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Modeling shocks in long-term equity returns

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Modeling shocks in long-term equity returns

Tiivistelmä/Referat/Summary
Kehitämme tässä raportissa mallin osaketuotoille, joka soveltuu erityisesti pitkän aikavälin ennusteisiin ja riskienhallintaan. Analysoimme aluksi S&P 500 kokonaistuottoindeksin vuosittaista aikasarjadataa ja luomme katsauksen joihinkin yleisiin osaketuottomalleihin. Tämän jälkeen kehitämme Gamma-hyppyjä sisältävän satunnaiskulkumallin osaketuotoille ja estimoimme sen käyttäen suurimman uskottavuuden menetelmää ja Markovin ketjujen simulointia (MCMC). Lopuksi esitämme mallin tuloksia.

I denna rapport utbildar vi en model för aktiepriser som är riktad till långfristiga prognoser och risk kontrol. Vi först analyserar det årliga S&P 500 totalavkastningindexet och granskar några allmänt tillämpade modeller för aktiepriser. Sedan utbildar vi en Gamma Jump Random Walk model för aktiepriserna och estimerar den med Maximum Likelihood och Markov Chain Monte-Carlo metoderna. Slutligen presenterar vi resultater från modellen.

In this paper we develop a model for equity returns that is aimed at long-term forecasting and risk management applications. We first analyse the yearly S&P 500 total return index data and review some common models for equity returns. Subsequently we develop a Gamma Jump Random Walk model for equity returns and estimate it through the Maximum Likelihood and Markov Chain Monte-Carlo methods. In the final section we present simulations of the model.

Avainsanat/Nyckelord/Keywords
Equity return shocks, jump random walk, risk management

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Modeling shocks in long term equity returns

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Abstract

In this paper we develop a model for equity returns that is aimed at long-term forecasting and risk management applications. We first analyse the yearly S&P 500 total return index data and review some common models for equity returns. Subsequently we develop a Gamma-jump random walk model for equity returns and estimate it through the Maximum Likelihood and Markov Chain Monte-Carlo methods. In the final section we present simulations of the model.

Contents

1 Introduction

Our goal in equity index modelling is to be able to generate long term simulations up to 75 years that have approximately similar distributional features that can be observed from the chosen reference data set for equity returns[1]. The reference data should form a suitable basis for long-term forecasting and risk management, and it should give an adequate approximation to a well-diversified equity portfolio that an insurance company and its clients (in case of unit linked business) may have.

In this paper we first analyse the S&P 500 equity market data and review the most common models for equity returns. Subsequently we develop a jump model for the equity returns and estimate it through the Maximum Likelihood and Markov Chain Monte-Carlo methods. In the final section we present simulations of the model.

[1]We restrict our attention solely to equity market data and do not consider any other explanatory economic variables.

2 Data on equity returns

2.1 General characteristics of long term equity returns

The data series we have chosen for the equity returns is the S&P 500 Total Return Index at the end of year 1925-2006 as given in Table 5-1 on pages 102-103 of Morningstar (2007). This index, denoted here by SP500, is expressed in nominal values (starting at 1.00 at the end of 1925 and reaching 3077.33 at the end of 2006), and it includes the effect of reinvested dividends. The SP500 consists of 500 large U.S. stocks, which are weighted by their market values monthly[1].

In Figure 2.5 the yearly return series of SP500 is given, approximated by the difference of successive values of natural logarithm of the index. A histogram of these so called log-returns is given in Figure 2.1. The summary statistics of SP500 log-returns are as follows: the sample mean is 0.099, standard deviation 0.192, skewness -0.853, and kurtosis 3.893. From the histogram and the skewness statistic we observe that that the returns are skewed to the left, and kurtosis indicates a higher probability than a normally distributed variable of extreme values.

The autocorrelations and partial autocorrelations of SP500 log-return series are given in Figures 2.3 and 2.4, and the autocorrelations of the squared log-returns in Figure 2.2. From these data we observe that autocorrelation is weak for the log-returns but significant for the squared log-returns.

Koskela et al. (2008) have analysed in chapter 4 the data in detail using various ARIMA and GARCH models. Their findings can be summarized as follows:

1. For the yearly data from 1925-2006 the following ARCH(1) model gives a better fit than any linear model when judged by the information criteria AICC or AIC or BIC[2]:

   $$p_t - p_{t-1} = 0.1163 + \epsilon_t, \tag{2.1}$$

   where p_t is the log of SP500 value for $t = 1, \ldots, 82$, $\epsilon_t = z_t\sqrt{h_t}$ with independent $z_t \sim N(0, 1)$, and $h_t = 0.0183 + 0.5829\epsilon_{t-1}^2$.

2. The yearly data from 1955-2006 appear to be uncorrelated and the best model is ARIMA(0,1,0):

   $$p_t - p_{t-1} = 0.1002 + \epsilon_t, \tag{2.2}$$

[1]For more details see Chapter 3 in Morningstar (2007).
[2]These terms are defined for instance in Brockwell & Davis (2002) on page 173.

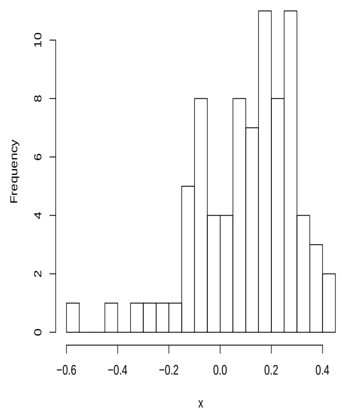

Figure 2.1: Histogram of SP500 log-returns for 1926-2006

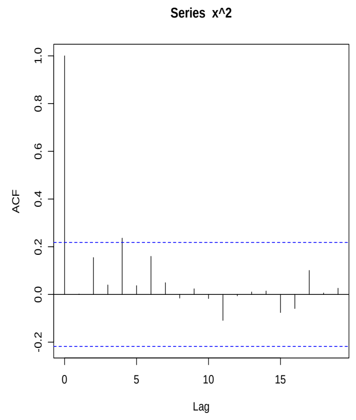

Figure 2.2: ACF of squared SP500 log-returns for 1926-2006

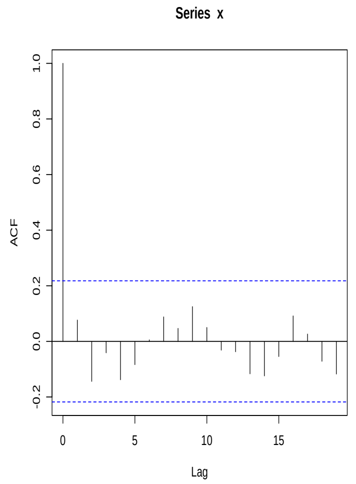

Figure 2.3: ACF of log-returns of SP500 for 1926-2006

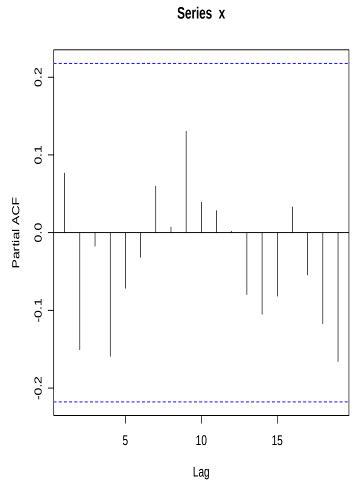

Figure 2.4: PACF of log-returns of SP500 for 1926-2006

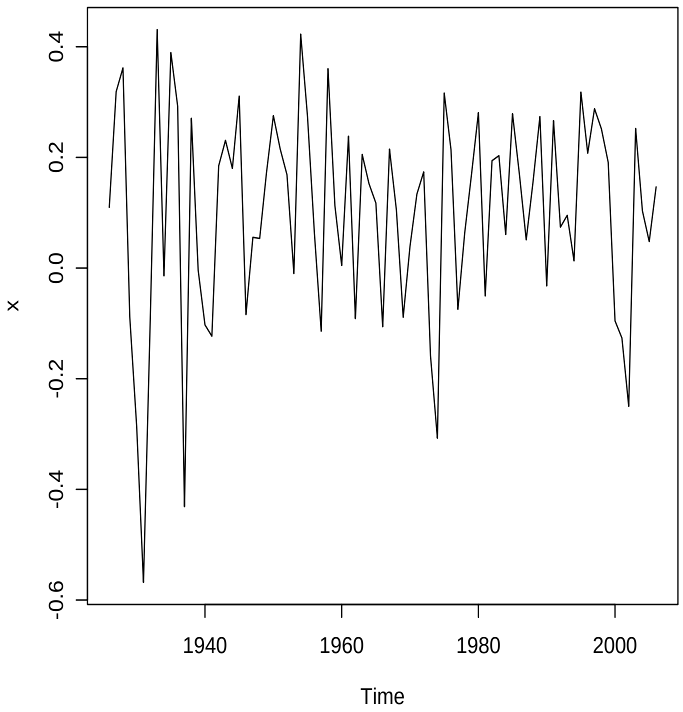

Figure 2.5: SP500 log-returns for 1926-2006

where $\epsilon_t \sim N(0, 0.0234)$. [3]

We conclude that the sample period matters. From 1955 onwards a linear model gives a good description of the data. On the other hand, by analysing the series where part of the data has been deleted, we find that it is approximately the first 10 years, i.e. the period from 1926 to 1935, that caused the ARCH(1) model to be chosen in point 1 above. However, the problem with the ARCH(1) model is its symmetry: it treats both the losses and the profits in the same way. This is not what we observe in the data (cf. Figures 2.5 and 2.1 and the skewness statistic). Therefore we conclude that the possibility of very bad losses is necessary to be taken into account for risk management purposes, but in our view it is better addressed by an asymmetric model than a symmetric ARCH-model. We now turn to models that are able to describe downward jumps.

2.2 Review of jump models for equity returns

Infrequent equity market crashes cause discontinuity in the data that can be modelled by a jump process. The classical example of this approach is the model of Merton (1976), which is specified in continuous time. It adds to a diffusion process lognormally distributed jumps according to a Poisson process. Maximum likelihood based comparative analysis of this and other jump model classes has been carried out for weekly and monthly equity market data

[3]For the monthly data from 1955-2006 a GARCH(1,1) model is the best.

from June 1973 to December 1983 in Jorion (1988). His analysis concludes that a simple diffusion model is chosen for monthly stock returns over a jump diffusion, a jump-ARCH and an ARCH model, and that for weekly data the jump-diffusion model is a significant improvement over the simple diffusion model.

A simplified modelling approach in discrete time has been suggested for stock returns by Ball & Torous (1983). In their approach the Bernoulli process is used for the jump times instead of the Poisson process, and the resulting model is a Bernoulli mixture of Gaussian densities for the daily stock returns.

Ramezani & Zeng (1998) apply in continous time an asymmetric jump-diffusion process to equity prices. Their model assumes that good news and bad news arrive according to two Poisson processes, and that the jumps sizes are Pareto and Beta distributed. Another, more recent jump model specification of this type is the double exponential Poisson jump diffusion, first proposed by Kou (2002) for option pricing applications. In the context of modelling the default risk in corporate bonds when the asset values may have jumps, Hilberink & Rogers (2002) model only negative jumps with an Exponential distribution.

Our model to be developed below is tailored in discrete time for the yearly equity return index, which is in contrast to the above mentioned models that are designed for shorter term applications and use more frequently sampled data. As in Ball & Torous (1983), we use the Bernoulli process for the jump times. We apply a similar idea as in Hilberink & Rogers (2002) in that we only consider negative returns in the jump term. However, in our model there is a coefficient to eliminate the effect of jump years from the normal years (cf. (1 - J_t) in (3.1)), which is not used in the above mentioned models. Moreover, our model is formulated for Gamma-distributed jumps.

3 Model specification and estimation

3.1 Definition of the model

Denote the log of SP500 index value with p_t, t = 1925,...,2006, so that yearly log-returns are $x_t = p_t - p_{t-1}$. We specify the jump model as follows:

$$x_t = (1 - J_t)(\mu + \sigma\epsilon_t) - J_t Y_t \tag{3.1}$$

where now t = 1,...,T, T=81, and μ is the mean, σ is the standard deviation, and ϵ_t are independent and identically distributed (iid) random variables from $N(0, 1)$ distribution. For the jump process we assume that the jump times are iid Bernoulli random variables: $J_t \sim Be(q)$, $0<q<1$; and that the jump sizes are iid Gamma variables: $Y_t \sim Gamma(\alpha, \beta), \alpha, \beta > 0$. Moreover, we assume that these three random variables are independent.

Because in the model the yearly returns $x_1, ..., x_T$ are assumed independent, we can write the likelihood function L as

$$L(\theta) = \prod_{t=1}^{T} f(x_t; \theta) \tag{3.2}$$

where

$$f(x_t; \theta) = (1 - q)\frac{1}{\sqrt{2\pi}\sigma}e^{-(x_t-\mu)^2/2\sigma^2} + q\frac{\beta^\alpha}{\Gamma(\alpha)}(-x_t)^{\alpha-1}e^{\beta x_t}1_{\{x_t<0\}} \tag{3.3}$$

Here $1_{\{x_t<0\}}$ is the indicator function that equals 1 when the yearly return is negative and is 0 otherwise as we only wish to model the negative jumps with the last term in (3.1). In addition we have used the independency of jump times and jump sizes, and the product rule of probability with the fact that the jump probability is $\mathrm{E}[J_t] = q$. Thus our model can be described as a Bernoulli-mixture of $N(\mu, \sigma^2)$ and $Gamma(\alpha, \beta)$ distributions.

In this model the mean is

$$E[x_t] = (1 - q)\mu - q\alpha/\beta \tag{3.4}$$

By taking expectations from the square of (3.1) we get

$$E[x_t^2] = (1 - q)(\mu^2 + \sigma^2) + q\alpha(1 + \alpha)/\beta^2 \tag{3.5}$$

and using $Var[x_t] = E[x_t^2] - E[x_t]^2$ we find that the variance is

$$Var[x_t] = \left\{(1-q)(\mu^2+\sigma^2) + q\alpha(1+\alpha)/\beta^2\right\}^2 - \{(1-q)\mu - q\alpha/\beta\}^2 \tag{3.6}$$

3.2 Maximum Likelihood Estimation

The log-likelihood function l corresponding to (3.2) is

$$l(\theta) = \sum_{t=1}^{T} ln[(1-q)\frac{1}{\sqrt{2\pi}\sigma}e^{-(x_t-\mu)^2/2\sigma^2} + q\frac{\beta^\alpha}{\Gamma(\alpha)}(-x_t)^{\alpha-1}e^{\beta x_t}1_{\{x_t<0\}}] \tag{3.7}$$

We have maximized (3.7) directly by R's unrestricted optimization function *optim*() with the Nelder-Mead simplex method[1].

The $Gamma(\alpha, \beta)$ distribution is able to produce a rich variety of functional shapes. This feature together with the observed bi-modality of the empirical distribution of SP500 log-returns (cf. Figure 2.1) have motivated us to analyse several specifications for the equity return model. The summary results of these calculations are given in Table 3.1.

Model 1 is based on the maximum likelihood estimates (MLE) with unrestricted α parameter. This model is highly bi-modal as is seen in Figure 3.2. This does not seem logical. Namely the probability density function has two spikes in its graph: a local maximum at -0.0227, and a local minimum at zero[2], and its first derivative is discontinuous at these points. We conclude that bi-modality is not easily explained, it brings undesirable features to the density, and thus it should not form a basic feature of the model. Indeed, this bi-modality is not observed in the mid-year index data. The same conclusions regarding the bi-modality problem apply to more parsimonuous Model 6, where α=1, corresponding to the Exponential distribution (cf. Figure 3.1).

To add more realism into our model, we now fix α and then carry out the maximum likelihood estimation. We note that when α is increased, the bi-modality decreases. In Figure 3.3 α is 2, and in Figure 3.4 it is 3. These models have less bi-modality but still seem implausible. When alpha is between 3.5 and 4, the bi-modality problem gradually disappears, as is observed in Figures 3.5 and 3.6.

In conclusion, the MLE result (Model 1), which has the best fit according to the likelihood value, as well as the simpler Exponential model (Model 6), have the serious problem of bi-modality. Therefore we have analysed above several specifications for the model, and we cannot say with certainty which would be the right specification to use. However, it seems to us that the *Gamma* distribution with α between 3.5 and 4 would provide a reasonable class of approximate models. Our preferred choice of α=4 also takes into account our original idea that the jumps should be relatively rare as then

[1] Using the version 2.6.0.

[2] We also note that f(0)=0.872, but f(-0.0001)=0.995.

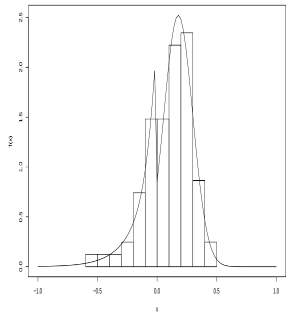

Figure 3.1: Log-returns of the model when α=1

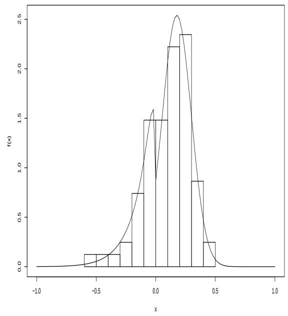

Figure 3.2: Log-returns of the model when α=1.41

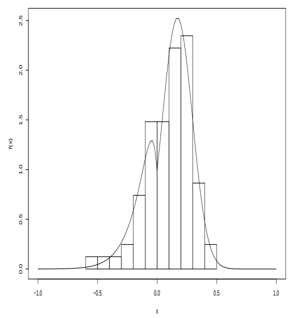

Figure 3.3: Log-returns of the model when α=2

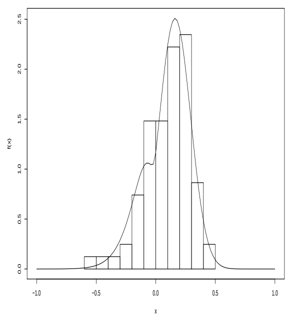

Figure 3.4: Log-returns of the model when α=3

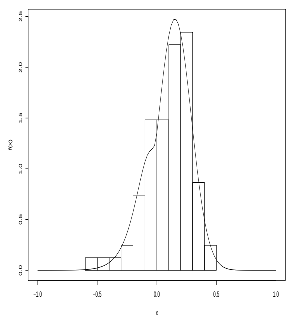

Figure 3.5: Log-returns of the model when α=3.5

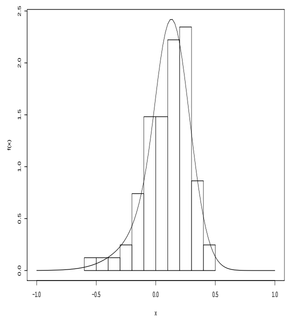

Figure 3.6: Log-returns of the model when α=4

Table 3.1: Comparison of 6 models with Gamma jumps.

Variables	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
μ	0.177	0.171	0.162	0.152	0.131	0.178
σ	0.121	0.125	0.131	0.138	0.153	0.121
q	0.229	0.208	0.176	0.143	0.071	0.235
α	1.41	2	3	3.5	4	1
β	8.65	11.4	15.45	16.17	12.79	6.29
l	24.77	24.5	23.58	23.04	22.86	24.36

q=0.071, i.e. on average there would be a negative jump of equity returns once in every 14 years. This feature of rare but large negative jumps is in our view desirable as it allows to model catastrophic losses for risk management purposes. Most of the time the returns thus come from the Normal distribution part according to an ARIMA(0,1,0) or a random walk with a drift process, which is the workhorse model for equity returns in economic theory and practice[3].

In the final part of our MLE procedure we have calculated the confidence intervals for the parameters. We have done this by the profile likelihood method, which is based on the likelihood ratio test and its asymptotic distribution. Using this approach we search for the lower and upper bound for each parameter such that $2(l(\widehat{\theta})$ - $l^*(\widetilde{\theta}))$ is approximately 3.84, i.e. the 5th percentile of the chi-squared distribution with 1 degree of freedom. Here the log-likelihood function l is evaluated at the maximum point $\widehat{\theta}$, and the other term, $l^*(\widetilde{\theta})$, is calculated by optimizing the log-likelihood for the remaining parameters while keeping one parameter fixed. By gradually changing the fixed value and re-running the optimization, the lower and upper bounds are found. This process is applied to each term of the parameter vector in turn. The optimization algorithm is the same as was applied to the maximum likelihood estimation.

To determine the profile likelihood 95 percent confidence intervals for the chosen Model 5, we first fixed α=4, and then calculated the following confidence intervals for the remaining parameters: $\mu \in$[0.09,0.19], $\sigma \in$[0.11,0.19], $q \in$[0.01,0.27], $\beta \in$[5.2,26]. We note from the size of confidence intervals that the parameter uncertainty is high. This is not particularly surprising when considering the complex nature of equity returns and their jump process, which depend not only on the economic climate but also on other exogenous factors and human behavior. Although the joint analysis of the parameter uncertainty is difficult, there are methods available for that purpose, which we consider next.

[3]For a practically oriented discussion on this topic see Malkiel (2007).

3.3 Parameter uncertainty via Markov Chain Monte-Carlo

Our goal is to simulate future equity returns and include the parameter uncertainty in the calculations. Resorting to the so called Markov Chain Monte-Carlo (MCMC) approach is the most convenient way to achieve that. We implement MCMC via the so called Gibbs sampler, which uses conditional distributions of the parameters to specify the Markov Chain having the target joint density as its stationary distribution. For a comprehensive discussion on MCMC see Gelman et al. (2004) and Gilks et al. (1996), and for an accessible introduction see Greenberg (2008).

We specify the Gibbs algorithm for our model in 2 steps as follows:

1. First we assume the jumps $J = J_t$ and the data $X = x_t$, t=1,...,T, known and write down the conditional likelihood L$|J$. Then the conditional distributions for the remaining parameters: L$(\beta|J, X, \mu, \sigma^2, q)$, L$(1/\sigma^2|J, X, \mu, \beta, q)$, L$(\mu|J, X, \sigma^2, \beta, q)$, L$(q|J, X, \mu, \sigma^2, \beta)$ are derived by picking only those terms that include the parameter in question (the other terms are constants and can be neglected in the Gibbs algorithm).

2. In the second step we generate new jumps when all the other parameters are known.

We assume the following independent priors: $q \sim Beta(a_q, b_q)$, $\mu \sim N(0, \sigma_\mu^2)$, $\tau = 1/\sigma^2 \sim Ga(a_\tau, b_\tau)$ and $\beta \sim Ga(a_\beta, b_\beta)$. The parameters should be chosen so that they allow appropriately wide range of values to occur. Based on both visual and empirical analysis we proceed as follows[4]:

- $a_q = b_q = 1$, which leads to a non-informative uniform distribution on (0,1).

- $\sigma_\mu = 0.8$, which is 4 times the observed standard deviation (0.192) of log-returns.

- $a_\tau = 3, b_\tau = 0.05$, which gives mean=0.15 and standard deviation=0.05 for σ, while its MLE was 0.15 and the 95 % profile likelihood confidence interval was [0.11, 0.19].

- $a_\beta = 1.5, b_\beta = 0.1$, which gives mean=15.0 and standard deviation=12.2, while the MLE was 12.8 and the 95 % profile likelihood confidence interval was [5.2, 26.0].

We conclude that the priors above are rather non-informative and thus cover a broad enough range of values for our purposes.

The conditional joint density for the observations and parameters is

[4]Testing with other reasonable parameters did not change the MCMC results.

$$
\begin{aligned}
\text{(3.8)} \qquad L|J = & \prod_{J_t=1} \frac{\beta^4}{\Gamma(4)} e^{\beta x_t} (-x_t)^3 1_{\{x_t<0\}} \\
& \times \prod_{J_t=0} \frac{\tau^{1/2}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(x_t-\mu)^2} \\
& \times \frac{1}{\sqrt{2\pi}\sigma_\mu} e^{-\mu^2/2\sigma_\mu{}^2} \\
& \times \frac{\tau^{a_\tau-1} e^{-\tau b_\tau}}{b_\tau^{a_\tau} \Gamma(a_\tau)} \\
& \times \frac{\beta^{a_\beta-1} e^{-\beta b_\beta}}{b_\beta^{a_\beta} \Gamma(a_\beta)}
\end{aligned}
$$

Note that α=4 by our assumption made earlier, and that the form of this likelihood is simpler and much better suited for simulation than (3.2) because here we use the conditional L$|J$, i.e. we assume J known.

Now the conditional distributions required for the Gibbs simulation can be derived in the following manner.

$$
\begin{aligned}
\text{(3.9)} \qquad L(\beta|J,X,\mu,\sigma^2,q) & \propto \left\{ \prod_{J_t=1} \beta^4 e^{\beta x_t} 1_{\{x_t<0\}} \right\} \beta^{a_\beta-1} e^{-\beta b_\beta} \\
& = \beta^{4n_1+a_\beta-1} exp \left\{ -\beta(b_\beta - \sum_{J_t=1} x_t 1_{\{x_t<0\}}) \right\}.
\end{aligned}
$$

Thus, $Ga(4n_1 + a_\beta, b_\beta - \sum_{J_t=1} x_t 1_{\{x_t<0\}})$ is the posterior of β.

$$
\begin{aligned}
\text{(3.10)} \qquad L(\tau|J,X,\mu,\beta,q) & \propto \left\{ \prod_{J_t=0} \tau^{1/2} e^{-\frac{\tau}{2}(x_t-\mu)^2} \right\} \tau^{a_\tau-1} e^{-\tau b_\tau} \\
& = \tau^{n_0/2+a_\tau-1} exp \left\{ -\tau(b_\tau + \frac{1}{2} \sum_{J_t=0} (x_t-\mu)^2 \right\}.
\end{aligned}
$$

Thus, $Ga(n_0/2 + a_\tau, b_\tau + \frac{1}{2}\sum_{J_t=0}(x_t-\mu)^2)$ is the posterior of τ.

$$
\begin{aligned}
\text{(3.11)} \qquad L(\mu|J,X,\sigma^2,\beta,q) & \propto \left\{ \prod_{J_t=0} e^{-\frac{\tau}{2}(x_t-\mu)^2} \right\} e^{-\mu^2/2\sigma_\mu{}^2} \\
& = exp \left\{ -\frac{\tau}{2} \sum_{J_t=0} (x_t-\mu)^2 - \frac{\mu^2}{2\sigma_\mu{}^2} \right\}.
\end{aligned}
$$

Thus, $N((\frac{1}{\sigma_\mu^2}+n_0\tau)^{-1}\tau\sum_{J_t=0}x_t,(\frac{1}{\sigma_\mu^2}+n_0\tau)^{-1})$ is the posterior of μ.

$$L(q|J,X,\mu,\sigma^2,\beta)\propto q^{a_q-1}(1-q)^{b_q-1}q^{n_1}(1-q)^{n_0}. \tag{3.12}$$

Thus, $Beta(a_q+n_1,b_q+n_0)$ is the posterior of q.

In the equations above n_0 and n_1 denote the number of no-jump ($J_t=0$) and jump ($J_t=1$) years respectively. The relation (3.11) follows from an analoguous result on page 76 in Gilks et al. (1996), or by using completing the square technique[5].

In the second step we update the jump process as follows. For $x_t<0$, t=1,...,81, we have by (3.3)

(3.13)
$$P(J_t=1|\mu,\tau,\beta,q,x_t)=\frac{q\beta^4e^{\beta x_t}(-x_t)^3/\Gamma(4)}{(1-q)\frac{\tau^{1/2}}{\sqrt{2\pi}}exp\left\{-\frac{\tau}{2}(x_t-\mu)^2\right\}+q\beta^4e^{\beta x_t}(-x_t)^3/\Gamma(4)}$$

Using these probabilities for each $x_t<0$ with updated parameters we generate a new jump process realisation from the Bernoulli distribution. Thereafter we start a new round of iteration (step 1 $\rightarrow$ step 2 etc) until the convergence of iteration is adequate.

[5]See e.g. the opening page of Finney (2001).

4 Generation of future equity returns

In this section we generate forecasts from our model with both the MLE and the MCMC parameters derived above and compare the results. The method to generate equity returns proceeds as follows. We take one parameter vector at the time from the MCMC sample (after the burn-in period) and plug them into the basic equation (3.1) as constants, while for the MLE-based forecast we use the MLE parameters. Using 750 000 sampled observations (corresponding to a forecast for 75 years repeated 10 000 times), we get the empirical distributions for the equity returns as shown in Figure 4.1. As initial values for the MCMC iteration we used the MLE results but also other values were tested and they did not change the outcome.

We note from the histogram that the two methods give rather similar distributional results, which indicates that parameter estimation is not the main source of uncertainty in the modelling. However, in the area where the returns are between -0.4 and 0, we see systematic difference as the MCMC method gives there more probability mass. In fact this seems to result in a return distribution that resembles the case where the value of α is between 3.5 and 4, as we suggested earlier. These findings are made more explicit in Table 4.1, where we have included the MLE figures of Model 4 (where α=3.5), and both the MLE and the mean MCMC estimates for our preferred model 5 (where α=4). We note that the jump probability q and *Gamma* rate parameter β are higher in the MCMC estimation, which imply that jumps occur more often and they are smaller. We have also included in the table the MLE 95 % confidence intervals calculated earlier by the profile likelihood method, and the 2.5th and 97.5th quantiles of the MCMC results. These confidence intervals are very similar. We also note asymmetry except for σ parameter.

Table 4.1: Comparison of Model 4 and the MLE and mean MCMC parameters of Model 5, and the lower and upper 95 % MLE confidence intervals (CI) and the respective MCMC quantiles for Model 5.

Variables	Model 4	MLE	MCMC	CI-left	CI-right	2.5%	97.5%
μ	0.152	0.131	0.14	0.09	0.19	0.09	0.19
σ	0.138	0.153	0.15	0.11	0.19	0.12	0.19
q	0.143	0.071	0.11	0.01	0.27	0.02	0.25
β	16.17	12.79	15.3	5.2	26	6.6	25.1

We conclude that both the MLE and MCMC methods are suitable for esti-

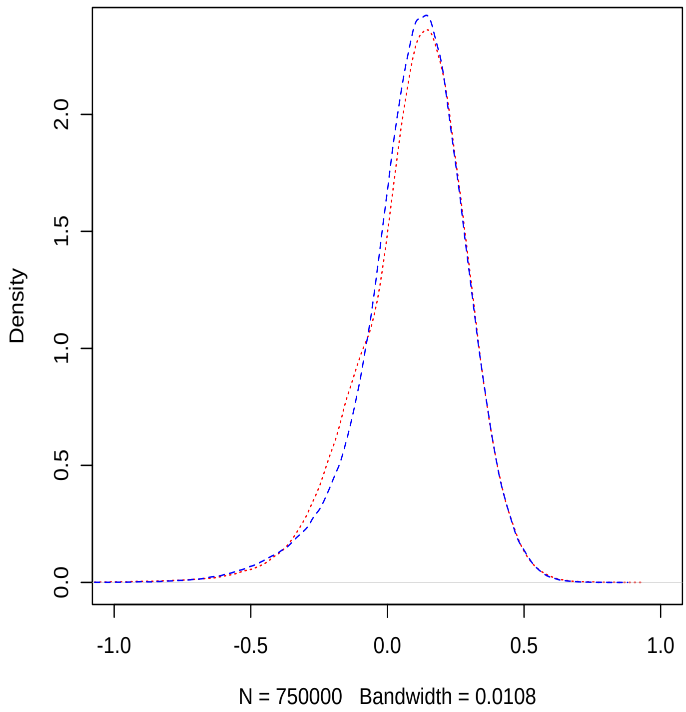

Figure 4.1: Comparison of MLE (dashed) and MCMC (dotted) based equity return simulations

Table 4.2: Posterior correlations of model parameters.

Variables	μ	σ	q	β
μ	1	-0.50	0.58	0.47
σ		1	-0.58	-0.40
q			1	0.60
β				1

mation and that they give consistent results. The MCMC parameters partially cancel the jump features that we subjectively preferred when choosing α=4. On the other hand MCMC allows more realistic dependence modelling. We have plotted in Figure 4.2 simulated μ against σ, in 4.3 μ against q, and in 4.4 q against β. The posterior correlations of simulated model parameters are listed in Table 4.2.

From these statistics we conclude that the model parameters generally are correlated. For instance we note from the high correlation (0.6) between q and β that the more frequent the jumps, the smaller the amount (cf. (3.4)). We also observe that μ, q, and β are pairwise positively correlated, while σ is negatively correlated with μ, q and β.

In the final Figure 4.5 we have compared the MCMC results with the histogram of empirical log-returns of SP500. We observe the smoothness and wider range of values of the modelled returns, which are desirable for the simulations.

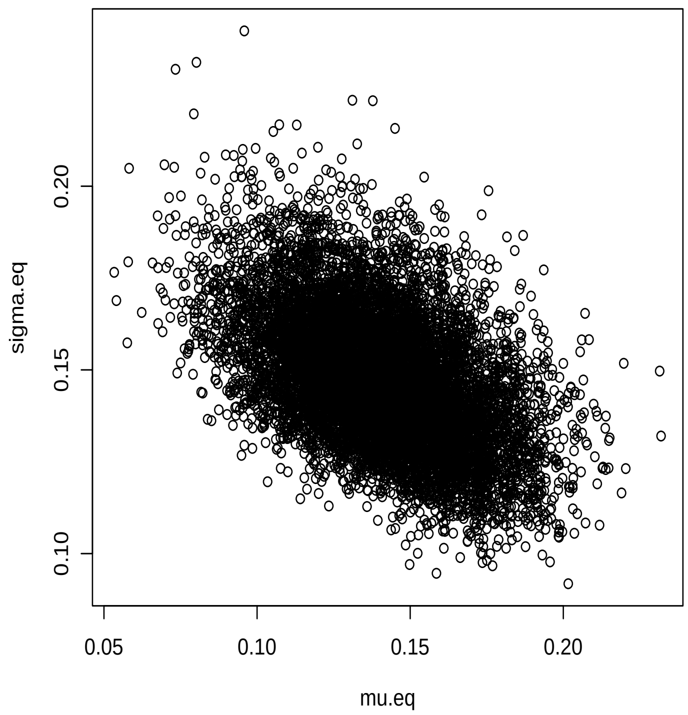

Figure 4.2: Simulated μ against σ

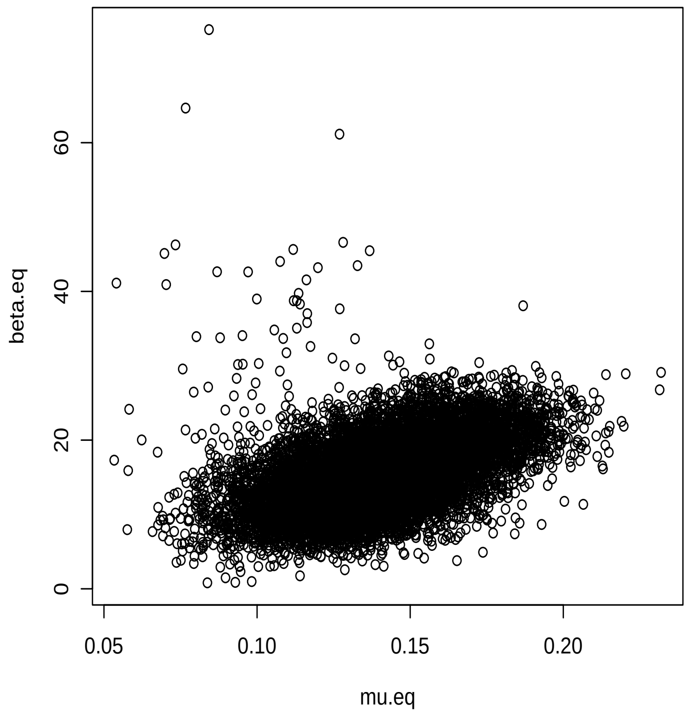

Figure 4.3: Simulated μ against q

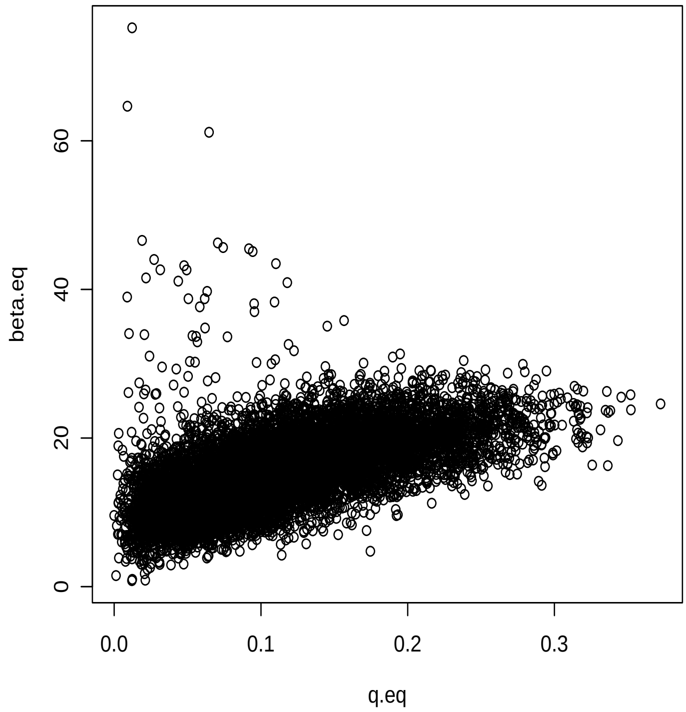

Figure 4.4: Simulated β against q

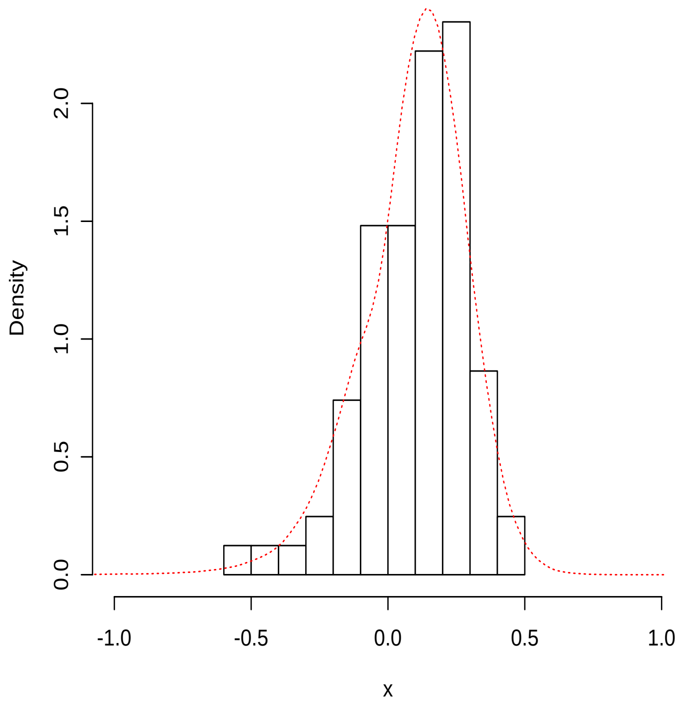

Figure 4.5: Comparison of the histogram of empirical log-returns and MCMC based simulations

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