



Heli Huhtala

# Along but beyond mean-variance: Utility maximization in a semimartingale model



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The views expressed in this paper are those of the author and do not necessarily reflect the views of the Bank of Finland.

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# Along but beyond mean-variance: Utility maximization in a semimartingale model

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## Abstract

It is well known that under certain assumptions the strategy of an investor maximizing his expected utility coincides with the mean-variance optimal strategy. In this paper we show that the two strategies are not equal in general and find the connection between a utility maximizing and a mean-variance optimal strategy in a continuous semimartingale model. That is done by showing that the utility maximizing strategy of a CARA investor can be expressed in terms of expectation and the expected quadratic variation of the underlying price process. It coincides with the mean-variance optimal strategy if the underlying price process is a local martingale.

Keywords: mean-variance portfolios, utility maximization, dynamic portfolio selection, quadratic variation

JEL classification numbers: G11, C61

# Keskiarvo-varianssiportfoliosta hyödyn maksimointiin: portfolion optimointi semimartingaalimallissa

Suomen Pankin keskustelualoitteita 5/2008

Heli Huhtala  
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## Tiivistelmä

Kuten hyvin tiedetään, tiettyjen oletusten vallitessa sijoittajan keskiarvo-varianssiltaan optimaalinen strategia on myös hyödyn maksimoiva strategia. Tässä tutkimuksessa osoitetaan, että nämä kaksi strategiaa eivät jatkuvassa semimartingaalimallissa ole yleisesti samat. Samalla löydetään yhteys strategioiden välillä. Tämä tehdään näyttämällä, että sijoittajan hyödyn maksimoiva allokaatio voidaan esittää tuoton odotusarvon ja odotetun kvadraattisen variaation funktiona.

Avainsanat: odotusarvo-varianssiportfolio, hyödyn maksimointi, dynaaminen portfolion valinta, kvadraattinen variaatio

JEL-luokittelu: G11, C61

# Contents

Abstract.....	3
Tiivistelmä (abstract in Finnish).....	4
<b>1 Introduction.....</b>	<b>7</b>
<b>2 Market model.....</b>	<b>9</b>
<b>3 Utility maximization problem.....</b>	<b>12</b>
<b>4 Recursive version of the problem.....</b>	<b>14</b>
<b>5 Solution for a sequence of simple strategies.....</b>	<b>18</b>
<b>6 Comparison of mean-variance optimal and utility maximizing strategies.....</b>	<b>21</b>
<b>7 Conclusions.....</b>	<b>24</b>
References.....	26



# 1 Introduction

Considering recent development in portfolio optimization methods in stochastic finance it is surprising how widely used the classical mean-variance optimization methods introduced by Markowitz (1952) still are among practitioners. Criticism on the mean-variance approach is well known (see f.ex. Philippatos, 1979) and wide spread especially among theorists, but the approach is still a standard in the financial industry. It is considered as a good enough an approximation of the true optimal portfolio because the trade-off between using a simplistic model that is computationally efficient and easy to understand and a sophisticated model that is hard or impossible to solve usually favors using the mean-variance model.

The assumption of log-normally distributed returns is a close approximation for many asset classes like government bonds or currencies of major industrial countries. But in recent years the development of financial markets has increased the use of more structured products, products with credit risk and even products whose structure is unknown to the investor at the time of investment. A typical example of the latter is a hedge fund investment where the investor gives the portfolio manager full discretion on the investment. When return distributions are skewed and fat tailed, or when the distribution is fully unknown to the investor, use of variance as a measure of risk is hardly justified anymore.

As an alternative to the mean-variance method the investor could try to use the results from stochastic finance, where the solution of the utility maximization problem is well understood even in a general semimartingale incomplete market setting (see Schachermayer, 2001, and references therein). Solution to the problem is presented as a density process of an equivalent martingale measure and as a consequence the stochastic finance literature has been concentrated on finding the optimal martingale measure (see Gundel, 2004, and references therein). Usually the portfolio optimization problem is first tried to be solved directly and if that fails, the dual problem is formulated. The solution of the problem involves specifying the true underlying market model and solving for the density of the equivalent martingale measure, which can be a burdensome task (see f. ex. Musiela, Rutkowski, 2005).

The aim of this paper is to some part fill the gap between mean-variance optimization and utility maximization. Firstly we want to introduce a criteria for portfolio selection (or asset allocation) that is easily applicable but at the same time theoretically more general and resulting in higher utility than the mean-variance approach. In the process of doing that we will reach our second goal which is to build a bridge between the two optimization approaches. The bridge turns out to be the measure of uncertainty, which in our setting will be revealed as expected quadratic variation of the underlying process in contrast to variance of the underlying model in the mean-variance approach.

This paper is mainly inspired by a recent paper of Xia (2005) where it is shown that in a semimartingale model with bankruptcy prohibition a portfolio minimizing a quadratic loss function coincides with a mean-variance optimal portfolio. In this paper we want to extend the analysis to a wider class of

utility functions and market models. By making the approach more general we will lose the equivalence of the two portfolios but, to compensate, we will find the connection between them.

Other recent comparisons of utility maximization and mean-variance optimality include work from Ottucsak and Vajda (2004) who have studied the difference in utility derived from the two optimization methods with logarithmic utility and certain assumptions on the price process using methods from asymptotic statistics. In another context Christensen and Platen (2005) take a different viewpoint to the problem and consider utility derived from different allocation rules. Their result is a general remark that Sharpe ratio based investment (that is essentially based on the mean-variance criteria) is not utility maximizing.

The main results of this paper are presented in Theorem 13 and in Corollary 16. The first result presents the optimal strategy of a utility maximizing investor as a function of the expected drift and expected quadratic variation of the underlying process. The result gives us an explicit way of describing the optimal asset allocation that does not require us to know the true stochastic structure of the underlying model. Instead, we can directly model quadratic variation of the price process, which presents us the potential of the process to move. The second result relates to the observation made by Xia (2005) by comparing the optimal strategies of mean-variance and utility maximizing investor in a semimartingale framework. Decomposing the underlying stochastic process into a martingale component and a predictable process of bounded variation enables us to prove that in our set up mean-variance optimal portfolio coincides with the (cara-) utility maximizing portfolio if the underlying price process is a local martingale.

The relevance of the results in this paper is both theoretical and practical. On theoretical side it becomes apparent that as the optimal allocation can be described by expected drift and quadratic variation of asset prices, we can focus on path properties of the underlying price **process** instead of the stochastic structure of the underlying **model**. On practical side the result provides for a compact and easily applicable way of expressing the optimal allocation. Using the method presented here one can increase expected utility of the terminal wealth without complicating the optimization procedure when compared to the mean-variance approach. In practice the only difference in optimization will be that instead of estimating variance of the underlying model one has to estimate quadratic variation of the path of the process. When estimating variance we have to make assumptions of the distribution of returns for all  $\omega \in \Omega$  whereas when estimating quadratic variation we are only dealing with one particular realization  $\hat{\omega}$  of the true model, which means that we need to make fewer assumptions to construct the optimal portfolio.

Because quadratic variation can be interpreted as squared length of the path of a process we are directly handling the uncertainty related to the underlying asset. By modelling and estimating quadratic variation we have means to treat the problem of higher moments. That gives new possibilities in managing risk borne from non-normally distributed asset classes like credit and event risk.

The structure of the paper is as follows. In section 2 we specify the model and present the basic framework of utility maximization that is used in stochastic finance. In section 3 we transform the general utility maximization problem into an equivalent asset allocation problem. Through the equivalence we will be able to specify the structure of the optimal strategy. In section 4 we show how the problem can be solved recursively by optimizing the value process of the portfolio. We also give a criterion according to which the optimal strategy will be selected. In section 5 we construct a sequence of simple strategies converging to the optimal strategy process. With the help of the simple strategies we are able to present the utility maximizing strategy in closed form as a mean – expected quadratic variation – optimal portfolio. In section 6 we study the relationship between variance and quadratic variation with some examples. Section 7 concludes.

## 2 Market model

We denote by  $\mathbf{S} = ((S_t^i)_{0 \leq t \leq T})_{0 \leq i \leq d}$  the price process of the  $d$  risky assets and suppose that the price of the riskless asset  $S^0$  is constant,  $S^0 \equiv 1$ . The process  $\mathbf{S}$  is assumed to be a semimartingale adapted to a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  satisfying the normal assumptions of completeness and right continuity. The time horizon is assumed to be finite and denoted by  $T$ .

A portfolio of assets is defined as a pair  $(x, H)$ , where the constant  $x \in \mathbb{R}_+^0$  is the initial wealth of the investor and  $H = ((H_t^i)_{0 \leq t \leq T})_{0 \leq i \leq d}$  is called the strategy process of the investor.  $H$  is assumed to be a predictable  $S$ -integrable process specifying the amount of each asset held in the portfolio. The strategy process is assumed to be self-financing so that the value process of the portfolio is given by

$$V_t = x + \int_0^t H_u dS_u, \quad 0 \leq t \leq T \quad (2.1)$$

We exclude doubling strategies by limiting the portfolio value process from the downside

$$(H \cdot S)_t := \int_0^t H_u dS_u \geq -C, \quad \text{for } 0 \leq t \leq T, \quad C \in \mathbb{R}_+ \quad (2.2)$$

The set of  $S$ -integrable strategies satisfying (2.1) and (2.2) is denoted by  $\mathcal{H}$ .

We denote by  $\mathcal{M}^e(S)$  the set of measures  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  (denote  $\mathbb{Q} \sim \mathbb{P}$ ) such that for each admissible integrand  $H$  the process  $H \cdot S$  is a local martingale under  $\mathbb{Q}$ . As is stated in Delbaen and Schachermayer (1994) the condition of no arbitrage in the market is equivalent to the condition that the set  $\mathcal{M}^e(S)$  is non-empty. So we use the words ‘no arbitrage’ in the meaning ‘no free lunch with vanishing risk’ in the terminology of Delbaen and Schachermayer. In this paper we assume that the set  $\mathcal{M}^e(S)$  is non-empty.

We define a function  $w \mapsto U(w)$  modeling the investor’s utility from wealth  $w$  at the terminal time  $T$ , with the following standard properties

**Definition 2.1** The utility function  $U : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  is assumed to be increasing, continuous on  $\{U > -\infty\}$ , differentiable and strictly concave in the interior of  $\{U > -\infty\}$  and satisfy the Inada conditions

$$U'(\bar{w}) := \lim_{w \rightarrow \bar{w}^+} U'(w) = \infty \quad (2.3)$$

and

$$U'(\infty) := \lim_{w \rightarrow \infty} U'(w) = 0 \quad (2.4)$$

where  $\bar{w}$  refers to the domain in which the utility function  $U$  is defined:  $\text{dom}(U) = ]\bar{w}, \infty[$ , that is the interior of  $\{U > -\infty\}$ .

**Definition 2.2** The dual function  $\tilde{U}$  related to a utility function  $U$  satisfying assumptions (2.3) and (2.4) in Definition 1 is defined by

$$\tilde{U}(x) = \sup_{\xi \in \mathbb{R}} (U(\xi) - x\xi), \quad x > 0 \quad (2.5)$$

The dual function  $\tilde{U}$  is finitely valued, differentiable, strictly convex on  $]0, \infty[$  and it satisfies

$$\tilde{U}'(0) = \lim_{x \downarrow 0} \tilde{U}'(x) = \lim_{x \downarrow 0} \sup_{\xi \in \mathbb{R}} (U'(\xi) - x\xi) = -\infty$$

$$\lim_{x \rightarrow \infty} \tilde{U}(x) = \lim_{v \rightarrow 0} U(v) \text{ and } \lim_{x \rightarrow \infty} \tilde{U}'(x) = 0, \text{ when } \text{dom}(U) = \mathbb{R}_+ \text{ and}$$

$$\lim_{x \rightarrow \infty} \tilde{U}(x) = \infty \text{ and } \lim_{x \rightarrow \infty} \tilde{U}'(x) = \infty, \text{ when } \text{dom}(U) = \mathbb{R}$$

The inverse function  $-\tilde{U}'(y)$  will be denoted by  $I$  following Karatzas, Lehocky, Shreve and Xu (1991).

We will present the results in this paper for a specific class of utility functions defined via the Arrow-Pratt coefficient of absolute risk aversion as follows

**Definition 2.3** The Arrow-Pratt coefficient of absolute risk aversion at level  $x$  is defined as

$$\alpha(x) = -\frac{U''(x)}{U'(x)}$$

For an investor with constant absolute risk aversion we have

$$\alpha(x) = c, \quad c > 0$$

which implies (up to affine transformations) a utility function called **cara utility** that is of the form

$$U(x) = -\frac{e^{-cx}}{c} \quad (2.6)$$

The class of utility functions of the form (2.6) will be denoted by  $\mathcal{U}^c$

Relating with a utility function  $U \in \mathcal{U}^c$  the conjugate function  $\tilde{U}$  and its inverse  $I$  are of the form

$$\begin{aligned}\tilde{U}(y) &= \frac{y}{c}(\ln(y) - 1), \quad c > 0 \\ I(y) &= -\tilde{U}'(y) = -\frac{1}{c} \ln(y)\end{aligned}\tag{2.7}$$

With notation defined above the general utility maximization problem can be stated as

$$\begin{aligned}\sup_{H \in \mathcal{H}} \mathbb{E}_{\mathbb{P}}[U(x + (H \cdot S)_T)] &\quad \text{subject to} \quad (2.8) \\ \mathbb{E}_{\mathbb{Q}}[(H \cdot S)_T] &= x\end{aligned}$$

Utility derived from optimizing (2.8) can be expressed by a value function

$$\hat{U}(x) := \sup_{H \in \mathcal{H}} \mathbb{E}_{\mathbb{P}}[U(x + (H \cdot S)_T)], \quad x \in \text{dom}(U), \quad U \in \mathcal{U}^c \tag{2.9}$$

For any utility function satisfying assumptions in definition (1) and  $H \in \mathcal{H}$  we know that  $\hat{U}$  in (2.9) is also a utility function (see Schachermayer, 2001).

For the analysis we shortly present two very fundamental concepts of stochastic analysis, stochastic exponent and quadratic variation of a semimartingale. They are both analyzed in much more detail in any standard textbook of modern probability like Protter (1990).

**Definition 2.4** For a continuous semimartingale  $X$ ,  $X_0 = 0$ , the **stochastic exponent** of  $X$  (denoted by  $\mathcal{E}(X)$ ) is the unique semimartingale  $Z_t$  satisfying the equation  $Z_t = 1 + \int_0^t Z_{s-} dX_s$ .  $\mathcal{E}(X)$  is given by

$$\mathcal{E}(X)_t = \exp(X_t - \frac{1}{2}[X, X]_t)$$

**Definition 2.5** For a continuous stochastic process  $X$  the process of **quadratic variation**  $[X, X] = [X, X]_{t, 0 \leq t \leq T}$  is defined as

$$[X, X]_t = X_t^2 - 2 \int_0^t X_s dX_s$$

It can be shown that for a cadlag adapted process  $X$ ,  $[X, X]$  is a cadlag, increasing and adapted process. If  $X$  is continuous so is  $[X, X]$ . If  $\pi_n$  is a sequence of partitions of  $[0, T]$  and  $0 = T_0^n \leq T_1^n \leq \dots \leq T_i^n \leq \dots \leq T_{k_n}^n = t$  a sequence of stopping times, then

$$X_0^2 + \sum_i (X^{T_{i+1}^n} - X^{T_i^n})^2 \rightarrow [X, X]_t \tag{2.10}$$

as  $n \rightarrow \infty$  for any  $t \in [0, T]$ . As expression (2.10) is for each  $\omega \in \Omega$ , it gives an intuitive meaning to quadratic variation as the squared length of the path of  $X$ .

To treat the case of multiple asset portfolio in what follows, we also define the quadratic covariation process

**Definition 2.6** Let  $X$  and  $Y$  be two continuous semimartingales. The process of **quadratic covariation** of  $X$  and  $Y$  is defined by

$$[X, Y]_t = XY_t - \int_0^t X_s dY_s - \int_0^t Y_s dX_s$$

We also have the *polarization identity*

$$[X, Y]_t = \frac{1}{2}([X + Y, X + Y]_t - [X, X]_t - [Y, Y]_t) \quad (2.11)$$

### 3 Utility maximization problem

In this section we will convert the general utility maximization problem into an equivalent asset allocation problem. In this paper the problem is solved in a continuous semimartingale model with one risky asset. The extension to the case of  $d$  risky assets is a straightforward extension from 1 to  $d$  dimensions by using polarization identity (2.11).

The asset allocation problem is stated in a form that can be related to an equivalent martingale measure using Girsanov theorem in reverse. In that way we will be able to start solving the asset allocation problem in a way that is consistent with the classical mean-variance portfolio selection problem introduced by Markowitz (1952).

**Lemma 3.1** *The general utility maximization problem*

$$\begin{aligned} \max_{H \in \mathcal{H}} \mathbb{E}_P[U(x + (H \cdot S)_T)] \\ \text{s.t. } \mathbb{E}_Q[(H \cdot S)_T] = x \end{aligned} \quad (3.1)$$

is equivalent to the problem

$$\max_{Q \in \mathcal{M}^e(P)} \mathbb{E}_P[U(x \mathbb{E}_Q[\frac{dP}{dQ} \mid \mathcal{F}_T])] \quad (3.2)$$

**Proof.** In the case of single risky asset strategy process  $H_t$  can be presented as a couple  $(\pi_t, \beta_t)$  where  $\pi_t$  denotes the number of risky assets in the portfolio and  $\beta_t$  denotes the number of riskless assets in the portfolio. As we work in discounted terms and use the riskless asset as a numeraire we can express the portfolio value process in the form

$$V_t(x) = x + \int_0^t \pi_u dS_u, \quad 0 \leq t \leq T$$

If we consider a problem of optimal asset allocation we have to optimize the portfolio weights  $\gamma_t^S$  and  $\gamma_t^B = 1 - \gamma_t^S$  defined by

$$\gamma_t = \gamma_t^S = \frac{\pi_t S_t}{V_t}$$

and write the portfolio value process as

$$V_t(x) = x + \int_0^t \frac{\gamma_u V_u}{S_u} dS_u, \quad 0 \leq t \leq T$$

If we assume that the underlying model for the risky asset is the solution of the stochastic differential equation

$$dS_t = S_t dX_t$$

where  $X_t$  is a semimartingale, we can write the portfolio value process as a stochastic exponent

$$\begin{aligned} V_t(x) &= x + \int_0^t \gamma_u V_u dX_u = x \mathcal{E}\left(\int_0^t \gamma_u dX_u\right)_t \\ &= x \exp\left(\int_0^t \gamma_u dX_u - \frac{1}{2} \left[\int_0^t \gamma_u dX_u, \int_0^t \gamma_u dX_u\right]_t\right) \\ &= x \exp\left(\int_0^t \gamma_u dX_u - \frac{1}{2} \int_0^t \gamma_u^2 d[X, X]_u\right) \end{aligned}$$

By the assumptions, the risky assets price process is a continuous semimartingale and the strategy process  $H$  is a bounded and adapted function of the underlying price process. Therefore process  $\gamma_t = \gamma(X_t)$  is bounded and adapted, and

$$z_t = \exp\left(\int_0^t \gamma(X_u) dX_u - \frac{1}{2} \int_0^t \gamma(X_u)^2 d[X, X]_u\right) \quad (3.3)$$

defines a continuous local martingale on  $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ . As it was assumed that the set of equivalent martingale measures  $\mathcal{M}^e(P)$  is non-empty, we can write

$$\begin{aligned} V_T(x) &= x z_T \\ &= x \mathbb{E}_Q\left[\frac{dP}{dQ} \mid \mathcal{F}_T\right] \end{aligned} \quad (3.4)$$

■

It has been shown (see f. ex. Schachermayer, 2001) that for a general utility maximization problem of the form (3.1) the solution is given as an optimal value process  $\hat{V}$  that is of the form

$$\hat{V}(y) = I\left(y \frac{d\hat{Q}}{dP}\right)$$

where  $I$  is the inverse function as defined in (2.7),  $y$  is related to initial wealth through  $y = \hat{U}'(x)$  and  $\hat{Q} \in \mathcal{M}^e(S)$ .

In a continuous model the set of measures with respect to which the underlying process is a martingale is equal to  $\mathcal{M}^e(S)$  (Delbaen, Schachermayer,

2002). For any two measures  $P \sim Q$  we have  $\tilde{z} = \frac{dP}{dQ} = \frac{1}{z}$ , where  $z = \frac{dQ}{dP}$ . Therefore

$$\begin{aligned}
\tilde{z}_t &= \mathbb{E}_Q\left[\frac{dP}{dQ} \mid \mathcal{F}_t\right] \\
&= \exp\left(\int_0^t \gamma(X_u) dX_u - \frac{1}{2} \int_0^t \gamma(X_u)^2 d[X, X]_u\right) \\
&= \frac{1}{z_t} = \frac{1}{\mathbb{E}_P\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right]} \\
&= \exp\left(-\int_0^t \gamma(X_u) dX_u + \frac{1}{2} \int_0^t \gamma(X_u)^2 d[X, X]_u\right)
\end{aligned}$$

for a measurable bounded  $\gamma$ .

Potential candidates for the optimal portfolio value process can thus be presented as a sequence of random variables of the form

$$\begin{aligned}
\hat{V}_t(x) &= I\left(y \frac{dQ_t}{dP_t}\right) \\
&= I\left(y \exp\left(-\int_0^t \gamma(X_u) dX_u + \frac{1}{2} \int_0^t \gamma(X_u)^2 d[X, X]_u\right)\right) \quad (3.5)
\end{aligned}$$

for  $Q \in \mathcal{M}^e(S)$  and a bounded measurable  $\gamma$ .

The question that remains is whether all measures  $Q \in \mathcal{M}^e(S)$  can be presented in the form (3.3). We do not actually have to show that since it will become apparent later that the optimal value  $\hat{U}$  can be reached using an allocation  $\gamma$  from above.

## 4 Recursive version of the problem

Instead of solving the problem for the whole period  $[0, T]$  at a time, we will now turn the attention to a local solution for a period defined by two stopping times  $\tau_1$  and  $\tau_2 \in [0, T]$ . For that we will present the problem in a recursive form, where the investor is choosing the optimal allocation process for any period  $[\tau_1, \tau_2] \subset [0, T]$  assuming that the expected utility will be maximized for the remaining period  $[\tau_2, T]$ . The allocation is then optimized by maximizing the expected value of the portfolio in the first period. We show that utility from optimizing this recursive problem equals the optimal value process of the general utility maximization problem.

For selecting  $\gamma$  we present a criteria that produces a strategy process that is optimal in the sense that the value process of a portfolio using the given allocation rule is asymptotically indistinguishable from the optimal value process of the general problem.

Let  $\tau_0, \dots, \tau_n$  be an increasing sequence of stopping times such that they form a partition of the interval  $[0, T]$ , that is,  $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{n-1} \leq \tau_n = T$ . We will refer to the partition by  $\pi_n = \{\tau_0, \tau_1, \dots, \tau_{n-1}, \tau_n\}$ . We know that for  $Q \sim P$  the Radon-Nikodym derivative  $\frac{dQ_T}{dP_T} = \mathbb{E}_P\left[\frac{dQ}{dP} \mid \mathcal{F}_T\right]$  can be presented

as a product

$$\frac{dQ_T}{dP_T} = h_{\tau_1} h_{\tau_2} \cdot \dots \cdot h_{\tau_n}$$

where  $h_{\tau_i} > 0$  and  $\mathbb{E}_P[h_{\tau_i} | \mathcal{F}_{\tau_{i-1}}] = 1$ . Specifically, in the model studied here we will be interested in the formulation

$$\begin{aligned} z_T &= \mathbb{E}_P\left[\frac{dQ}{dP} \mid \mathcal{F}_T\right] \\ &= \exp\left(-\int_0^T \gamma_u dX_u + \frac{1}{2} \int_0^T \gamma_u^2 d[X, X]_u\right) \\ &= \exp\left(-\int_0^{\tau_1} \gamma_u dX_u + \frac{1}{2} \int_0^{\tau_1} \gamma_u^2 d[X, X]_u\right) \cdot \dots \\ &\quad \dots \cdot \exp\left(-\int_{\tau_{n-1}}^T \gamma_u dX_u + \frac{1}{2} \int_{\tau_{n-1}}^T \gamma_u^2 d[X, X]_u\right) \\ &= \frac{z_{\tau_1}}{z_{\tau_1}} \cdot \frac{z_{\tau_2}}{z_{\tau_1}} \cdot \dots \cdot \frac{z_T}{z_{\tau_{n-1}}} \end{aligned}$$

that is,  $h_i = \frac{z_i}{z_{i-1}}$ . If needed, we will make explicit the function  $\gamma$  driving  $z$  and  $h$  by notation  $z_t^\gamma$  and  $h_t^\gamma$  respectively.

Corresponding to the set of allowed strategies  $\mathcal{H}$  we denote by  $\mathcal{A}_\tau$ ,  $\tau \in \pi_n$ , the set of admissible allocations such that for a random variable  $\xi \in \mathcal{F}_\tau$

$$\gamma \in \mathcal{A}_\tau \Leftrightarrow \xi z_t \geq \epsilon, \quad \text{for } \tau \leq t \leq T, \quad \epsilon > 0$$

Because we are working with a continuous model we can use the results from Ankirchner (2004) and assume that the process  $\gamma$  is a stopped process fulfilling the downside constraint.

**Lemma 4.1** Denote  $\hat{U}_{\tau,T}(\xi) = \sup \mathbb{E}_P[U(V_{\tau,T}(\xi)) | \mathcal{F}_\tau]$ , where  $0 \leq \tau < \kappa \leq T$  are two stopping times from the partition  $\pi_n$ . Then

$$\hat{U}_{\tau,T}(\xi) = \sup_{\gamma \in \mathcal{A}_\tau} \mathbb{E}_P[\hat{U}_{\kappa,T}(V_{\tau,\kappa}(\xi)) | \mathcal{F}_\tau] \quad (4.1)$$

**Proof.** Let  $\gamma^{\tau,T} \in \mathcal{A}_\tau$  and  $\gamma^{\kappa,T} \in \mathcal{A}_\kappa$  be two strategies defined on intervals  $[\tau, T]$  and  $[\kappa, T]$  respectively. Then a strategy

$$\tilde{\gamma} = \gamma^{\tau,T} 1_{[\tau,\kappa[} + \gamma^{\kappa,T} 1_{[\kappa,T[}$$

is also admissible, that is  $\tilde{\gamma} \in \mathcal{A}_\tau$ . We denote  $h_{\tau,\kappa} = \frac{z_\kappa}{z_\tau} = \exp\left(-\int_\tau^\kappa \gamma(X_u) dX_u + \frac{1}{2} \int_\tau^\kappa \gamma(X_u)^2 d[X, X]_u\right)$ . Using the definition of the value process from (3.4) and with notation defined here we have for any  $\gamma^{\tau,T} \in \mathcal{A}_\tau$ ,  $\gamma^{\kappa,T} \in \mathcal{A}_\kappa$

$$\begin{aligned} \hat{U}_{\tau,T}(\xi) &= \sup_{\gamma} \mathbb{E}_P[U(\xi h_{\tau,T}) | \mathcal{F}_\tau] \\ &\geq \sup_{\tilde{\gamma} \in \mathcal{A}_\tau} \mathbb{E}_P[U(\xi h_{\tau,T}^{\tilde{\gamma}}) | \mathcal{F}_\tau] \\ &= \sup_{\gamma^{\tau,T} \in \mathcal{A}_\tau, \gamma^{\kappa,T} \in \mathcal{A}_\kappa} \mathbb{E}_P[U(\xi h_{\tau,\kappa}^{\gamma^{\tau,T}} h_{\kappa,T}^{\gamma^{\kappa,T}}) | \mathcal{F}_\tau] \\ &= \sup_{\gamma^{\tau,T} \in \mathcal{A}_\tau, \gamma^{\kappa,T} \in \mathcal{A}_\kappa} \mathbb{E}_P[\mathbb{E}_P[U(\xi h_{\tau,\kappa}^{\gamma^{\tau,T}} h_{\kappa,T}^{\gamma^{\kappa,T}}) | \mathcal{F}_\kappa] | \mathcal{F}_\tau] \end{aligned}$$

So specifically, as  $X$  is continuous and locally bounded

$$\begin{aligned}
\hat{U}_{\tau,T}(\xi) &\geq \sup_{\gamma^{\tau,T} \in \mathcal{A}_\tau} \mathbb{E}_P \left[ \sup_{\gamma^{\kappa,T} \in \mathcal{A}_\kappa} \mathbb{E}_P [U((\xi h_{\tau,\kappa}^{\gamma^{\tau,T}}) h_{\kappa,T}^{\gamma^{\kappa,T}}) \mid \mathcal{F}_\kappa] \mid \mathcal{F}_\tau \right] \\
&= \sup_{\gamma \in \mathcal{A}_\tau} \mathbb{E}_P [\hat{U}_{\kappa,T}(\xi h_{\tau,\kappa}) \mid \mathcal{F}_\tau] \\
&= \sup_{\gamma \in \mathcal{A}_\tau} \mathbb{E}_P [\hat{U}_{\kappa,T}(V_{\tau,\kappa}(\xi)) \mid \mathcal{F}_\tau]
\end{aligned}$$

On the other hand, let  $\gamma^n = \gamma 1_{] \kappa, T ]} 1_{\{z_\kappa^\gamma \leq n\}}$ . Then  $\gamma^n \leq \gamma 1_{] \kappa, T ]}$  and  $\limsup \gamma^n = \gamma 1_{] \kappa, T ]}$  so

$$\begin{aligned}
\hat{U}_{\tau,T}(\xi) &= \sup_{\gamma \in \mathcal{A}_\tau} \mathbb{E}_P [U(\xi h_{\tau,T}) \mid \mathcal{F}_\tau] \\
&\leq \sup_{\gamma^{\tau,T} \in \mathcal{A}_\tau, \gamma^{\kappa,T} \in \mathcal{A}_\kappa} \mathbb{E}_P [\mathbb{E}_P [U(\xi h_{\tau,\kappa}^\gamma h_{\kappa,T}^{\gamma^n}) \mid \mathcal{F}_\kappa] \mid \mathcal{F}_\tau] \\
&\leq \sup_\gamma \mathbb{E}_P [\sup_n \mathbb{E}_P [U(\xi h_{\tau,\kappa}^\gamma h_{\kappa,T}^{\gamma^n}) \mid \mathcal{F}_\kappa] \mid \mathcal{F}_\tau] \\
&\leq \sup_\gamma \mathbb{E}_P [\hat{U}(\xi h_{\tau,T}^\gamma) \mid \mathcal{F}_\tau]
\end{aligned}$$

because  $\mathbb{E}_P [U(\xi h_{\tau,\kappa}^\gamma h_{\kappa,T}^{\gamma^n}) \mid \mathcal{F}_\kappa] \leq \sup \mathbb{E}_P [U(\xi h_{\tau,\kappa}^\gamma h_{\kappa,T}^\gamma) \mid \mathcal{F}_\kappa]$  for all  $n$ . ■

Next we show that the solution to the recursive problem is identical with the solution of the general problem at any period  $[\tau, T]$  defined by a stopping time  $\tau \leq T$ .

**Lemma 4.2** *Let  $\gamma^*$  be the optimal allocation in the general utility maximization problem and denote by  $\hat{\gamma}$  the optimal allocation in the recursive utility maximization problem (4.1). Then we have*

$$\hat{\gamma} = \gamma^* 1_{[\tau, T]} \tag{4.2}$$

**Proof.** Let  $H^*$  the the optimal strategy process of the general utility maximization problem (3.1) and  $\hat{H}$  the optimal strategy of the corresponding recursive utility maximization problem. We proof the claim in the form  $\hat{H} = H^* 1_{[\tau, T]}$ , that is, we show that

$$\sup_H \mathbb{E}_P [U(\xi + \int_\tau^T H_u dS_u) \mid \mathcal{F}_\tau] = \mathbb{E}_P [U(\xi + \int_\tau^T (H^* 1_{[\tau, T]})_u dS_u) \mid \mathcal{F}_\tau]$$

As the model is continuous we can work the proof in the set  $\xi \in [-C, M]$  where  $\hat{H}$  is admissible if  $H^*$  is admissible. So we have

$$\begin{aligned}
\mathbb{E}_P [U(\xi + \int_\tau^T \hat{H}_u dS_u) \mid \mathcal{F}_\tau] &= \mathbb{E}_P [U(\xi + \int_\tau^T (H^* 1_{[\tau, T]})_u dS_u) \mid \mathcal{F}_\tau] \\
&\leq \sup_H \mathbb{E}_P [U(\xi + \int_\tau^T H_u dS_u) \mid \mathcal{F}_\tau]
\end{aligned}$$

On the other hand we can represent  $\xi \in \mathcal{F}_\tau$  with the help of an admissible strategy  $L$  as  $\xi = l + \int_0^\tau L_u dS_u$ ,  $l \in \mathbb{R}$ , and we have

$$\begin{aligned}
\sup_H \mathbb{E}_P[U(\xi + \int_\tau^T H_u dS_u) \mid \mathcal{F}_\tau] &= \sup_H \mathbb{E}_P[U(l + \int_0^\tau L_u dS_u + \int_\tau^T H_u dS_u) \mid \mathcal{F}_\tau] \\
&= \sup_H \mathbb{E}_P[U(l + \int_0^T (L_u 1_{[0,\tau[} + H_u 1_{[\tau,T]}) dS_u) \mid \mathcal{F}_\tau] \\
&\leq \mathbb{E}_P[U(l + \int_0^T H_u^* dS_u) \mid \mathcal{F}_\tau] \\
&= \mathbb{E}_P[U(l + \int_0^\tau H_u^* dS_u + \int_\tau^T H_u^* dS_u) \mid \mathcal{F}_\tau] \\
&= \mathbb{E}_P[U(\xi + \int_\tau^T H_u^* dS_u) \mid \mathcal{F}_\tau]
\end{aligned}$$

for an  $l \in \mathbb{R}$  s.t.  $l + \int_0^\tau H_u^* dS_u = \xi$ . As the model is continuous and locally bounded, we have corresponding to any strategy  $H \in \mathcal{H}$  a unique allocation  $\gamma \in \mathcal{A}$  and thus the claim is proved. ■

We now move to the structure of the actual strategy that gives us the optimal level of utility at the terminal time  $T$ , or as we have just seen, at any stopping time  $0 \leq \tau \leq T$ . We first characterize the strategy as a general decision rule and then calculate the strategy explicitly in the next section.

**Lemma 4.3** *Let  $\gamma^* \in \mathcal{A}$  be such that*

$$\mathbb{E}_P[V_{\tau,\kappa}^\gamma(\xi) \mid \mathcal{F}_\tau] \leq \mathbb{E}_P[V_{\tau,\kappa}^{\gamma^*}(\xi) \mid \mathcal{F}_\tau]$$

for any  $\gamma \in \mathcal{A}$ . Then  $\gamma^*$  is an utility maximizing strategy.

**Proof.** For any fixed  $\xi \in \mathcal{F}_\tau$ , any strategy  $\delta \in \mathcal{A}$  produces a payoff that is less or equal the optimal value, and on the other hand,

$$\begin{aligned}
\hat{U}_{\tau,T}(\xi) &= \sup_{\gamma \in \mathcal{A}} \mathbb{E}_P[\hat{U}_{\kappa,T}(V_{\tau,\kappa}(\xi)) \mid \mathcal{F}_\tau] \\
&= \sup_{\gamma \in \mathcal{A}} \mathbb{E}_P[\mathbb{E}_P[\hat{U}_{\kappa,T}(V_{\tau,\kappa}(\xi)) \mid \mathcal{F}_\tau] \mid \mathcal{F}_\tau] \\
&\leq \sup_{\gamma \in \mathcal{A}} \mathbb{E}_P[\hat{U}_{\kappa,T}(\mathbb{E}_P[(V_{\tau,\kappa}(\xi) \mid \mathcal{F}_\tau]) \mid \mathcal{F}_\tau)] \\
&\leq \sup_{\gamma \in \mathcal{A}} \mathbb{E}_P[\hat{U}_{\kappa,T}(\sup \mathbb{E}_P[(V_{\tau,\kappa}(\xi) \mid \mathcal{F}_\tau]) \mid \mathcal{F}_\tau)]
\end{aligned}$$

As  $\gamma \in \mathcal{A}$ , solution to  $\mathbb{E}_P[V_{\tau,\kappa}(\xi) \mid \mathcal{F}_\tau] \rightarrow \max!$  is admissible, and because the model is continuous  $\mathcal{F}_t = \sigma(S_t)$  is continuous and so  $\gamma^*$  is a strategy. ■

## 5 Solution for a sequence of simple strategies

Solution of the utility maximization problem is a stochastic process that will later be referred to as the dynamic solution. In practice portfolio managers are never able to adjust their portfolios continuously so what is required is an – even locally – static solution of the problem. In stochastic finance that is done with the help of simple strategies.

In this section we will define a sequence of simple strategies  $\gamma^n$  defined on the partition  $\pi_n$  that will converge to the continuous dynamic strategy  $\gamma$  as  $n \rightarrow \infty$ . Using the piecewise defined process we can define a value process that will converge to the value process of the dynamic portfolio. We then solve the optimization problem for one piece (or a period defined by two stopping times  $\tau_i$  and  $\tau_{i+1} \in [0, T]$ ) at a time. Theorem (13) presents the solution with a proof that combines the lemmas from above.

In the following we use extensively the results for simple strategies from Ankirchner (2005).

**Definition 5.1** *A strategy  $\delta \in \mathcal{A}_0(S, \pi_n)$  is called **simple** (with respect to a partition  $\pi_n$ ) if it can be presented in the form*

$$\delta = \sum_{i=0}^{n-1} \delta_i 1_{] \tau_i, \tau_{i+1} ]}$$

$\tau_i \in \pi_n$ .

Let  $V_{\tau_i, \tau_j}^n(\xi) = I(\xi \exp(-\int_{\tau_i}^{\tau_j} \delta_u dX_u + \frac{1}{2} \int_{\tau_i}^{\tau_j} \delta_u d[X, X]_u))$ , where  $\delta$  is a simple strategy defined on the partition  $\pi_n$  and  $\xi \in \mathcal{F}_{\tau_i}$ . In the following we want to show that there exists a sequence of simple strategies that is asymptotically optimal in the sense that

$$P\{\hat{V}_{0,T}^n(\xi) - \hat{V}_{0,T}(\xi) > \epsilon\} \rightarrow 0$$

when the partition of  $[0, T]$  is infinitely refined. In a special case we present the sequence in a way that is very similar to the mean-variance optimal strategy.

**Lemma 5.2** *There is a sequence of simple strategies  $\gamma^n = \{\gamma_0^n, \gamma_1^n, \dots, \gamma_{n-1}^n\}$  such that*

$$\begin{aligned} \hat{V}_{0,T}^n(x) &= \sup \mathbb{E}_P[V_T^n] \\ &\rightarrow \hat{V}_{0,T}(x) \end{aligned}$$

**Proof.** *Let us define a sequence of simple strategies  $\gamma^n$  by*

$$\gamma_i^n = \gamma_{\tau_i} I_{] \tau_i, \tau_{i+1} [}$$

for  $i = 0, 1, \dots, n-1$  and  $\tau_i \in \pi_n$ . As  $S$  is continuous and locally bounded we have

$$(\gamma^n \cdot X)_t \rightarrow (\gamma \cdot X)_t$$

and

$$((\gamma^n)^2 \cdot [X, X])_t \rightarrow (\gamma^2 \cdot [X, X])_t$$

as  $n \rightarrow \infty$  for a bounded measurable  $\gamma$ . Thus for all  $t \in [0, T]$

$$\begin{aligned} z_t^{\gamma^n} &= \exp\left(-\int_0^t \gamma_u^n dX_u + \frac{1}{2} \int_0^t (\gamma_u^n)^2 d[X, X]_u\right) \\ &\rightarrow \exp\left(-\int_0^t \gamma_u dX_u + \frac{1}{2} \int_0^t (\gamma_u)^2 d[X, X]_u\right) \\ &= z_t^\gamma \end{aligned}$$

By the assumptions that  $I$  is continuous and  $S$  is locally bounded we have

$$\begin{aligned} \hat{V}_T^n(x) &= \sup_{\gamma^n} \mathbb{E}_P[V_T^n] \\ &= \sup_{\gamma^n} \mathbb{E}_P[I(yz_T^{\gamma^n})] \\ &\rightarrow \sup_{\gamma} \mathbb{E}_P[I(yz_T^\gamma)] \\ &= \hat{V}_{0,T}(x) \end{aligned}$$

■

Now we are finally ready to present the major result of this paper.

**Theorem 5.3** *For an investor with cara utility function the utility maximizing allocation  $\hat{\gamma}_i$  in the risky asset at any given stopping time  $\tau_i \in \pi_n$  is given by*

$$\hat{\gamma}_i = \frac{\mathbb{E}_P[X_{\tau_{i+1}} - X_{\tau_i} \mid \mathcal{F}_\tau]}{\mathbb{E}_P[[X, X]_{\tau_{i+1}} - [X, X]_{\tau_i} \mid \mathcal{F}_\tau]} \quad (5.1)$$

**Proof.** We consider an asset allocation problem defined in (3.1). According to lemma 7 the solution to the problem can be given by an optimal utility  $\tilde{U}$  that can be attained by recursively optimizing value process  $\hat{V}_{\tau, \kappa}(\xi)$ . Let us choose a partition  $\pi_n$  of the interval  $[0, T]$  such that  $\pi_n$  can be infinitely refined. Corresponding to the partition we define a sequence of simple strategies by  $\gamma_i^n = \gamma_{\tau_i} I_{[\tau_i, \tau_{i+1}[}$  for  $\tau_i \in \pi_n$ ,  $i = 0, \dots, n-1$ . According to lemma(10) the optimal strategy can be found by maximizing the conditional expectation of the portfolio value for a period  $[\tau_i, \tau_{i+1}[$ . For any  $\varsigma \in \mathcal{F}_{\tau_i}$  with  $U(\varsigma) \in L^1(P)$ . Set

$$f(\gamma_i) = \mathbb{E}_P\left[I(\varsigma \exp\left(-\int_{\tau_i}^{\tau_{i+1}} \gamma_i^n dS_u + \frac{1}{2} \int_{\tau_i}^{\tau_{i+1}} (\gamma_i^n)^2 d[S, S]_u\right)) \mid \mathcal{F}_{\tau_i}\right]$$

As  $X$  is continuous, we have  $\Delta X_\tau = X_\tau - X_\kappa \rightarrow 0$  and  $\Delta[X, X]_\tau = [X, X]_\tau - [X, X]_\kappa \rightarrow 0$  for any two stopping times  $\tau$  and  $\kappa$  as  $\tau \rightarrow \kappa$ . According to lemma (12) there is a sequence of simple strategies such that  $\hat{V}_{0,T}^n(x) \rightarrow \hat{V}_{0,T}(x)$ . Therefore the utility maximization problem can be reduced to a sequence of static maximization problems

$$f_i(\gamma_i) \rightarrow \max! \quad \forall i \in \{0, \dots, n-1\}$$

where  $f_i = \mathbb{E}_P[I(\varsigma \exp(-\gamma_i^n \Delta X_{\tau_i} + \frac{1}{2}(\gamma_i^n)^2 \Delta[X, X]_{\tau_i})) \mid \mathcal{F}_{\tau_i}]$ . Derivating with respect to  $\gamma_i$  gives

$$\begin{aligned}
& \frac{\partial}{\partial \gamma_i} \mathbb{E}_P[I(\varsigma \exp(-\gamma_i^n \Delta X_{\tau_i} + \frac{1}{2}(\gamma_i^n)^2 \Delta[X, X]_{\tau_i}))] = 0 \\
\iff & \mathbb{E}_P[I'(Z_{\tau_i}) Z_{\tau_i} \frac{\partial}{\partial \gamma} (-\gamma_i^n \Delta X_{\tau_i} + \frac{1}{2}(\gamma_i^n)^2 \Delta[X, X]_{\tau_i})] = 0 \\
\iff & -\frac{1}{c} \mathbb{E}_P[-\Delta X_{\tau_i} + \gamma_i^n \Delta[X, X]_{\tau_i}] = 0 \\
\iff & \gamma_i^n = \frac{\mathbb{E}_P[\Delta X_{\tau_i}]}{\mathbb{E}_P[\Delta[X, X]_{\tau_i}]} \tag{5.2}
\end{aligned}$$

where we used the fact that for a cara utility function  $I'(Z_{\tau_i}(\omega))Z_{\tau_i}(\omega) = -\frac{1}{c}$  for all  $\omega \in \Omega$ , and denoted  $Z_{\tau_i} = \varsigma \exp(-\gamma_i^n \Delta X_{\tau_i} + \frac{1}{2}(\gamma_i^n)^2 \Delta[X, X]_{\tau_i})$ . The result follows. ■

**Corollary 5.4** *The optimal static allocation for a period  $[0, T]$  is given by*

$$\hat{\gamma} = \frac{\mathbb{E}_P[X_T - X_0]}{\mathbb{E}_P[[X, X]_T]}$$

Utility of the static allocation can be increased by refining the partition of the interval so that

$$\lim_{n \rightarrow \infty} \sup_{\gamma^n} \mathbb{E}_P[U(\hat{V}_T^{\gamma^n})] = \hat{U}(x)$$

**Proof.** *The form of the allocation is a straightforward calculation with respect to  $\mathcal{F}_0$ . The latter claim follows from the concavity of the utility function  $U$  as we have*

$$\sup_{\gamma^n} \mathbb{E}_P U(\hat{V}_T^{\gamma^n}) \leq \sup_{\gamma^n} U(\mathbb{E}_P[\hat{V}_T^{\gamma^n}]) \leq U(\lim_{n \rightarrow \infty} \sup_{\gamma^n} \mathbb{E}_P[\hat{V}_T^{\gamma^n}]) \tag{5.3}$$

■

What theorem 13 says is that the risk-return trade-off of a utility maximizing investor is really a trade-off between return and quadratic variation of the return. It is a common practice to use variance as a measure of risk, but as variance and quadratic variation do not generally coincide the mean-variance optimal portfolio is not in general a utility maximizing portfolio.

Condition (5.1) also tells us that given a level of target return, the investor should not be interested on the stochastic properties of the underlying model, but instead he should select the portfolio according to the expected path properties of the underlying process. Specifically, different assets can be divided into separate classes according to the quadratic variation of their paths. That approach would be very similar to the one proposed by Bender et al (2006) in the context of asset pricing.

Like in this paper, quadratic variation is recognized as the true measure of uncertainty in Andersen, Bollerslev, Diebold and Labys (2000) and their subsequent work. The difference of their approach to the one introduced here

is that they use realized volatility as an estimate of variance of the underlying (Gaussian) model. As is shown in the next section the estimate is in general unbiased only for instantaneous variance or local martingales. We approach the same problem – and come to almost the same solution – from a different angle by showing that in continuous model we do not need to make the assumption of normality if we change the measure of uncertainty from variance to quadratic variation.

## 6 Comparison of mean-variance optimal and utility maximizing strategies

In this section we make explicit the relationship between variance and quadratic variation. Connection between the two is presented in lemma 15 in the proof of which the basic Doob decomposition for semimartingales is utilized. As a corollary we get the second major result of this paper that makes explicit the relationship between a mean-variance optimal strategy and a utility maximizing strategy. With that we are able to generalize the results from Xia (2005) to a wider class of utility functions and asset price processes.

We make the presented results more concrete by giving examples of two different market models and calculating the risk measures variance and quadratic variation. We then express the optimal strategies for both optimality criterion. With the examples it will be clarified why the utility maximizing portfolio is superior to the mean-variance optimal portfolio.

To relate variance and quadratic variation of a continuous stochastic process we prove the following simple lemma:

**Lemma 6.1** *Let  $X$  be a semimartingale with a decomposition  $X_t = X_0 + M_t + A_t$ . If  $M$  and  $A$  are uncorrelated, variance of  $X$  is related to the quadratic variation of  $X$  as*

$$\text{var}(X_t) = \mathbb{E}[X, X]_t + \text{var}(A_t)$$

**Proof.** Assuming that  $X_0 = 0$ ,  $X$  has a decomposition  $X_t = M_t + A_t$ , where  $M_t$  is a local martingale and the compensator  $A_t$  is predictable and of bounded variation. If  $M$  and  $A$  are uncorrelated the variance of the process  $X$  is

$$\begin{aligned} \text{var}(X_t) &= \text{var}(M_t + A_t) \\ &= \text{var}(M_t) + \text{var}(A_t) \\ &= \mathbb{E}(M_t - \mathbb{E}M_t)^2 + \text{var}(A_t) \\ &= \mathbb{E}(M_t)^2 + \text{var}(A_t) \\ &= \mathbb{E}[M, M]_t + \text{var}(A_t) \end{aligned} \tag{6.1}$$

Using polarization equality we get

$$\begin{aligned} \mathbb{E}[X, X]_t &= \mathbb{E}[M + A, M + A]_t \\ &= \mathbb{E}[M, M]_t + \mathbb{E}[A, A]_t \\ &= \mathbb{E}[M, M]_t \end{aligned} \tag{6.2}$$

because  $A$  as a compensator is predictable and of bounded variation and thus has a quadratic variation of zero. ■

**Corollary 6.2** *Mean-variance optimal portfolio is a utility maximizing portfolio for an investor with cara utility if in the underlying market model the process of the risky asset  $X_t$  is a local martingale.*

**Proof.** Decomposition  $X_t = X_0 + M_t + A_t$  reduces to  $X_t = X_0 + M_t$ . ■

To make the difference between variance and quadratic variation more intuitive let us consider two simple market models that prove to be very illustrative in the analysis. In the first example the price process of the risky asset is modelled with geometric Brownian motion which will be slightly modified in the second example to include a stochastic drift.

**Example 6.3** *First, let us assume that stochastics of the underlying model is generated by Brownian motion  $W$ . The risky asset follows a diffusion*

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad (6.3)$$

Variance of the return series is

$$\begin{aligned} \text{var}\left(\frac{dS_t}{S_t}\right) &= \sigma^2 \text{var}(W_t) \\ &= \sigma^2 \mathbb{E}(W_t^2) \\ &= \sigma^2 t \end{aligned} \quad (6.4)$$

and quadratic variation of the model is

$$\begin{aligned} \left[\frac{dS}{S}\right]_t &= [\sigma W, \sigma W]_t \\ &= \sigma^2 [W, W]_t \\ &= \sigma^2 t \end{aligned} \quad (6.5)$$

So we have seen that in the case of Brownian motion the theoretical variance and quadratic variation process are equal. The optimal strategy  $\gamma^*$  of a utility maximizing investor is given by the condition

$$\begin{aligned} \gamma^* &= \frac{\mathbb{E}_{\mathbb{P}} S_T}{\mathbb{E}_{\mathbb{P}} [S, S]_T} \\ &= \frac{\mu T}{\sigma^2 T} \\ &= \frac{\mu}{\sigma^2} \end{aligned} \quad (6.6)$$

which coincides with the mean-variance optimal strategy.

**Example 6.4** *To incorporate the widely recognized stylized fact of fat tails into the underlying market model and still keep the model continuous we add to the basic model an integrated compound Poisson process  $U_t$  defined by*

$$U_t = \int_0^t C_s ds \quad (6.7)$$

where  $C_t = \sum_{n=1}^{\infty} X_n \mathbf{1}_{\{\tau_n \leq t\}}$ ,  $X_n \sim x^{-\alpha x}$  ( $\alpha > 0$ ),  $\tau_n \sim \text{Poisson}(\lambda)$  ( $\lambda > 0$ ). The modified market model is defined via the price process of the risky asset

$$\frac{d\hat{S}_t}{\hat{S}_t} = \mu dt + \sigma dW_t + U_t \quad (6.8)$$

where for simplicity we assume that  $W_t$  and  $U_t$  are independent. The model belongs to the class of semimartingale models because it admits a canonical decomposition  $\frac{dS_t}{S_t} = M_t + A_t$ , where  $M_t = \sigma dW_t$  is a martingale and  $A_t = \mu dt + U_t$  is continuous and of finite variation and predictable process. The process  $U_t$  is of zero quadratic variation because

$$\begin{aligned} \lim_{|\pi_n| \rightarrow 0} \sum (U_{t_{n_k}} - U_{t_{n_{k-1}}})^2 &= \lim_{|\pi_n| \rightarrow 0} \sum \left( \int_0^{t_{n_k}} C_s ds - \int_0^{t_{n_{k-1}}} C_s ds \right)^2 \\ &= \lim_{|\pi_n| \rightarrow 0} \sum \left( \int_{t_{n_{k-1}}}^{t_{n_k}} C_s ds \right)^2 \\ &= 0 \end{aligned}$$

where  $\pi_n$  is a partition of the interval  $[0, T]$  as in (13). From the polarization identity (15) we get

$$\begin{aligned} \left[ \frac{d\hat{S}}{\hat{S}} \right]_t &= [\sigma W + U, \sigma W + U]_t \\ &= [\sigma W, \sigma W]_t \\ &= \sigma^2 t \end{aligned} \quad (6.9)$$

Calculating theoretical variance gives us

$$\begin{aligned} \text{var}\left(\frac{dS_t}{S_t}\right) &= \sigma^2 \text{var}(W_t) + \text{var}(U_t) \\ &= \sigma^2 t + \text{var}(U_t) \end{aligned} \quad (6.10)$$

Because  $U_t$  is a stochastic process with non-constant paths the variance term  $\text{var}(U_t)$  is strictly positive and so the variance of the process is strictly greater than the quadratic variation of the process. In this case the optimal strategy  $\gamma^*$  of a utility maximizing investor is given by the condition

$$\begin{aligned} \gamma^* &= \frac{\mathbb{E}_{\mathbb{P}} \hat{S}_T}{\mathbb{E}_{\mathbb{P}}[\hat{S}, \hat{S}]_T} \\ &= \frac{\mathbb{E}_{\mathbb{P}} \hat{S}_T}{\sigma^2 T} \\ &= \frac{\mu T + \mathbb{E}_{\mathbb{P}}[U_T]}{\sigma^2 T} \end{aligned} \quad (6.11)$$

That utility maximizing strategy is clearly different from the mean variance optimal strategy  $\gamma^{MV}$  that would be given by

$$\gamma^{MV} = \frac{\mu T + \mathbb{E}_{\mathbb{P}}[U_T]}{\sigma^2 T + \text{var}(U_T)} \quad (6.12)$$

*Intuitively the difference between the two risk measures can be explained as noting that in the case of variance also the predictable part of variation would be considered as risk. If the investor is able to adjust the allocation continuously he can constantly hedge away the predictable part of the variation.*

In the above examples we have assumed that we know the underlying market model to make clear the difference between variance and quadratic variation. From the practitioners point of view, however, the most appealing feature in the optimal strategy given in Theorem 13 is that one does not have to know the true market model. Instead, the investor can directly focus on the potential variability of the process that is specified as quadratic variation.

Quadratic variation as a measure of uncertainty has been previously studied by Andersen et al (2003) and its estimation has been considered on one hand in stochastics by Dzhaparidze and Spreij (1994) and in finance lately among others by Ait-Sahalia and Mancini (2007) and references therein. In finance literature quadratic variation is often referred to as realized volatility (RV).

We want to stress that quadratic variation is used here as a measure of uncertainty and not as a measure of risk in the sense of Artzner et al (1999). As a risk measure quadratic variation would fulfill the requirement of convexity but it is not coherent as it punishes also for upward movements. Still, compared to variance quadratic variation works better as a measure of uncertainty because rewards diversification better (see f.ex. Föllmer and Schied, 2001). That might give us a reason to think that risk measures, i.e. functions on measures of uncertainty, based on quadratic variation would do better than risk measures based on variance (like VaR). But as the question is far beyond the scope of this paper it is left for further research.

## 7 Conclusions

In this paper we have shown that the optimal allocation can be presented in terms of expected drift and quadratic variation of the underlying price process. As a by-product we also show why mean-variance optimal portfolio is not in general a utility maximizing one. Heuristically that can be understood via the canonical decomposition of a semimartingale, as part of the variability of a stochastic process comes from a predictable component. As variance also accounts for that predictable (and thus hedgeable) part of variation of the process it is in a sense overestimating true uncertainty related to the underlying process. The second major drawback of variance is that by using it as a measure of uncertainty the investor assumes that the distribution of returns can be fully described by its first two moments. It is well documented that that is not the case and so variance does not capture all of the risk borne from non-normal distributions. Quadratic variation instead is model independent and it exists for any stochastic process that is a semimartingale. By counting the squared length of path of a process it is able to capture phenomena like fat tails and skewed distributions. With the help of polarization identity we can also treat portfolios with different types of risk.

The result leads to the same direction as some earlier research (Bender et al, 2006) saying that the true measure of uncertainty is quadratic variation and not variance (or any other stochastic property of the asset price process). It is also in agreement with the fact from stochastic analysis that a semimartingale can be fully presented by a triplet consisting of measures of drift, quadratic variation and jumps. As the model studied here is continuous, the result that optimal asset allocation can be presented in terms of drift and quadratic variation seems logical.

For practioneers the result should come as welcome news as expecting something from the behaviour of the path of a stochastic process is much less than knowing the whole distribution of the process. So by using the allocation rule presented here the investor can increase his utility by assuming less. On the practical side, estimation of quadratic variation – or realized variance – is very simple as one does not have to have anything else than the market prices because quadratic variation is model-free and non-parametric.

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