BAYESIAN ANALYSIS OF PARTICIPATING LIFE INSURANCE CONTRACTS WITH AMERICAN-STYLE OPTIONS

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ABSTRACT

In this paper a Bayesian approach is utilized to analyze the role of the underlying asset and interest rate model in the market consistent valuation of life insurance policies. The focus is on a novel application of advanced theoretical and computational methods. A guaranteed participating contract embedding an American-style option is considered. This option is valued using the regression method. We exploit the flexibility inborn in Markov Chain Monte Carlo methods in order to deal with a fairly realistic valuation framework. The Bayesian approach enables us to address model and parameter error issues. Our empirical results support the use of elaborated instead of stylized models for asset dynamics in practical applications. Furthermore, it appears that the choice of model and initial values is essential for risk management.

KEYWORDS

American-style option, Metropolis algorithm, Model error, Risk neutral valuation, Solvency II, Stochastic interest rate

1 INTRODUCTION

Most participating life insurance policies include implicit options representing a significant risk to the company issuing these contracts. Concern over implicit options is also reflected in recent regulatory processes: one of the key objectives of the Solvency II project is to encourage and provide incentive for insurance companies to measure and manage their risks better. Also financial reporting requires an evaluation of the market value of implicit options at fair value, c.f. e.g. European Comission (2008), Gatzert and Schmeiser (2006), and Ronkainen et al. (2007). Not surprisingly, market consistent valuation of life insurance contracts has become a popular research area among actuaries and financial mathematicians; see e.g. Tanskanen and Lukkarinen (2003), Bernard et al. (2005), Ballotta et al. (2006), Bauer et al. (2006) and Grosen and Jorgensen (2000). However, most valuation models allowing for sophisticated bonus distribution rules and the inclusion of frequently offered options assume a simplified set-up. One of the aims of this paper is to present a more realistic framework in which participating life insurance contracts including guarantees and options can be valuated and analyzed.

Assumptions on the price dynamics of underlying assets lead to a partial differential equation characterizing the price of the option. Several features may, however, limit the suitability of calculating option prices directly by solving partial differential equations. The reason for this is that apart from "vanilla options", most calculations involve the evaluation of high-dimensional integrals. For instance, if the asset price dynamics are sufficiently complex (the payoff of an option depends on the paths of the underlying assets) or if the number of underlying assets required by the replicating strategy is large (greater than three), a partial differential equation characterizing the option price may be difficult to solve. Instead, Monte Carlo methods are used routinely in pricing this kind of derivatives (Glasserman, 2003). Nonetheless, pricing American-style options via Monte Carlo simulation still remains a very challenging task. The problem lies in the estimation of the early exercise decisions available.

Applications of Monte Carlo methods in life insurance are more scarce. Zaglauer and Bauer (2008) present a framework in which participating life insurance contracts can be valuated and analyzed in a stochastic interest rate environment using Monte Carlo and discretization methods. Bacinello et al. (2008) describe an algorithm based on the Least Squares Monte Carlo method to price American options. Their framework allows e.g. randomness in mortality. Hardy (2002) uses Bayesian MCMC methods for a different problem, i.e. the risk management of equity-linked insurance.

The price of an option depends on the model describing the behavior of the underlying instrument. Most approaches specify a particular stochastic process to represent the price dynamics of the underlying asset and then derive an explicit pricing model. However, neither the true model, nor its parameter values are known. A common practice is to assume a relatively simple model, and to use point estimates of the model parameters. Yet many options in practice require an elaborate time-series specification for the price dynamics of the underlying asset, since a too simple model might fail to explain the price of its derivative (see, e.g., Brigo and Mercurio, 2001). Hence, it becomes difficult at best to derive explicit pricing formulae. Furthermore, with the additional complexity of a rich time-series specification, estimation uncertainty becomes a genuine concern.

Participating life insurance contracts are characterized by an interest rate guarantee and some bonus distribution rules. One of the most most common options available is the possibility to exit (surrender) the contract before maturity and receive a lump sum reflecting the insurer's past contribution to the policy minus some charges. These American-style options are called surrender options. In the related research the emphasis has been on the mathematics of pricing and on Monte Carlo experiments.

In this article we describe in detail how to apply Bayesian statistics to value participating life insurance contracts including surplus options using a fairly realistic model for assets and interest rates. However, we ignore the risk from mortality in this analysis. We follow Bunnin et al. (2002), who use Bayesian numerical techniques to price a European Call option on a share index. When estimating the option price, they simulate the posterior predictive distribution of the underlying asset by averaging over alternative models and their parameters, thus taking into account the uncertainty related to them. In order to value American-style options we use the Longstaff and Schwartz (2001) regression approach, which approximates the value of the option against a set of basic functions. We address questions about: a) implementation of MCMC and regression methods for option pricing, b) statistical modelling and analysis of financial time series, c) model and parameter errors. The two major benefits from using Bayesian techniques are that we can explicitly acknowledge the risks associated to model choice and parameter estimation.

The paper is organized as follows. Section 2 introduces the framework and model, Section 3 presents the estimation and evaluation procedures and Section 4 the empirical results. The final Section 5 concludes.

2 THE FRAMEWORK

2.1 The participating life insurance contract

Our goal is to price a participating life insurance, which consists of two parts. The first part is a guaranteed interest and the second part a bonus depending on the yield of some equity index. We denote the amount of savings in the insurance contract at time t_i by $Y(t_i)$. Then its growth during a time interval of length $\delta = t_{i+1} - t_i$ is given by

$$\log \frac{Y(t_{i+1})}{Y(t_i)} = g\,\delta + b \max\left(0, \log \frac{X(t_{i+1})}{X(t_i)} - g\,\delta\right), \tag{1}$$

where $X(t_i) = \sum_{j=0}^{q} S(t_{i-j})/(q+1)$ is the moving average of the equity index total return $S(t_i)$. The guaranteed rate g is set to be less than the riskless interest rate. The bonus rate b is the proportion of the excessive equity index yield that is returned to the customer. In this study we use the time interval $\delta = 1/255$, where 255 is approximately the number of the days in a year on which the index is quoted. The model also incorporates a surrender (early exercise) option and the possibility for a penalty, which occurs if the customer reclaims the contract before the final expiration date.

In the following, we will consider the cases when (i) the riskless interest rate is fixed at a predetermined value r and (ii) it is assumed to be stochastic. For the constant interest rate r the guaranteed rate g is set at kr throughout the entire contract period for some constant $k < 1$. In the case of stochastic interest rate the guaranteed rate is fixed for one year at a time. It is set annually at kr_t, where r_t is the short-term interest rate at time t. In this framework the penalty for early exercise and the parameters k, g and b are predefined by the insurance company. We determine the market consistent bonus rate such that the price of the contract will be equal to the initial savings. This gives the contract a simple structure and makes its costs and returns visible and predictable for the insurer and the customer. Our main interest is to study the effects of the expiration date, guarantee rate and penalty rate on the fair bonus rate in both constant and stochastic interest rate cases.

2.2 Model with constant interest rate

The constant elasticity of variance (CEV) model introduced by Cox and Ross (1976) is used to model the equity index process. It is defined by the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \nu S_t^{1-\alpha}\mathrm{d}W_t, \tag{2}$$

where μ, ν and α are fixed parameters and W_t is a standard Brownian motion under the real-world probability measure. If $\alpha = 0$, the model (2) becomes a geometric Brownian motion. The model may also be written in the form

$$\mathrm{d}S_t = rS_t \mathrm{d}t + \nu S_t^{1-\alpha}\mathrm{d}Z_t, \tag{3}$$

where r is the riskless short-term interest rate and Z_t the standard Brownian motion under a risk-neutral probability measure. Parameters ν and α are unknown and will be estimated.

2.3 Model with stochastic interest rate

Let $Z_t^{(i)}, i = 1, 2, 3$ be standard Brownian motions under a risk-neutral probability measure Q, where the Brownian motions $Z_t^{(1)}$ and $Z_t^{(3)}$ are assumed to be independent and $Z_t^{(1)}$ and $Z_t^{(2)}$ correlated through

$$Z_t^{(2)} = \rho Z_t^{(1)} + \sqrt{1+\rho^2} Z_t^{(3)}.$$

We assume that the riskless short-term interest rate r_t follows the process

$$\mathrm{d}r_t = \kappa(\xi - r_t)\mathrm{d}t + \sigma r_t^{\gamma} \mathrm{d}Z_t^{(1)}, \tag{4}$$

where κ, ξ, σ and γ are unknown parameters, which will be estimated. This process was introduced by Chan et al. (1992), who provide a good summary of short-term interest rate models in their paper. The two most commonly used models which may be derived from (4) by parameter restriction are the following: If $\gamma = 0$, the model becomes the Ornstein-Uhlenbeck process proposed by Vasiček (1977) as a model of the short rate, and, if $\gamma = \frac{1}{2}$, it becomes the square-root diffusion referred to as the Cox-Ingersoll-Ross (CIR) model (Cox et al., 1985).

Similarly to the fixed rate case, we assume that the stock index follows the CEV process and that its stochastic differential equation under Q is

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \nu S_t^{1-\alpha} \mathrm{d}Z_t^{(2)}. \tag{5}$$

Now the discounted price $\tilde{S}_t = S_t \exp(-\int_0^t r_s ds)$ is a martingale under Q.

To our knowledge, this system of equations does not have a closed form solution. Therefore, we will use its Euler discretization for estimation and simulation purposes. In order to obtain numerical stability in estimation, we reparametrize model (4) as

$$\mathrm{d}x_t = (\beta - \kappa x_t)\mathrm{d}t + \tau x_t^{\gamma} \mathrm{d}Z_t^{(1)},$$

where $x_t = 100\ r_t$ (the interest rate given in percentages), $\beta = 100\ \kappa\xi$ and $\tau = (100)^{1-\gamma}\sigma$. Assuming that the bivariate process has been observed at equally-spaced time points $0, \delta, ...N\delta$, the likelihood function can be written in the form

$$\begin{aligned} p(y|\theta) &= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\tau^2 x_{(i-1)\delta}^{2\gamma}\delta}} \exp\left(-\frac{(\Delta x_{i\delta} - (\beta - \kappa x_{(i-1)\delta})\delta)^2}{2\tau^2 x_{(i-1)\delta}^{2\gamma}\delta}\right) \\ &\times \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\nu^2 S_{(i-1)\delta}^{2(1-\alpha)}(1-\rho^2)\delta}} \exp\left(-\frac{\left(\Delta S_{i\delta} - \mu S_{(i-1)\delta}\delta - \nu S_{(i-1)\delta}^{1-\alpha}\rho\Delta Z_{i\delta}^{(1)}\right)^2}{2\nu^2 S_{(i-1)\delta}^{2(1-\alpha)}(1-\rho^2)\delta}\right), \end{aligned} \tag{6}$$

where y is data, $\theta = (\nu, \alpha, \beta, \kappa, \tau, \gamma, \rho)$, $\Delta x_{i\delta} = x_{i\delta} - x_{(i-1)\delta}$, $\Delta S_{i\delta} = S_{i\delta} - S_{(i-1)\delta}$ and

$$\Delta Z^{(1)}_{i\delta} = \frac{x_{i\delta} - x_{(i-1)\delta} - (\beta - \kappa x_{(i-1)\delta})\delta}{\tau x^{\gamma}_{(i-1)\delta}}.$$

3 ESTIMATION AND EVALUATION PROCEDURES

3.1 The Metropolis algorithm

The unknown parameters of the stock index and interest rate models are estimated using Bayesian methods. This makes it possible to take parameter uncertainty into account when evaluating the fair prices of derivatives. We follow Bunnin et al. (2002) and simulate the paths of the underlying asset using their posterior predictive distribution. However, we do not average over models, since we assume that model uncertainty can be taken into account by using a sufficiently general, continuously parametrized family of distributions (see Gelman et al., 2004). In both fixed and stochastic interest rate cases we use the Metropolis algorithm introduced by Metropolis et al. (1953) to simulate the joint posterior distribution of the unknown parameters.

The Metropolis algorithm is a Markov Chain Monte Carlo (MCMC) method and can be used to simulate Markov chains with given stationary distributions. MCMC methods are especially useful when direct sampling from a probability distribution is difficult. The Metropolis algorithm is based on an acceptance/rejection sampling and is thus more flexible than the Gibbs sampler, which presumes the full conditional distributions of the target distribution to be known. In order to implement the Metropolis algorithm, one only needs to know the joint density function of the target distribution up to a constant of proportionality.

Suppose that we wish to simulate a (multivariate) distribution with density $p(\theta)$. The algorithm works as follows: We first assign an initial value θ^0 such that $p(\theta^0) > 0$ from the starting distribution $p_0(\theta)$. Then, assuming that vectors $\theta^0, \theta^1, ..., \theta^{t-1}$ have been generated, we generate a proposal θ^* for θ^t from a suitable jumping distribution $J(\theta^*|\theta^{t-1})$. In the case of the Metropolis algorithm it is assumed that the jumping distribution is symmetric in the sense that $J(\theta_a|\theta_b) = J(\theta_b|\theta_a)$ for all θ_a and θ_b. Finally, iteration t is completed by calculating the ratio

$$r = \frac{p(\theta^*)}{p(\theta^{t-i})}$$

and by setting the new value at

$$\theta^t = \begin{cases} \theta^* & \text{with probability } \min(r, 1) \\ \theta^{t-1} & \text{otherwise.} \end{cases}$$

It can be shown that, under mild conditions, the algorithm produces an ergodic Markov Chain, whose stationary distribution is $p(\theta)$. We see that the transition kernel $T(\theta^t|\theta^{t-1})$ is a mixture of discrete probability at $\theta^t = \theta^{t-1}$ and the jumping density $J(\theta^*|\theta^{t-1})$.

As mentioned above, we use the Metropolis algorithm to simulate the posterior distribution. The posterior density is proportional to the product of the prior density and the likelihood,

$$p(\theta|y) \propto p(\theta)p(y|\theta).$$

We use an improper uniform prior distribution

$$p(\theta) \propto \begin{cases} 1 & \text{when } |\rho| < 1 \text{ and } \min(\kappa, \xi, \sigma, \nu, \alpha) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The posterior function is thus proportional to the likelihood (6) in a feasible region of parameters.

3.2 Procedure to evaluate the fair bonus rate

3.2.1 Pricing American options with regression methods

The participating life insurance contract we want to price is in practice an American option with a path-dependent moving average feature. An American option gives the holder the right to exercise the option at any time up to the expiry date T. In pricing we adopt the least squares method introduced by Longstaff and Schwartz (2001). It is a simple but powerful approximation method for American-style options. In the following brief introduction we follow Glasserman (2003).

The pricing of an American option is based on an optimal exercising strategy. Let us assume that the relevant underlying security prices of the economy follow a d-dimensional Markov process $X(t)$ and that the payoff value of the option at time t is given by $\tilde{h}(X(t))$. The process $X(t)$ may be augmented to include a stochastic interest rate $r(t)$ and, in the case of path-dependent options, past values of the underlying processes as well.

Furthermore, let $\{\mathcal{F}_t\}$ denote the natural filtration of $X(t)$ and let $\mathcal{T}$ denote the set of all stopping times with respect to it. We will assume that the decision whether to stop at time t is a function of $X(t)$. The goal in optimal exercising is to find a stopping time maximizing the expected discounted payoff of the option. The price of the option is given by

$$\sup_{\tau\in\mathcal{T}} \mathrm{E}\left[\exp\left(-\int_0^{\tau} r(s)ds\right)\tilde{h}(X(\tau))\right],$$

where the expectation is taken with respect to a risk-neutral probability measure.

It is assumed that the option can only be exercised at the m discrete times $0 < t_1 \le t_2 \le \cdots \le t_m = T$. If desirable, one can improve the approximation to continuously exercisable options by increasing m. To simplify notation we will write $X(t_i)$ as X_i. Let $\tilde{h}_i$ denote the payoff function for exercise at t_i and $\tilde{V}_i(x)$ the value of the option at t_i given $X_i = x$. One can then represent pricing algorithms recursively as follows:

$$\begin{aligned}
\tilde{V}_m(x) &= \tilde{h}_m(x)\\
\tilde{V}_{i-1}(x) &= \max\{\tilde{h}_{i-1}(x), \mathrm{E}[D_{i-1,i}(X_i)\tilde{V}_i(X_i)|X_{i-1} = x]\},\\
& \quad\; i = 1, ..., m,
\end{aligned}$$

where $D_{i-1,i}(X_i)$ is the discount factor from t_{i-1} to t_i. We thus assume that the discount factor is a function of X_i, which may be achieved by augmenting X_i, if necessary. Typically, it is of the form $D_{i-1,i}(X_i) = \exp\left(-\int_{t_{i-1}}^{t_i} r(u)du\right)$. One can show that equivalent to the procedure described above is to deal with time zero values $h_i(x) = D_{0,i}(x)\tilde{h}_i(x)$ and $V_i(x) = D_{0,i}(x)\tilde{V}_i(x),\ i = 0, 1, ..., m$; see Glasserman (2003). Then, at time t_i, one compares the immediate exercise value $h_i(x)$ and the continuation value $C_i(x) = \mathrm{E}[V_{i+1}(X_{i+1})|X_i = x]$.

In regression methods it is assumed that the continuation value may be expressed as the linear regression

$$\mathrm{E}[V_{i+1}(X_{i+1})|X_i = x] = \sum_{r=1}^{M} \beta_{ir}\psi_r(x),$$

for some basis functions $\psi_r : \mathcal{R}^d \to \mathcal{R}$ and constants $\beta_{ir},\ r = 1, ..., M$. In order to estimate the coefficients one first generates B independent paths $\{X_{1j}, ..., X_{mj}\},\ j = 1, ..., B$, and sets $\hat{V}_{mj} = h_m(X_{mj}),\ j = 1, ..., B$ at terminal nodes. Then one proceeds backward in time and, using ordinary least squares, fits at time t_i the regression model

$$\hat{V}_{i+1,j}(X_{i+1,j}) = \sum_{r=1}^{M} \beta_{ir}\psi_r(X_{i,j}) + \epsilon_{i,j},\ j = 1, ..., B, \tag{7}$$

where $\epsilon_{i,j}$ are residuals. The estimated value of the option for path j at time t_i is $\hat{V}_{ij} = \max\{h_i(X_{ij}), \hat{C}_i(X_{ij})\}$, where $\hat{C}_i(X_{ij})$ is the fitted value from equation (7). Finally, the estimate of the option price is given by $\hat{V}_0 = (\hat{V}_{11} + ... + \hat{V}_{1B})/B$.

3.2.2 Implementation: Choosing the regression variables

In our application, the continuation values of the option depend on the path of the underlying index value in a complicated way. However, we think that the current value of the index, its moving average and the first index value appearing in the moving average may be used to predict the continuation value reasonably well. The use of the moving average may be motivated by observing that the growth of savings in the insurance contract depends on the path of the moving average; see equation (1). The current index value and the first value appearing in the moving average help predict the future evolution of the moving average. The current amount of savings also helps predict the continuation value, but it is not included in the regression variables. Instead, it is subtracted from the regressed value before fitting the regression and subsequently added to the fitted value.

To avoid under- and overflows in the computations, the regression variables are scaled by the first index value. Thus, the following state variables are used: $X_1(t_i) = S(t_i)/S(0)$, $X_2(t_i) = \left[\sum_{j=0}^{q} S(t_{i-j})/(q+1)\right]/S(0)$ and $X_3(t_i) = S(t_{i-q})/S(0)$. However, multicollinearity problems would occur, if all the variables X_1, X_2 and X_3 were used at all time points. In fact, X_3 would be equal for all simulations paths for $i \leq q$ and the moving averages X_2 would be very close to each other for small values of i. Therefore, we apply the following rule: The variable X_1 alone is used for $i < q/2$, X_1 and X_2 are used for $q/2 \leq i < 3q/2$ and all variables are used for $i \geq 3q/2$. In this study the lag length in the moving average was chosen to be $q = 125$ (that is, half a year).

We use Laguerre polynomials, suggested by Longstaff and Schwartz (2001), as basis functions. More specifically, we use the first two polynomials

$$\begin{aligned} L_0(X) &= \exp(-X/2) \\ L_1(X) &= \exp(-X/2)(1-X) \end{aligned}$$

for the variables X_1, X_2 and X_3. In addition, we use the cross-products $L_0(X_1)L_0(X_2)$, $L_0(X_1)L_1(X_2)$, $L_1(X_1)L_0(X_2)$, $L_0(X_1)L_0(X_3)$ and $L_0(X_2)L_0(X_3)$. Thus, we have altogether 11 explanatory variables in the regression. At time points, where only X_1 is used, we have only two explanatory variables, $L_0(X_1)$ and $L_1(X_1)$.

3.2.3 Implementation: Inverse problem

Using the procedure described above we can determine the option price (that is, the price of the insurance contract) when the bonus rate b and the guaranteed rate g have been given. However, we are interested to determine the bonus rate such that the price of the contract will be equal to the initial savings. This would give the contract a transparent structure. The problem of determining b is a kind of inverse prediction problem and we need to estimate the option value for various values of b. Since there are several sources of uncertainty involved in the estimation of the option price, we also need to repeat it several times for fixed values of b. We end up estimating a regression model, where option price estimates are regressed on bonus rates. We found the third degree polynomial curve to be flexible enough for this purpose. After fitting the curve, we solve the bonus rate b for which the option price is equal to 100, which we assume to be the initial amount of savings.

Prior to fitting the polynomial, it is, however, necessary to determine an interval, where the solution is situated. For this purpose we developed a modified bisection method. In the method, one first specifies initial upper and lower limits for the bonus rate; we use the values $l = 0$ and $u = 1$. Then one estimates the option price at $(l+u)/2$. If the price is greater than 100, the lower limit is set at $l + (u - l)/4$; if the price is smaller than 100, the upper limit is set at $l + 3(u - l)/4$. This procedure is continued until $u - l = 0.15$. Note that the new limit is not set at the middle of the interval, as is done in the ordinary bisection method, since this might lead to missing the correct solution due to the randomness of price estimates.

Figure 1 illustrates the estimation procedure. The option price is estimated for 25 different bonus rates and the estimation is repeated 10 times for each bonus rate, which produces 250 points to the scatter plot. Each estimation is based on 1000 simulated paths. The smallest and largest bonus rate were determined using the modified bisection method described above. When producing this figure, the time to maturity was assumed to be 3 years, the guaranteed rate 0 and the interest rate constant and equal to 0.04. We can see that the fair bonus rate is approximately 0.32.

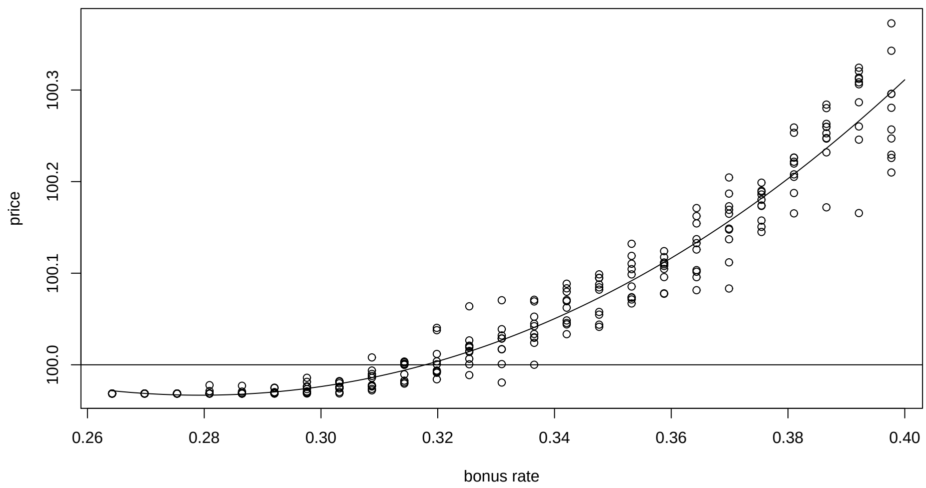

Figure 1: Option price estimates vs. bonus rates.

As mentioned above, the bonus rate is solved from the equation $y = f(x)$, where y is the price of the contract and

$$f(x) = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 x^2 + \hat{\beta}_3 x^3 = \mathbf{x}'\hat{\boldsymbol{\beta}}, \tag{8}$$

where $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)'$ is the OLS estimate of the regression model and $\mathbf{x} = (1, x, x^2, x^3)'$ a regression vector. Using the delta method, one also obtains an approximate variance for the estimate of x:

$$Var(\hat{x}) \approx \frac{1}{[f'(x)]^2} Var(f(x)) \approx \frac{1}{\left(\hat{\beta}_1 + 2\hat{\beta}_2\hat{x} + 3\hat{\beta}_3\hat{x}^2\right)^2} \hat{\mathbf{x}}' Var(\hat{\boldsymbol{\beta}}) \hat{\mathbf{x}}.$$

4 EMPIRICAL RESULTS

4.1 Estimation of the parameters

In order to experiment with actual data and to estimate the unknown parameters of the models (3), (4) and (5), we chose the following data sets: As an equity index we use the Total Return of Dow Jones EURO STOXX Total Market Index (TMI), which is a benchmark covering approximately 95 per cent of the free float market capitalization of Europe. The objective of the index is to provide a broad coverage of companies in the Euro zone including Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, the Netherlands, Portugal and Spain. The index is constructed by aggregating the stocks traded on the major exchanges of Euro zone. Only common stocks and those with similar characteristics are included, and any stocks that have had more than 10 non-trading days during the past three months are removed. In estimation, we use daily quotes from March 4th, 2002 until December 6th, 2007.

As a proxy for riskless short-term interest rate, we use Eurepo, which is the benchmark rate of the large Euro repo market. Eurepo is the rate at which one prime bank offers funds in euro to another prime bank if in exchange the former receives from the latter Eurepo GC as collateral. It is a good benchmark for secured money market transactions in the Euro zone. In the estimation of the interest rate model we use the 3 month Eurepo rate, since it behaves more regularly than the rates with shorter maturities. Both the index and interest series are presented in Figure 2.

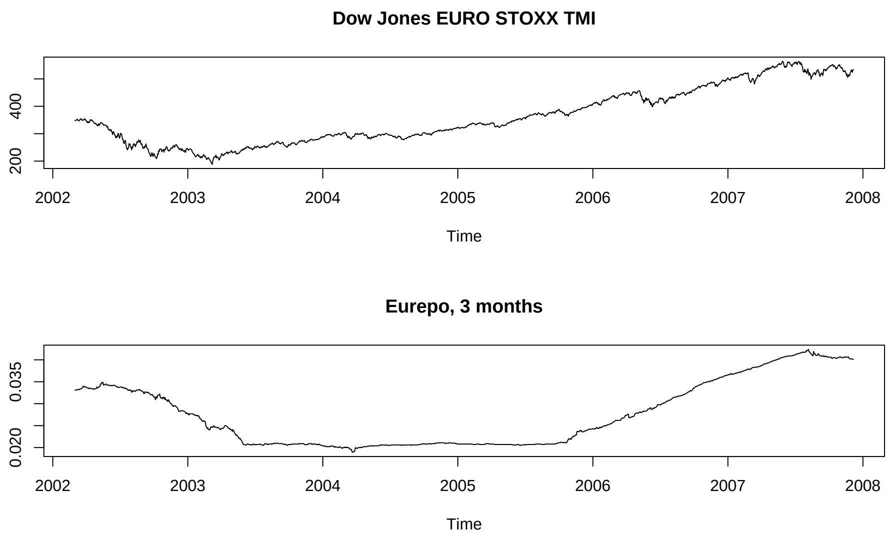

Figure 2: The equity index and interest series

We had no remarkable convergence problems when estimating the model parameters. We used three chains in MCMC simulation and all chains converged rapidly to their stationary distributions. The summary of the estimation results, as well as Gelman and Rubin's diagnostics (see Gelman et al., 2004), are given in the Appendix. The values of the diagnostic are close to 1 and thus indicate good convergence. All computations were made and figures produced using the R computing environment (see http://www.r-project.org).

The posterior distributions of the parameters α (equation (2)) and γ (equation (4)) are shown in Figure 3. As already noted in Section 2.2, the CEV model becomes the geometric Brownian

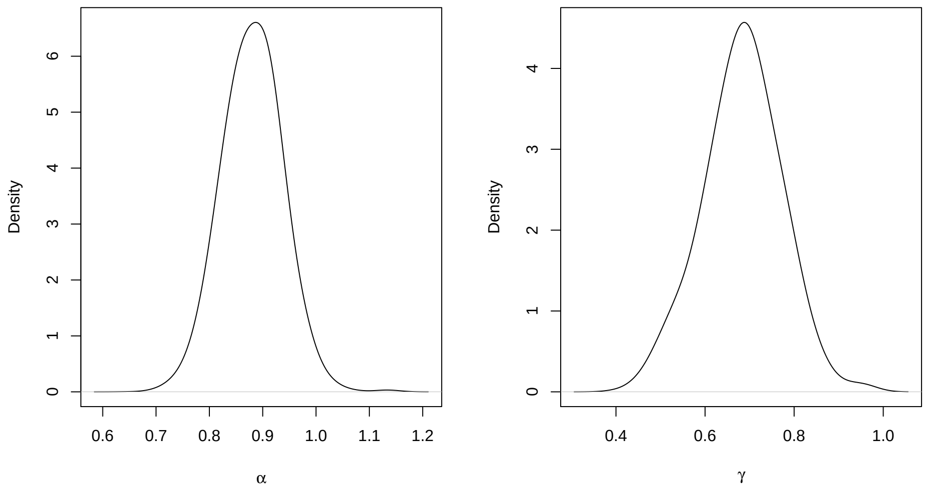

Figure 3: Posterior distributions of the parameters α (index model) and γ (interest rate model).

motion when $\alpha = 0$. The figure reveals clearly that the posterior probability of α being around zero is vanishingly small, which makes the geometric Brownian motion highly improbable. We also tested its use in the pricing of the contract and found that it gave considerably lower bonus rates than the more general alternative. This illustrates how our approach to use general models efficiently prevents the model error resulting from the use of a too simple model. On the other hand, we see that $\gamma = 1/2$ is not highly improbable in the interest rate model, so the model error would not be large if the CIR model were used instead of the more general model.

4.2 Evaluation of fair bonus rate

There are several parameters which may be varied in the participating life insurance contract described by equation (1). These include the duration of the contract T, the lag length of the moving average of the index and the guaranteed rate g. Furthermore, the number of simulated paths needs to be decided when estimating the contract price, as well as the number of estimation repetitions when determining the fair bonus rate. In the case of the constant interest rate model, the interest rate must be fixed at some level, and in the case of stochastic interest rate, the starting level of the interest rate must be given. Our model also incorporates the possibility of a penalty rate. The effect of the penalty is that the insurance company detains a certain percentage of the savings, if the customer reclaims the contract before the final expiration date. When the penalty rate is set at a high level, the price of the contract is determined like that of a European option, since the customer probably wishes to keep the contract until the final expiration date.

We compared the accuracy of fair bonus rate estimation in the following two cases: first, we simulated 1000 paths to estimate the contract price and repeated the estimation 250 times to estimate the fair bonus price using the regression model (8), and, second, used 500 simulation paths and repeated it 500 times. We found that the standard error of the bonus rate estimate was in the second case almost twice as large as in the first case. This indicates that it is more important to increase the number of paths in the option price calculation than the number of repetitions in the bonus rate calculation. However, the differences in the bonus rate estimates were very small; the maximum difference was 0.6 percentage units in our simulations.

The estimates of the fair bonus rate in the cases of constant interest rates 0.04 and 0.07 are shown in Tables 1 and 3, respectively. The guarantee rate was set at 0, 1/3 and 2/3 of the interest rate, that is, 0, 0.013 and 0.027 for $r = 0.04$, and 0, 0.023 and 0.047 for $r = 0.07$. The corresponding results for the stochastic interest rate case with the starting interest rate levels 0.04 and 0.07 are shown in Tables 2 and 4, respectively. The guarantee rate was not fixed at a constant value throughout the entire contract period but it was fixed for one year at a time. More specifically, it was set at 0, 1/3 and 2/3 of the short-term interest rate prediction at intervals of one year. In all cases, the lag length of the moving average was 125 days, the number of simulated paths 1000 and the number of estimations 250.

Table 1: Fair bonus rate and its standard error in the case of constant interest rate $r = 0.04$.

length of the contract	guarantee rate	penalty rate	bonus rate	SE of bonus rate
3	0	0	0.308	0.01
3	1/3	0	0.218	0.012
3	2/3	0	0.117	0.012
3	0	0.03	0.498	0.006
3	1/3	0.03	0.368	0.004
3	2/3	0.03	0.205	0.002
10	0	0	0.308	0.017
10	1/3	0	0.218	0.011
10	2/3	0	0.133	0.016
10	0	0.03	0.493	0.003
10	1/3	0.03	0.373	0.002
10	2/3	0.03	0.212	0.001

Table 2: Fair bonus rate and its standard error in the case of stochastic interest rate with $r = 0.04$ as the starting level.

length of the contract	guarantee rate	penalty rate	bonus rate	SE of bonus rate
3	0	0	0.303	0.01
3	1/3	0	0.216	0.013
3	2/3	0	0.117	0.014
3	0	0.03	0.493	0.006
3	1/3	0.03	0.365	0.004
3	2/3	0.03	0.202	0.002
10	0	0	0.305	0.013
10	1/3	0	0.221	0.012
10	2/3	0	0.125	0.011
10	0	0.03	0.488	0.003
10	1/3	0.03	0.368	0.002
10	2/3	0.03	0.211	0.001

When comparing the results in Tables 1 and 2 one can see that the fixed and stochastic interest rate models are similar in that the estimated bonus rates do not have significant differences. The estimation errors are also similar in these cases. However, one can note that the estimation error is smaller in the cases where the penalty is included in the contract. This is since the penalty changes the contract to an European-style option, which removes the uncertainty related to optimal stopping.

We see from Tables 1, 2, 3 and 4 that the duration of the contract does not generally affect the bonus rate. Only in the cases where there is penalty and the guarantee rate is 2/3 of the interest rate, the bonus rate of 10 years seems to be larger than that of 3 years, the difference being most obvious when the interest rate is fixed at 7 percents. This phenomenon is not easy to explain and might be due to an estimation error related to the regression method.

Table 3: Fair bonus rate and its standard error with constant interest rate $r = 0.07$.

length of the contract	guarantee rate	penalty rate	bonus rate	SE of bonus rate
3	0	0	0.481	0.014
3	1/3	0	0.363	0.016
3	2/3	0	0.205	0.017
3	0	0.03	0.72	0.008
3	1/3	0.03	0.572	0.006
3	2/3	0.03	0.345	0.004
10	0	0	0.484	0.02
10	1/3	0	0.366	0.017
10	2/3	0	0.207	0.012
10	0	0.03	0.716	0.005
10	1/3	0.03	0.584	0.004
10	2/3	0.03	0.372	0.002

Table 4: Fair bonus rate and its standard error in the case of stochastic interest rate with $r = 0.07$ as the starting level.

length of the contract	guarantee rate	penalty rate	bonus rate	SE of bonus rate
3	0	0	0.476	0.012
3	1/3	0	0.355	0.012
3	2/3	0	0.204	0.015
3	0	0.03	0.701	0.006
3	1/3	0.03	0.55	0.006
3	2/3	0.03	0.322	0.003
10	0	0	0.477	0.016
10	1/3	0	0.36	0.013
10	2/3	0	0.202	0.012
10	0	0.03	0.678	0.003
10	1/3	0.03	0.538	0.004
10	2/3	0.03	0.329	0.002

When the initial interest rate is larger ($r = 0.07$), there seems to be a systematic difference between the constant and stochastic interest rate models, which is seen from Tables 3 and 4. The estimated bonus rates tend to be smaller in the stochastic interest rate model. The difference is especially large when there is penalty and the duration is 10 years. The reason is probably the mean-reverting property of the interest rate model, which causes the interest rate to decrease during the contract period. This property also makes the bonus of the 3 years contract larger than that of the 10 years contract when the interest rate is stochastic and the penalty and guarantee rates are 0.

Finally, one should note that the regression methods used in determining the prices of American options are approximative and that there may occur modelling errors related to the choice of regressors. Therefore, the standard errors presented in the above tables do not tell the actual accuracy of the fair bonus rate estimates. It is possible to determine reliable lower and upper bounds for the prices of American options (see Andersen and Broadie, 2004, Haugh and Kogan, 2004, or Glasserman, 2003), but we have not calculated them here.

5 CONCLUSIONS

This paper has attempted to provide a full Bayesian analysis of participating life insurance contracts in a way which leads to fair valuation. The Bayesian approach enables us to exploit Markov Chain Monte Carlo methods and analyze estimation and model errors. A guaranteed participating contract embedding an American-style option was valued using the regression method. Both fixed and stochastic interest rate environments were studied. In the analysis we focused on financial risks and ignored the risk from mortality. As Ballotta et al. (2006) note, in this type of model one can interpret the policyholder to survive until the maturity of the contract. The model thus provides an upper bound for insurance liabilities. As a concrete problem we quantified the effect of the discount rate, guarantee rate and penalty rate on the fair bonus rate.

The adopted approach leads to a fairly realistic valuation framework, which accommodates the main empirical features of the contracts. The equity index yield was modelled with the CEV model and the interest rate with a generalization of the Vasiček and CIR models. The equity index and interest rate processes were assumed to be correlated. For the sake of comparison, we also estimated the equity index yield independently of the interest rate and valuated the contract assuming a fixed interest rate. When the methodology was applied to financial time series, we found clear evidence that the CEV model, which explicitly allows departures from the geometric Brownian motion, provides a better fit to data. This is an important feature, since we can avoid a modelling error by using a more general model.

One important algorithmic feature, which we found, is that in order to improve estimation accuracy it is more important to increase the number of paths in the option price calculation than the number of repetitions in the bonus rate calculation. Another interesting finding was that the duration of the contract did not have a significant effect on the fair bonus rate when there was no penalty for reclaiming the savings prior to the date of maturity. When the penalty rate was set at 3% there were some differences, but it is not clear if these differences are actual or if they are due to a possible modelling error related to the regression method. It is possible to determine reliable lower and upper limits for the prices of American options but we have not calculated them here.

We also found that there were no significant differences in the fair bonus rate between the stochastic and fixed interest rate models when the initial interest rate was set at 4%. When the initial interest rate level was set at a higher level of 7% the fair bonus rate was estimated to be lower in the stochastic interest rate case. This result is probably a consequence of the mean-reverting property of the interest rate model.

The practical import of the results presented here relates to the extent to which the better fit and error evaluations associated with the Bayesian analysis translate into better risk management and more accurate valuation.

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APPENDIX

The posterior simulations were performed using the R computing environment. The following output was obtained using the summary function of the add-on package MCMCpack:

Table 5: Estimation results of the index model with constant interest rate

```
Number of chains = 3
Sample size per chain = 5000

1. Empirical mean and standard deviation for each variable,
   plus standard error of the mean:

          Mean      SD  Naive SE Time-series SE
mu     0.08723 0.06768 0.0005526       0.001766
log nu 3.41621 0.32460 0.0026503       0.009327
alpha  0.88321 0.05544 0.0004526       0.001592


2. Quantiles for each variable:

          2.5%     25%     50%    75%  97.5%
mu     -0.0448 0.04134 0.08835 0.1316 0.2185
log nu  2.7876 3.20125 3.41310 3.6349 4.0524
alpha   0.7766 0.84630 0.88314 0.9205 0.9914


Gelman and Rubin's diagnostics
(Potential scale reduction factors):

      Point est. 97.5% quantile
mu          1.00           1.01
log nu      1.01           1.03
alpha       1.01           1.03
```

Table 6: Estimation results of the index model with stochastic interest rate

Number of chains = 3
Sample size per chain = 5000

1. Empirical mean and standard deviation for each variable,
plus standard error of the mean:

	Mean	SD	Naive SE	Time-series SE
mu	0.079225	0.067939	5.547e-04	0.0042595
log nu	3.402534	0.327204	2.672e-03	0.0219129
alpha	0.880626	0.055834	4.559e-04	0.0037286
kappa	0.052439	0.045232	3.693e-04	0.0018596
beta	0.221869	0.132709	1.084e-03	0.0060462
tau^2	0.009487	0.001697	1.386e-05	0.0001046
gamma	0.683214	0.087154	7.116e-04	0.0051778
rho	0.091389	0.025618	2.092e-04	0.0016489

2. Quantiles for each variable:

	2.5%	25%	50%	75%	97.5%
mu	-0.048659	0.026263	0.079805	0.12508	0.21444
log nu	2.769493	3.176798	3.401768	3.61407	4.04341
alpha	0.772006	0.843013	0.881094	0.91714	0.98700
kappa	0.001606	0.018515	0.039505	0.07354	0.16534
beta	0.035210	0.126786	0.200871	0.29001	0.53552
tau^2	0.006695	0.008333	0.009257	0.01045	0.01355
gamma	0.504586	0.627500	0.687341	0.74042	0.84705
rho	0.041503	0.075450	0.090498	0.10661	0.14419

Gelman and Rubin's diagnostics
(Potential scale reduction factors):

	Point est.	97.5% quantile
mu	1.01	1.04
log nu	1.01	1.02
alpha	1.01	1.02
kappa	1.01	1.03
beta	1.01	1.03
tau^2	1.02	1.06
gamma	1.04	1.10
rho	1.01	1.03