bids to be accepted). For example, if the central bank told the counterparties, that the most probable allotment ratio is $\frac{1}{k}$, and that it will deviate from the announced ratio only to make use of the superior knowledge it has of the evolution of the

[68] However, even if the allocation of reserves does not meet the real demand for them, it should not be a problem as long as the overnight market is efficient and the banks are interested only on the expected profits they make.

autonomous liquidity factors, there would be a unique equilibrium where each bank bids k times its real demand. With liquidity targeting the overbidding would be similar to that of under the interest rate targeting, if the overnight rate were expected to be realized at the level of the tender rate. However, if the expected overnight rate were above the tender rate, there would not be equilibrium bidding at all (as the optimal bid of a single bank would be infinite, if the size of it were not limited) or when the bid size was limited the equilibrium bidding would consist of every bank placing the maximum bid.

We would like to conclude by stating that in light of our model the full allotment procedure seems to be a very market oriented liquidity policy rule, where the monetary policy stance is uniquely determined by the tender rate, and the banks are bidding according to their real liquidity demand. The stochastic volatility of the liquidity and the overnight rate of interest is lowest with the liquidity targeting policy rule. However, the uniqueness of monetary policy is somehow questionable with this procedure. The interest rate targeting procedure shares many of the good qualities of the full allotment procedure, at least if the policy is made explicit and is believed by the counterparties. Also, the stochastic liquidity volatility (of the full allotment) could perhaps be limited with this procedure in some cases (eg if the central bank has superior knowledge over the evolution of the autonomous liquidity factors and it can properly estimate the effect of interest rate expectations on the liquidity demand). However, the drawback of this policy seems to be the multiplicity of the bidding equilibria. Furthermore, the estimation of the real demand for liquidity might be extremely hard task, if a rate change were expected.

Finally, one should notice that most of the differences in the formation of the expected overnight rate and in its volatility arise from the expected changes in the central bank rates during the remainder of the ongoing reserves maintenance period. Thus, the equilibria under all these procedures would be very similar to each other, if the central bank changed the tender rate (or the target rate) only in the first operation of each maintenance period. In such a case there would never be speculative front- or backloading of reserves in order to profit from the intraperiod arbitrage opportunity.

References

[1] Ayuso, J. – Haldane, A.G. – Restoy, F. (1994) **Volatility Transmission along the Money Market Yield Curve.** Banco de España – Servicio de Estudidios, Documente de Trabajo No. *9403*.

[2] Ayuso, J. – Repullo, R. (2000) **A Model of the Open Market Operations of the European Central Bank**. Banco de España – Servicio de Estudidios, Documente de Trabajo No. *0014*.

[3] Bartolini, L. – Bertola, G. – Prati, A. (1998) **Day-To-Day Monetary Policy and the Volatility of the Federal Funds Interest Rate**. European University Institute working paper 98/35.

[4] Ehrhart, K-M. (2000) **Repo Games: Experimental Investigation of the Fixed Rate Tender.** An unpublished working paper presented in the ZEI Conference: Central Bank Operations: Theory and Practice in Frankfurt on 22.–23. September 2000.

[5] European Central Bank (1998) **The Single Monetary Policy in Stage Three; General Documentation on ESCB Monetary Policy Instruments and Procedures.** Frankfurt am Main.

[6] European Central Bank (2000) **Annual Report.** Frankfurt am Main.

[7] Federal Reserve Bank of New York (2000) **Open Market Operations During 1999.** Federal Reserve Bank of New York, Markets Group, February 2000.

[8] Furfine, G. (1998) **Interbank Payments and the Daily Federal Funds Rate.** Board of Governors of the Federal Reserve System. Finance and Economics Discussion Series. No. 1998–31.

[9] Hamilton, J. (1996) **The Daily Market for Federal Funds.** Journal of Political Economy 104, 26–56.

[10] Miyanoya, A. (2000) **A Guide to Bank of Japan's Market Operations**. Bank of Japan Financial Markets Department Working paper series 00-E-3.

[11] Perez-Quiros, G. – Rodriguez, H. (2000) **The Daily Market for Funds in Europe: Has Something Changed with the EMU?** Pompeu Fabra working paper 474/2000.

[12] Poole, W. (1968) **Commercial Bank Reserve Management in a Stochastic Model: Implications for Monetary Policy.** Journal of Finance 23, 769–791.

[13] Välimäki, T. (1998) **The Overnight Rate of Interest under Averaged Reserve Requirements: Some Theoretical Aspects and the Finnish Experience**. Bank of Finland Discussion Papers 7/98.

A Proof of equation (11)

The proof of equation (11) and the relationship between $G(\cdot)$ and $E[F(\cdot)]$.[69]

Let $F_\mu(\cdot)$, $F_\varepsilon(\cdot)$and $G(\cdot)$ be the cumulative distribution functions of independent stochastic variables μ, ε and ν ($\nu = \mu + \varepsilon$) respectively.

From the very definition of a distribution function we have:

$$G(s) = P\{\nu \leq s\}. \tag{39}$$

Now, it can be shown that the distribution function of the sum ν has the representation

$$G(s) = \int F_\varepsilon(s - \mu) dF_\mu(\mu) \tag{40}$$

or, assuming continuous distributions with density functions f_ε and f_μ respectively

$$G(s) = \int f_\varepsilon(s - \mu) f_\mu(\mu) d\mu. \tag{41}$$

But note that by the definition of an expectation, we have

$$G(s) = \mathrm{E}_{f_\mu}[F(s - \mu)] \tag{42}$$

ie the distribution function of the sum can be obtained as an expected value, where expectation is taken over the distribution of μ.

This result can be proved by the use of the convolution theorem in association with characteristic functions: the characteristic function (chf) of a distribution (with density f), $\psi_f(t)$, say, is defined by

$$\psi_f(t) = \mathrm{E}\left(e^{itx}\right) = \int e^{itx} f(x)\, dx. \tag{43}$$

For independent random variables x and y, the characteristic function of the distribution of their sum, $\nu = \mu + \varepsilon$, is the product of the individual characteristic functions:

$$\psi_g(t) = \mathrm{E}\left(e^{its}\right) = \mathrm{E}\left(e^{itx}\right) \mathrm{E}\left(e^{ity}\right) = \psi_{f_x}(t) \psi_{f_y}(t). \tag{44}$$

On the other hand, we know that the chf of the convolution

$$(f * h)(\nu) = \int f(\nu - x)\, h(x)\, dx \tag{45}$$

[69] I am indebted to Jouko Vilmunen for the derivation of this appendix .

equals the product of the chf:s of the distributions with densities f and h respectively, ie

$$\psi_{f*h}(t) = \psi_f(t)\psi_h(t). \tag{46}$$

Since the chf determines the distribution uniquely, we must consequently have

$$\psi_{f_x * f_y}(t) = \psi_g(t) \tag{47}$$

so that the density function of the distribution of the sum ν, g must be the convolution of the densities

$$g(u) = \int f_y(u-x) f_x(x)\, dx. \tag{48}$$

Integrating on both sides and changing the order of integration on the RHS we obtain:

$$G(s) = \int \left[\int^s f_y(u-x)\, du \right] f_x(x)\, dx = \int F_y(s-x) f_x(x)\, dx. \tag{49}$$

Now apply all this to equation (8):

$$\begin{aligned} r^T &= r^m G(-eER_{i,T}^{neutral}) + r^d \left[1 - G(eER_{i,T}^{neutral})\right] \qquad (50) \\ &= r^m \mathrm{E}_{f_\mu} \left[F_\varepsilon \left(-eER_{i,T}^{neutral} - \mu_T \right) \right] + r^d \left\{ 1 - \mathrm{E}_{f_\mu} \left[F_\varepsilon \left(eER_{i,T}^{neutral} + \mu_T \right) \right] \right\} \\ &= r^m \mathrm{E}_{f_\mu} \left[F_\varepsilon \left(-ER_{i,T}^{neutral} \right) \right] + r^d \left\{ 1 - \mathrm{E}_{f_\mu} \left[F_\varepsilon \left(ER_{i,T}^{neutral} \right) \right] \right\} \\ &= \mathrm{E}_{f_\mu} \left[r^m \mathrm{E}_{f_\mu} \left[F_\varepsilon \left(-ER_{i,T}^{neutral} \right) \right] + r^d \left\{ 1 - \mathrm{E}_{f_\mu} \left[F_\varepsilon \left(ER_{i,T}^{neutral} \right) \right] \right\} \right] \\ &= \mathrm{E}_{f_\mu} \left[r^{on} \right]. \end{aligned}$$

That is, the expected overnight rate, where the expectation is taken w.r.t. the distribution of the shock μ (ie $\mathrm{E}_{f\mu}[\cdot]$) equals the tender rate.

B Profit maximization problem at T-1

The overnight rate of interest at T-1 as a function of liquidity is determined by the first order condition to the banks' profit maximization problem:

$$
\begin{aligned}
\max_{b_{i,T-1}} \mathrm{E} \quad & [\Pi_{T-1}] = r^m_{T-1} \int\limits_{-\infty}^{-OB_{i,T-1}-b_{i,T-1}} (OB_{i,T-1} + b_{i,T-1} + \varepsilon_{T-1}) f(\varepsilon_{T-1}) d\varepsilon_{T-1} \\
& + \mathrm{E}_{T-1}\left[r^T_T\right] \int\limits_{-OB_{i,T-1}-b_{i,T-1}}^{\infty} (OB_{i,T-1} + b_{i,T-1} + \varepsilon_{T-1}) f(\varepsilon_{T-1}) d\varepsilon_{T-1} \\
& + (r^d_{T-1} - \mathrm{E}_{T-1}\left[r^T_T\right]) \int\limits_{IB_{i,T-1}-b_{i,T-1}}^{\infty} (-IB_{i,T-1} + b_{i,T-1} + \varepsilon_{T-1}) f(\varepsilon_{T-1}) d\varepsilon_{T-1} \\
& - r^{on}_{T-1} * b_{i,T-1},
\end{aligned} \tag{51}
$$

where the difference between the amount of reserves needed to fulfill the reserve requirement for the whole remainder of the maintenance period and the overnight balances before the interbank lending at $T-1$ is denoted by $IB_{i,T-1}$ (ie $IB_{i,T-1} = 2RDB_{i,T-1} - OB_{i,T-1}$).

By using the Leibniz's rule, we get the FOC:

$$
\begin{aligned}
\frac{\partial \mathrm{E}\left[\Pi_{T-1}\right]}{\partial b_{i,T}} = \quad & r^m_{T-1} F\left(-OB_{i,T-1} - b_{i,T-1}\right) \\
& + r^d_{T-1}\left(1 - F\left(2RDB_{i,T-1} - OB_{i,T-1} - b_{i,T-1}\right)\right) \\
& + \mathrm{E}_{T-1}\left[r^T_T\right]\{1 - F\left(-OB_{i,T-1} - b_{i,T-1}\right) \\
& - \left[1 - F(2RDB_{i,T-1} - OB_{i,T-1} - b_{i,T-1})\right]\} \\
- \quad & r^{on}_{T-1} = 0,
\end{aligned}
$$

which, by aggregating over unitary mass gives us the result:

$$
\begin{aligned}
r^{on}_{T-1} = \quad & \mathrm{E}_{T-1}\left[r^T_T\right]\{1 - F(-OB_{T-1}) - [1 - F(2RDB_{T-1} - OB_{T-1})]\} \\
& + r^m_{T-1} F(-OB_{T-1}) + r^d_{T-1}\left[1 - F(2RDB_{T-1} - OB_{T-1})\right]
\end{aligned}
$$

C Profit maximization problem at t

The profit maximization problem of a bank at the interbank overnight market at t is:

$$\begin{aligned}
\max_{b_{i,t}} \quad & \mathrm{E}\left[\Pi_t\right] = r_t^m \int_{-\infty}^{-OB_{i,t}-b_{i,t}} (OB_{i,t}+b_{i,t}+\varepsilon_t) f(\varepsilon_t) d\varepsilon_t \\
& + \mathrm{E}\left[r_f^T\right] \int_{-OB_{i,t}-b_{i,t}}^{\infty} (OB_{i,t}+b_{i,t}+\varepsilon_t) f(\varepsilon_t) d\varepsilon_t \\
& + \left(r_t^d - \mathrm{E}\left[r_f^T\right]\right) \int_{IB_{i,t}-b_{i,t}}^{\infty} (-IB_{i,t}+b_{i,t}+\varepsilon_t) f(\varepsilon_t) d\varepsilon_t \\
& + \sum_{j=t+1}^{T-1} \left(\mathrm{E}_t\left[r_j^m - r_f^T\right] \left[\int_{-\infty}^{-eOB_{i,j}^*} (eOB_{i,j}^*+\nu_j) g(\nu_j) d\nu_j \right.\right. \\
& \left. - \int_{-\infty}^{-eOB_{i,j}^{*\prime}} (eOB_{i,j}^{*\prime}+\nu_j) g(\nu_j) d\left(\nu_j\right) \right] \\
& \left. + \mathrm{E}_t\left[r_j^d - r_f^T\right] \left[\int_{JB_{i,j}^*}^{\infty} (-JB_{i,j}^*+\nu_j) g(\nu_j) d\upsilon_j - \int_{JB_{i,j}^{*\prime}}^{\infty} (-JB_{i,j}^{*\prime}+\nu_j) g(\nu_j) d\nu_j \right] \right) \\
& - r_{T-1}^{on} b_{i,T-1},
\end{aligned} \tag{52}$$

where

$\nu_j = \mu_j + \varepsilon_j$
$IB_{i,t} = (T-t+1)RBD_{i,t} - OB_{i,t}$
$JB_{i,t} = (T-j+1)eRBD_{i,j} - eOB_{i,j}^*$
$JB_{i,t}^{*\prime} = (T-j+1)eRBD_{i,j}^{\prime} - eOB_{i,j}^{*\prime}$

and the equilibrium liquidity at j ($eOB_{i,j}^*$) is a function of the expected required daily balances for the remaining period at j ($eRBD_{i,j}$), and $eRBD_{i,j}$ itself depends on the expected equilibrium reserve holdings up to $j-1$:

$$eRBD_{i,j} = \frac{(T-t+1)RDB_t - \left[\begin{array}{l} \int_{-OB_{i,t}-b_{i,t}}^{\infty} (OB_{i,t}+b_{i,t}+\varepsilon_t) f(\varepsilon_t) d\varepsilon_t \\ - \int_{IB_{i,t}-b_{i,t}}^{\infty} (IB_{i,t}+b_{i,t}+\varepsilon_t) f(\varepsilon_t) d\varepsilon_t \end{array} \right] - eOB_{t+1}^* - \ldots - eOB_{j-1}^*}{(T-j+1)}.$$

$eOB_{i,j}^{*\prime}$ is the expected equilibrium overnight balances, if the bank did not participate the interbank overnight market at t. Thus, $eOB_{i,j}^{*\prime}$ is a function of $eRBD_{i,j}^{\prime}$ (the expected required daily balances for the remaining period at j, without participation into the overnight market at t);

$$eRBD'_{i,j}=\frac{(T-t+1)RDB_t-\left[\begin{array}{c}\int\limits_{-OB_{i,t}-b_{i,t}}^{\infty}(OB_{i,t}+b_{i,t}+\varepsilon_t)f(\varepsilon_t)d\varepsilon_t\\ -\int\limits_{IB_{i,t}-b_{i,t}}^{\infty}(IB_{i,t}+b_{i,t}+\varepsilon_t)f(\varepsilon_t)d\varepsilon_t\end{array}\right]-eOB^{*\prime}_{t+1}-\ldots-eOB^{*\prime}_{j-1}}{(T-j+1)}.$$

Consequently, $eOB^*_{i,j}$ and $eRBD_{i,j}$ are functions of b_t, while $eOB^{*\prime}_{i,j}$ and $eRBD'_{i,j}$ are not.

Now, once again applying the Leibniz rule and aggregating over the unitary mass, we will get the FOC stating:

$$\begin{aligned}&\mathrm{E}_t\left[r^T_f\right]+\left(r^m_t-\mathrm{E}_t\left[r^T_f\right]\right)F(-OB_t)\\&+\left(r^d_t-\mathrm{E}_t\left[r^T_f\right]\right)\{[1-F(T-t+1)RDB_t-OB_t]\}\\&+\sum_{j=t+1}^{T-1}\Bigg(\mathrm{E}\left[r^m_j-r^T_f\right]\frac{\partial eOB^*_j}{\partial b_t}G(-eOB^*_j)\\&+\mathrm{E}\left[r^d_j-r^T_f\right]\left(-(T-j+1)\frac{\partial eRDB_j}{\partial b_t}+\frac{\partial eOB^*_j}{\partial b_t}\right)\\&\times\left\{1-G\left[(T-j+1)eRDB_j-eOB^*_j\right]\right\}\Bigg)-r^{on}_t=0\end{aligned}\tag{53}$$

The equilibrium condition for the money market ($\mathrm{E}_t\left[r^{on}_t\right]=r^T_t$) will consequently be (note that $\frac{\partial eOB^*_j}{\partial b_t}=\frac{\partial eOB^*_j}{\partial eRDB_j}\frac{\partial eRDB_j}{\partial b_t}$):

$$\begin{aligned}&\mathrm{E}_t\left[r^{on}_t\right]-\mathrm{E}_t\left[r^T_f\right]=r^T_t-\mathrm{E}_t\left[r^T_f\right]=\left(r^m_t-\mathrm{E}_t\left[r^T_f\right]\right)G(-eOB^*_t)\\&+\left(r^d_t-\mathrm{E}_t\left[r^T_f\right]\right)\{1-G\left[(T-t+1)RDB_t-eOB^*_t\right]\}\\&+\sum_{j=t+1}^{T-1}\Bigg(\mathrm{E}_t\left[r^m_j-r^T_f\right]\frac{\partial eOB^*_j}{\partial eRDB_j}\frac{\partial eRDB_j}{\partial b_t}G(-eOB^*_j)\\&+\mathrm{E}_t\left[r^d_j-r^T_f\right]\left(\frac{\partial eOB^*_j}{\partial eRDB_j}-(T-j+1)\right)\frac{\partial eRDB_j}{\partial b_t}\\&\times\left\{1-G\left[(T-j+1)eRDB_j-eOB^*_j\right]\right\}\Bigg).\end{aligned}\tag{54}$$

We can open equations (54) and (53) a little bit further by taking the partial derivative $\frac{\partial eRDB_j}{\partial b_t}$:

$$\frac{\partial eRBD_j}{\partial b_t}=\frac{-1}{T-j+1}\left\{[F(IB_t)-F(-OB_t)]+\sum_{k=t+1}^{j-1}\frac{\partial eOB^*_k}{\partial eRDB_k}\frac{\partial eRDB_k}{\partial b_t}\right\}.$$

Taking it one step further we will have:

$$\begin{aligned}\frac{\partial eRBD_{i,j}}{\partial b_t}=&\frac{-1}{T-j+1}\Bigg([F(IB_t)-F(-OB_t)]\\&+\sum_{k=t+1}^{j-1}\frac{\partial eOB^*_k}{\partial eRDB_k}\frac{-1}{T-k+1}\\&\times\left\{[F(IB_t)-F(-OB_t)]+\sum_{l=t+1}^{k-1}\frac{\partial eOB^*_l}{\partial eRDB_l}\frac{\partial eRDB_l}{\partial b_t}\right\}\Bigg)\end{aligned}\tag{55}$$

And repeating the partial derivatives in succession, equation 55 will become:

$$\frac{\partial eRBD_{i,j}}{\partial b_t} = \frac{-\left[F(IB_t) - F(-OB_t)\right]}{T-j+1} \times \left(1 + \sum_{k=t+1}^{j-1} \frac{\partial eOB_k^*}{\partial eRDB_k} \frac{-1}{T-k+1}\right.$$
$$\times \left\{1 + \sum_{l=t+1}^{k-1} \frac{\partial eOB_l^*}{\partial eRDB_l} \frac{-1}{T-l+1} \times \left[1 + \sum_{m=t+1}^{l-1} \frac{\partial eOB_m^*}{\partial eRDB_m} \frac{-1}{T-m+1}\right.\right.$$
$$\left.\left.\left.\times \left(... \times \left\{1 + \frac{\partial eOB_{t+2}^*}{\partial eRDB_{t+2}} \frac{-1}{T-t-1} \times \left[1 + \frac{\partial eOB_{t+1}^*}{\partial eRDB_{t+1}} \left(\frac{-1}{T-t}\right)\right]\right\}\right)\right]\right\}\right)$$

We easily see that the equilibrium bidding at t depends on the optimal path of reserve holding during the rest of the maintenance period. Thus, one has to solve the optimal bidding recursively by first deriving OB_{T-1}^* as a function of RDB_{T-1}, and using that information to calculate OB_{T-2}^* as a function of RDB_{T-2} and so forth.

BANK OF FINLAND DISCUSSION PAPERS

ISSN 0785-3572, print; ISSN 1456-6184, online

1/2001 Risto Herrala **An assessment of alternative lender of last resort schemes.** 2001. 28 p. ISBN 951-686-701-4, print; ISBN 951-686-702-2, online. (TU)

2/2001 Esa Jokivuolle – Karlo Kauko **The new basel accord: some potential implications of the new standards for credit risk.** 2001. 23 p. ISBN 951-686-703-0, print; ISBN 951-686-704-9, online. (RM)

3/2001 Mika Kortelainen **Actual and perceived monetary policy rules in a dynamic general equilibrium model of the euro area.** 2001. 76 p. ISBN 951-686-705-7, print; ISBN 951-686-706-5, online. (TU)

4/2001 Martin Ellison **Stabilisation bias in monetary policy under endogenous price stickiness.** 2001. 20 p. ISBN 951-686-707-3, print; ISBN 951-686-708-1, online. (TU)

5/2001 Erkki Koskela – Rune Stenbacka **Equilibrium unemployment with credit and labour market imperfections.** 2001. 35 p. ISBN 951-686-709-X, print; ISBN 951-686-710-3, online. (TU)

6/2001 Jarmo Pesola **The role of macroeconomic shocks in banking crises.** 2001. 61 p. ISBN 951-686.713-8, print; ISBN 951-686-714-6, online. (TU)

7/2001 Anssi Rantala **Does monetary union reduce employment?** 2001. 36 p. ISBN 951-686-715-4, print; ISBN 951-686-716-2, online. (TU)

8/2001 Tuomas Välimäki **Fixed rate tenders and the overnight money market equilibrium.** 2001. 72 p. ISBN 951-682-717-0, print; ISBN 951-686-718-9, online (TU)