A THEORETICAL ANALYSIS OF GROWTH AND CYCLES

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HELSINKI

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PREFACE

This study was originally published in Finnish in mimeographed form in 1965 (Bank of Finland Institute for Economic Research Publications, Series D:6). My sincere thanks are due to the staff of the Institute who assisted me in a variety of ways during all the phases of this work. I wish to mention in particular Mrs. Lea Honkanen, M. A., Miss Annikki Leukkunen, M. A., Mr. Raimo Heiskanen, M. A., and Mrs. Aira Kasanko who between them undertook numerous tasks involved in the computations, computer programs, printing and so on.

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Helsinki, June 1969.

J. J. Paunio

INTRODUCTION

During the past two decades or so, interest in macro-economics has largely centred on the problems of growth. Important contributions to the theory of the trade cycle have become increasingly infrequent, whereas the literature on growth has continued to expand rapidly. The trade cycle itself, however, has not disappeared. True, the mechanism of the market economy has been tamed, in a sense; yet, economic activity continues to display cyclical fluctuations, though these are considerably less sharp than they used to be.

The institutional changes that have taken place during the past few decades also necessitate of course a new approach to the theory of the trade cycle. It may indeed be most appropriate to approach the trade cycle phenomenon from the aspect of growth. Signs of such a shift in the centre of emphasis in macro-theory have been in evidence for some length of time. However, the foundations of this theory should also be reconsidered in the light of the altered circumstances. It is in fact reasonable to ask: Don't the economic policies pursued in mixed-type economies exert an influence on the expectations and policy decisions of those operating in the market sector of the economy?

Assumptions implying a short-run analysis are inappropriate where the intention is, as in the present study, to consider both growth and the trade cycle as essential features of the economic process. It is imperative to replace them by other, more adequate ones. The first step in this direction consists in abandoning the assumption that the changes in the capital stock are so small, in relative terms, that they can be neglected in the analysis. The capital stock must be included in the model from the beginning, both as an independent and as a dependent variable. This is done in the present study, whereas otherwise it rests on assumptions typical of short-run analysis.

Explicit consideration of the capital stock may also be important from another point of view. Monetary developments are obviously determined to a decisive extent by impulses originating in decisions concerning the disposition of wealth. Thus, an endeavour to develop the theoretical analysis of the monetary and real sectors into an integral whole would seem to presuppose the treatment of the most important form of wealth, that is, real capital, even in the analysis of the real sector — with which the present study is concerned.

The purpose of this study is the development and analysis of a certain theoretical macro-dynamic model. The basic model is constructed in Chapter

I, and an effort is made to incorporate features characteristic of economic growth, as well as those characteristic of the trade cycle, in the model. The investment function constructed for the model is of prime importance in this respect. Despite the strongly simplifying assumptions underlying the model, it is rather complicated in its final form.

Chapter II is concerned with applications of the model. Certain problems which have been extensively discussed in recent years in growth theory are analyzed in terms of the model. The focus will, however, be on the question concerning the preconditions of steady growth, in the form raised by the so-called Harrod—Domar model.

The results of this analysis of growth are utilized in the final chapter, where the basic model is flexibly employed as a tool in the analysis of the cyclical fluctuation of economic activity.

I. THE MACRO-DYNAMIC MODEL

The intention in the present chapter is to develop a macro-economic model in terms of which cyclical fluctuations and growth could be accounted for simultaneously. The structure of the model is determined largely by the view that not only total output but also capital stock should be dealt with as dependent variables.

1. INTRODUCTORY REMARKS ON CAPITAL FORMATION

The development of a market economy depends decisively on how strongly motivated the entrepreneurs are to increase the productive capacity and to alter the pattern of output. Both types of activity usually lead to capital formation. Capital formation, in turn, is a time-consuming process, and thus the entrepreneur can only know in the future whether any investment decision he made today was economically correct or not. In consequence it is of prime importance for the development of the economy what kinds of expectation the entrepreneurs harbour concerning their own special field and the economy at large. Obviously, economic growth depends in large measure upon whether the entrepreneurs believe in the economy's future growth.[1] In other words, economic growth has its *subjective* determinants.

On the other hand, natural resources, the population, and technological development constitute an *objective* framework for the growth of an economy. Though this framework can be manipulated in some degree, and though the technological development, for example, depends on the economic process itself to some extent, it may be legitimate to regard this framework as given where the analysis of economic growth is confined to a certain narrowly circumscribed period.

In the following analysis an attempt will be made to incorporate these distinctive features of the growth process in a trade cycle model.

1. See especially Nicholas Kaldor »Hicks on the Trade Cycle», Essays on Economic Stability and Growth, London 1960, p. 203.

2. STRUCTURAL ELEMENTS OF THE MODEL: THE PRODUCTION FUNCTION

The model to be built here will be one of a closed economy without the public sector. The firms sector is assumed to produce goods that can be used for both consumption and investment. The value of the total output produced through the capital and labour inputs is represented by the production function[2]

(1) $Y_t = F\ (K_t, N_t)$,

where $Y =$ the volume of total output (net national product[3]), $K =$ the capital stock, $N =$ the labour input, and the subscript t ($= 0, 1, 2, \ldots$) indicates the time period. Following HAAVELMO, for example, the flow of capital inputs is interpreted as being due to the presence of the capital stock in the productive process. And it is postulated that the problem of the measurement of capital has been solved in one way or another.

As already mentioned, the model will be built so as to be capable of explaining the changes in the capital stock. On the other hand, the determinants of the supply of labour will be treated as exogenous to the model. The supply of labour function will be

(2) $\bar{L}_t = \bar{L}_o\, \nu^t$,

where $\bar{L}_t =$ the supply of labour, and ν is the growth coefficient of the labour force. In the initial period of the analysis ($t = 0$) the growth factor ν^t equals unity. In the spirit of Keynesian macro-theory it is assumed that no substitution between capital and labour is possible.

The volume of output may grow, as a result of technological progress, even when the quantities of capital and labour remain unchanged. Here, technical progress is supposed to be *embodied* in labour and capital uniformly or *neutrally*, and thus it is not represented explicitly in the production function (1).

The non-substitutability assumption indeed makes it possible to present the production function in a still simpler form, or

(1:1) $Y_t = F_n\ (N_t)$.

If the quantity of manpower, in equation (2), is interpreted as measured, say, in man-hours, the available labour input, L, may be written

2. For the problems associated with this kind of production function, see HAAVELMO A Study in the Theory of Investment, Chicago 1960, pp. 78—83.

3. The problem of depreciation will not be dealt with in the model.

$$(2{:}1) \quad L_t = \tau^t \, (\bar{L}_o \, \nu^t) \, .$$

In this equation the parameter τ represents the technical progress taking place in one unit period and regarded here as exogenously determined.

Thus it is possible to define the maximum level of output permitted by the available labour input, or the *full-employment ceiling* of output, in terms of equation (1:1):

$$(3) \quad \bar{\bar{Y}}_t = F_n \, (L_t)$$

where $\bar{\bar{Y}}$ = the full-employment ceiling.

Technological progress was assumed to be embodied in capital and labour neutrally. This implies that it affects old capital equipment in the same way as new capital equipment. This assumption is in line with the traditional production-function analysis, which has been criticized in recent years mainly because of its failure to account for the fact that technological development tends mainly to raise the productivity of new rather than of old capital goods.[4] Adoption of the traditional assumption is not, however, likely to do much harm to the present analysis: here the centre of emphasis does not lie in the production function. If the volume of the capital stock created in the course of time through investment is denoted by $\bar{K}$ and the capital input by K, the interrelation between the two can be expressed as

$$(4) \quad K_t = \tau^t \, \bar{K}_t.$$

Though the capital stock was assumed to yield the capital input simply through its presence in the productive process, this does not exclude the possibility of the capital stock setting an upper limit to output. If this upper limit, which will be referred to as the *capital-stock ceiling*[5] of output, is denoted by $\bar{Y}$, we may write

$$(5) \quad \bar{Y}_t = F_k \, (K_t).$$

The assumptions concerning technological progress and non-substitutability can be used to specify the form of the functions F_n and F_k in equations (3) and (5), which can respectively be written as

$$(3{:}1) \quad \bar{\bar{Y}}_t = \varphi_n L_t$$

and

4. See especially Robert M. Solow Investment and Technical Progress, Mathematical Methods in the Social Sciences, Editors: Kenneth J. Arrow, Samuel Karlin, and Patric Suppes, Stanford, California 1960.

5. Cf. Haavelmo A Study . . ., pp. 88—90.

(5:1) $\bar{Y}_t = \varphi_k K_t$.

The labour and capital coefficients, φ_n and φ_k, involved in these equations, differ somewhat in character. The capital stock, $\bar{K}$, is invariably »present», and capital input, K_t, is a single-valued function of $\bar{K}_t$. Thus, the capital coefficient φ_k only relates the capital stock to the capital-stock ceiling. The labour coefficient, by contrast, is valid for any level of output. In other words, rather than using the production functions expressed by equations (1) and (1:1), we may write

(1:2) $Y_t = \varphi_n N_t$.

It should be borne in mind, however, that here the changes in labour input, N, do not exclusively reflect changes in employment, as measured in man-hours; as in equation (2:1), technological advances are also embodied. The supply of labour, in terms of man-hours, was expressed through equation (2). The demand for labour, in man-hours, $\bar{N}$, can be written

(6) $\bar{N}_t = \frac{1}{\tau^t} : N_t$.

Thus it is assumed that during the initial period $N_o = N_o$. In the subsequent periods, however, different values are obtained for employment, depending on whether it is measured in man-hours or in terms of the output produced.

According to the foregoing argument, both the existing labour force and the existing capital stock are factors restricting the growth of output. The labour force must, however, be considered the primary constraint: it depends exclusively upon exogeneous factors. The capital-stock ceiling, on the other hand, may change markedly as a result of the economic process itself. The full-employment ceiling and the capital-stock ceiling may, of course, coincide. In an industrial country, however, the full-employment ceiling is likely to be the principal constraint, whereas in a developing country the effective constraint is likely to be the capital stock. As a rule, it is impossible for a developing country, almost by definition, to raise the capital-stock ceiling sufficiently over a relatively short period.

Now the *full equilibrium* will be defined in terms of the full-employment ceiling, the capital-stock ceiling and total output as follows:

(7) $Y_t = \bar{Y}_t = \bar{\bar{Y}}_t$.

In the light of the model developed so far, a state of equilibrium defined in terms of this double equation has the following two

First, the demand for and supply of labour will be equal, since $Y_t = \bar{\bar{Y}}_t$; this condition can also be written, by equations (2) and (6), as

$$\bar{N}_t = L_t, \tag{7:1}$$

or, by equations (2:1) and (6), as

$$N_t = L_t \tag{7:2}$$

Second, the input due to the »presence» of the capital stock will be fully utilized, since $Y_t = \bar{Y}_t$. In this sense, the demand for (or, perhaps more correctly, the utilization of) and the supply of the capital stock are equal.

3. STRUCTURAL ELEMENTS OF THE MODEL: THE CAPITAL STOCK AND INVESTMENT

The next task is to elaborate the model further, so as to enable analysis of the development of the capital-stock ceiling $(\bar{Y}_t)$ and total output (Y_t). The model will be built in such a way that the factors affecting the capital stock will also have a decisive bearing on the development of total output.

It is not infrequent — and this applies to the Keynesian macro-theory in particular — that no very definite stand is taken as regards the supply side, that is, the conditions under which the production of goods takes place, as a factor bearing on economic development: analysis is customarily confined to the factors on the demand side. HAAVELMO, more than anyone else, has called attention to this obvious bias.[6]

The present analysis also adheres to the »demand tradition», in that the production of consumer goods is supposed to respond instantaneously, i.e., during the same unit period, to a change in the demand for them. On this assumption, demand wholly explains the changes in supply and production — of course within the limits permitted by the existing productive factors.

On the other hand, the problems associated with the production of capital goods will not be fully disregarded. In several cases the production of capital goods is incontestably a rather lengthy process, in terms of calendar time. Let us consider the construction of a power plant, for example. The time interval elapsing between the date when the investment decision is made and the date when the project is completed does not exclusively consist of the period requisite for the process of construction itself; account must also

6. See HAAVELMO A Study . . ., p. 196.

be taken of the time required to frame the plans necessary for the project: this time may also be long.

Let $\bar{I}$ stand for the value of net investment and $\bar{K}$ (as in equation (4)) for the capital stock, both measured disregarding the technological progress that has taken place since the initial period of the analysis. The relationship between these two variables is defined through

$$\bar{K} = \bar{I}_{t-1} + \bar{K}_{t-1}\,, \tag{8}$$

which has certain notable consequences as regards the entire model.

The first thing needing attention is the role played by technological progress in determining the interrelation of investment and capital input.

From equations (4) and (8) we have

$$K_t = \tau^t\, \bar{I}_{t-1} + K_{t-1}\,, \tag{8:1}$$

and, if we write $I_t = \tau^t\, \bar{I}_t$,

$$K_t = I_{t-1} + K_{t-1}\,. \tag{8:2}$$

Thus, investment in any unit period is technologically up-to-date, and technological progress is supposed to be reflected in the pre-existing capital stock as well. As already pointed out, the latter assumption is rather strong: to justify it, it is obviously necessary to imagine that technological advances are organizational in nature.

In equation (8:2), K_t refers to the capital stock at the beginning of period t. Thus the investment amounting to an increment in the capital stock has taken place during the period immediately preceding this. For the subsequent analysis it is important to state that the investment decision, the formulation of the investment plans and the production of capital goods all take place during one and the same period. It is indeed necessary to define the *unit period* so as to make it sufficiently long to embrace all these phases of investment. As the analysis relates to the entire economy, rather than to a single firm, the average length of time it takes to carry out an investment must be chosen as the unit.

4. STRUCTURAL ELEMENTS OF THE MODEL: THE INVESTMENT FUNCTION

At the beginning of this chapter it was asserted that a basic subjective factor associated with entrepreneurs' expectations underlies economic growth. The

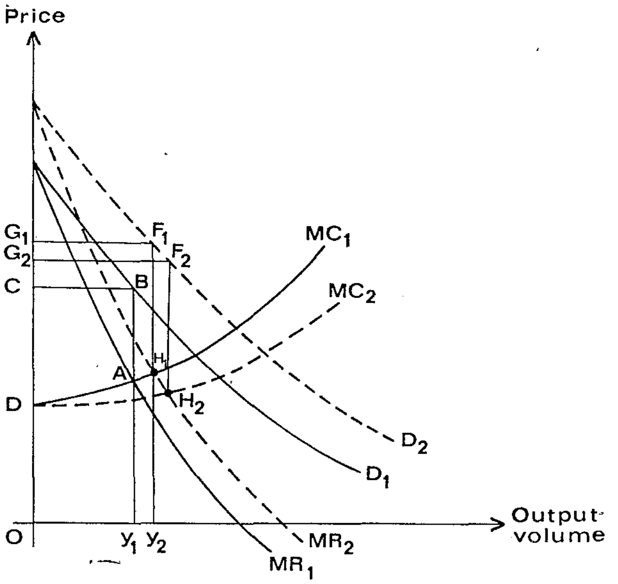


Figure I.

next task will in fact be to incorporate such subjective elements into the model via the investment component.

The expectations relevant to the firms' investment decisions relate to the demand and supply factors that affect production. Initially, investments will be considered from the point of view of demand factors, by employing Figure I.

The figure relates to a »representative» monopolistic firm seeking to maximize its profits.[7] The product price is measured along the vertical axis and the output volume along the horizontal axis. When the demand curve facing the firm is D_1, its marginal cost[8] MC_1 and the marginal revenue curve MR_1, the firm is supposed to adjust its capital equipment accordingly, i.e., with a view to producing the output Y_1. By the *adjustment* of capital equipment is meant that the average total cost reaches a minimum at the output level Y_1. Then, of the total revenue Y_1BCO of the »representative» firm, the proportion $ABCD$ is its profit, and this is defined as the *normal profit*.

Let us now assume that demand increases in such a way that the demand curve rises to the position D_2 and the marginal revenue curve shifts, in consequence, from the position MR_1 to MR_2. The profit of the firm will then

7. Cf. Duesenberry's geometric investment analysis in DUESENBERRY Business Cycles and Economic Growth, New York 1958, Chapter 4.

8. The variable costs consist in wages.

increase to $DH_1F_1G_1$ and its output to Y_2. It should be noted that an increase in demand is interpreted here as a parallel shift in the demand curve. In consequence, the firm's profit becomes over-normal. Over-normal profits are assumed to provide the firm with a stimulus to new investment. Provided that the level of demand remains unchanged for a sufficient length of time, the marginal cost curve will shift to a new position MC_2 in the way represented in the figure (the representative marginal cost curve MC is assumed to enjoy a minimum at the point where it intersects the vertical axis), as the technical efficiency of the new capital equipment and the old was assumed to be the same.[9] The normal profit will increase to $DH_2F_2G_2$. Hence, it can be argued that investment activity will occur only on condition that demand continues to increase.[10]

The diagrammatic analysis allows two alternative interpretations. First, it is of course possible to suppose that the firms make their investment decisions only when profits have already grown over-normal; that is, when a realized increase in demand in fact calls for expansion of the capital stock.

The second alternative is to assume that the entrepreneurs make the investment decisions on the basis of their expectations; on this interpretation, the curves D_2 and MC_2 in the figure depict the expectations of the »representative» entrepreneur.

The latter alternative would appear to be more realistic in the case of a growth-oriented economy. And, since the present model is primarily intended for such an economy, the subsequent elaboration of the model will be based on that assumption. Let us suppose first that the »representative» firm is capable of adjusting its capital stock, so as to gain a normal profit continuously. A precondition for this assumption to be realistic is that demand continues to increase steadily: it is only under such circumstances that the management of the firm can form a definite idea of the movement of demand — that is, a relatively certain expectation concerning a continued growth in demand — upon which it can base its investment decisions. The firm considered here was assumed to be »representative», and thus it may also be legitimate to assume that the expectations it has concerning the demand for its output represent the prevailing expectations concerning the development of the whole economy in general.

9. See Duesenberry op. cit., pp. 58—59.

10. This analysis exceeds the boundaries of the model to some extent, in the sense that (1:2) was assumed to be a fixed-coefficient production function. The above considerations must, in fact, be interpreted in the first place as a qualitative analysis of the factors affecting investment.

Here it will be assumed that under such conditions the entrepreneurs expect — on the basis of the uniform growth experienced — that demand will continue to grow at a constant annual rate (of, e.g., 3 to 4 per cent). This will be termed the *expected growth rate of demand.*

This postulate of single-valued growth-of-demand expectations rests on the assumption of a steady growth, which enables the entrepreneur to adjust his capital stock to the level of output. In the world of reality, development does not take place so smoothly: an entrepreneur's expectations often prove erroneous, and he has to experience that his productive capacity is sometimes insufficient and sometimes excessive. Nevertheless, under realistic conditions, too, it is plausible that, mainly on the basis of the past long-run development of the economy, the entrepreneurs form a rather definite idea of how demand will move in the future. Because they rest on long-run developments, such expectations are also likely to relate primarily to future long-run developments.

Thus, the expected growth rate of demand would be a sort of »core» in the entrepreneurs' short-term expectations, into which the element of uncertainty would be introduced by the short-term prospects proper. We are consequently led to this view: the expected long-run rate of growth in demand, in combination with the uncertainty factor, determines, on the demand side, the firms' investment decisions; an increase in uncertainty tends to diminish, and a decrease tends to reinforce the willingness of entrepreneurs to invest.

Are there variables with which the short-term expectations of the firms can be connected? Contrary to the case with long-term expectations, the short-term expectations must obviously be assumed to be closely tied up with the current economic situation. More specifically, the »current economic situation» should be interpreted to refer to the level of output (and, *ex post,* the level of demand) during the period concerned: the volume of output may correspond to the existing capital stock, and then there is likely to be little uncertainty about the increase in demand in the near future; or output may run below the capacity level, and then the uncertainty regarding demand in the near future is likely to be greater. To put the same more precisely: uncertainty is supposed to be an increasing function of excess capacity.

On the basis of the above it is now possible to construct an *expectation function,* affecting investment decisions on the demand side, and to write it as

$$g_t = \bar{\alpha}\,\frac{Y_t}{\bar{Y}_t} - \bar{\beta}\,. \tag{9}$$

The equation has to be interpreted as follows: g, that is, the expectations concerning the growth rate of demand, is influenced on the one hand by the long-term growth expectations, assumed to be constant and denoted by $(\bar{\alpha}-\bar{\beta})$; and, on the other hand, by the degree of utilization of the capital stock, in accordance with the term $\bar{\alpha}\frac{Y_t}{\bar{Y}_t}$, $\bar{Y}_t$ being the capital-stock ceiling defined by equation (5). When the capital stock is fully utilized — i.e., when $\frac{Y_t}{\bar{Y}_t}$ equals unity — g equals the expected long-term growth rate, but it is less than the expected long-term growth rate if $\frac{Y_t}{\bar{Y}_t}$ is less than unity.

In the interpretation of equation (9) attention should be drawn to the fact that the value of the expected long-term growth rate is expressed by means of two parameters, $\bar{\alpha}$ and $\bar{\beta}$. This is the consequence of an additional assumption, still to be introduced: it is postulated that the excess capacity and, at the same time, the uncertainty among entrepreneurs concerning the future may be so great as to exclude expectations inducing the entrepreneurs to investment. Such a state of affairs prevails if

$$\text{(9:1)} \qquad \bar{\alpha}\frac{Y_t}{\bar{Y}_t} \gtreqless \bar{\beta} \ .$$

If it is found desirable to confine the analysis to a growth-oriented economy, this can be done through postulating that $g \geqq 0$, which constraint in fact allows making the concept of a 'growth-oriented' economy considerably more precise. On this condition, and recollecting that $\frac{Y_t}{\bar{Y}_t} \leqq 1$, it follows from (9) that

$$\text{(9:2)} \qquad 1 \geqq \frac{Y_t}{\bar{Y}_t} \geqq \frac{\bar{\beta}}{\bar{\alpha}} \ .$$

Thus, in a growth-oriented economy, total output may vary between $\bar{Y}$ and $\frac{\bar{\beta}}{\bar{\alpha}}\bar{Y}$ $(\bar{\alpha}, \bar{\beta} > 0; \ \bar{\alpha} > \bar{\beta})$.

Figure II illustrates the linear g-function given by equation (9). The assumption that $\bar{\beta} > 0$ only signifies that g equals zero at a certain positive value of the degree of capacity utilization; this is also apparent from the situation of the point at which the graph of the g-function intersects the

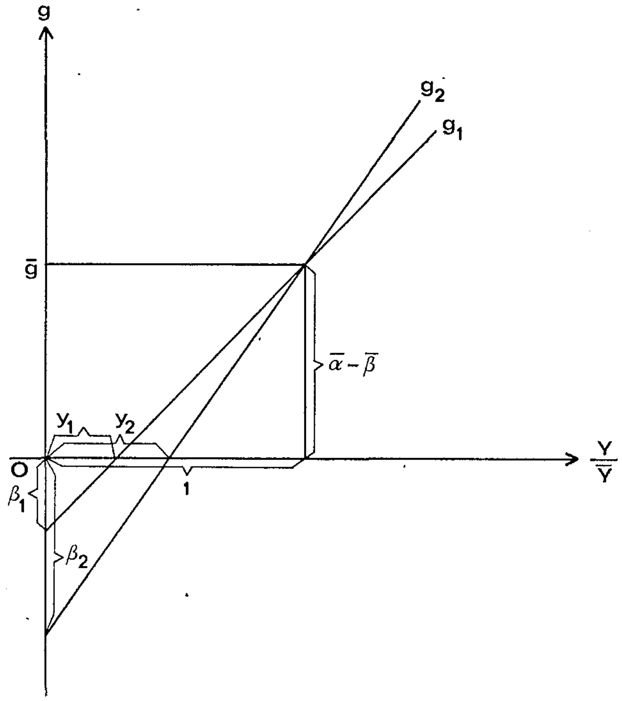


Figure II.

horizontal axis in the figure. If the expected long-term rate of growth is constant — $\bar{g}$ in the figure. — the interrelation between $\bar{\alpha}$ and $\bar{\beta}$ is uniquely determined:

(9:3) $\bar{g} = \bar{\alpha} - \bar{\beta}.$

This interrelation is illustrated in the figure. For example, if the g-function is turned from the position g_1 to the position g_2, both $\bar{\alpha}$ (or the slope of the graph of the function) and β (the negative of its g-axis intercept) increase. In economic terms this means that a decline in the degree of capacity utilization has a greater bearing on the uncertainty of expectations than before; and thus the minimum level of capacity utilization compatible with an expected growth in demand, and with investment, will rise, from y_1 to y_2 in the figure.

The expectation function given by equation (9) and the analysis of this function were related to the expectations affecting the firms' investment on the demand side. Our next task is to consider the impulses coming from the supply side, or, more specifically, from the factors of production.

As already emphasized, a continued growth of human capital is a distinc-

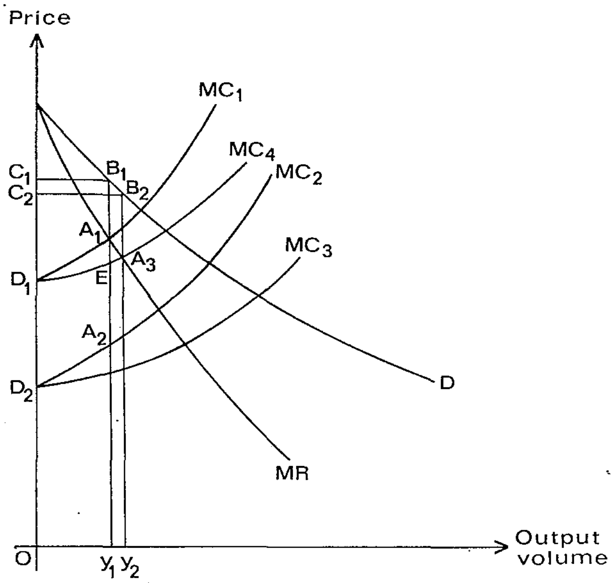


Figure III.

tive characteristic of advancing society, and this, in turn, helps the firms to increase the efficiency of their productive activity. In the foregoing it was postulated that society's human capital resources suffice for technological progress in production. The assumption that technological advances are embodied uniformly, and at a constant rate, in the capital stock and the labour force alike was based implicitly on this underlying postulate. From the assumption, again, that technological development is embodied uniformly in all productive factors — in the »old» as well as in the »new» ones — it follows implicitly that, in the present model, technological progress cannot *directly* induce the entrepreneurs to increase their capital stock. New technological knowledge can indeed be applied even in the absence of investment.[11]

Just as the firms can draw on society's growing stock of human capital for technological know-how, they can utilize the increasing labour force reserves to expand their productive activity. Neither an increase in the labour force nor technological progress is likely to be the proximate reason for the firms' desire to expand their capital stock. Thus it is necessary to make an attempt at outlining the mechanism through which technological development and

11. See above, p. 12.

the increase in the labour force might affect the entrepreneurs' investment decisions.

To this end, use will again be made of the concept of »representative» firm; and, here too, the point of departure will be a situation where the firm is gaining a normal profit from production. In Figure III, $A_1B_1C_1D_1$ is normal profit corresponding to the demand curve D (the marginal revenue curve MR) and the marginal cost curve MC_1. Though prices and costs are not dealt with explicitly in the model itself, the price and cost mechanism is supposed to work, and affect the entrepreneurs' behaviour, in a certain specific way. (The same was already the case with the analysis of the demand side; cf. Figure I).

Technological progress continues to diminish the costs of production. If the wages react with a lag to the increase in productivity resulting from technological progress, the marginal cost curve will shift downwards, retaining its shape. Provided the lag in the wage rates is one unit period, the change in marginal cost *(ΔMC)* is, according to the assumptions underlying the model,

$$(10) \qquad \Delta MC_t = \frac{w_{t-1}}{\varphi_n} - \frac{w_{t-1}}{\varphi_n \tau} = \left(1 - \frac{1}{\tau}\right) \frac{1}{\varphi_n} w_{t-1}$$

where w = the wage per employment unit and φ_n = the marginal physical productivity of labour. The rate of change in marginal cost is constant and independent of the level of output. The assumption that wages react to productivity changes with a lag can be substantiated by the fact that the increase in the labour force prevents the employees from reinforcing their position vis-à-vis the employers.

Figure III represents a case where the marginal cost curve has shifted from the position MC_1 to the position MC_2 as the result of technological progress.[12] When the marginal cost curve is in the position MC_2, the profit of the firm has increased to $A_2B_1C_1D_2$ provided that the volume of its output continues to be Y_1. Just as in the analysis of the demand side, an increase in profit due to technological progress is assumed to provide the firm with a stimulus to expand its capital stock.

As a result of consequent investment the marginal cost curve again changes, its new position being MC_3. However, the length of the unit period was determined by the length of time necessary for the production of capital goods, and thus we have already passed to the next period. Provided that wages adjust completely to the changes in productivity, there will occur an upward shift in the marginal cost curve that is exactly equal, in absolute terms, to the down-

12. Cf. again Duesenberry op. cit., pp. 60—61.

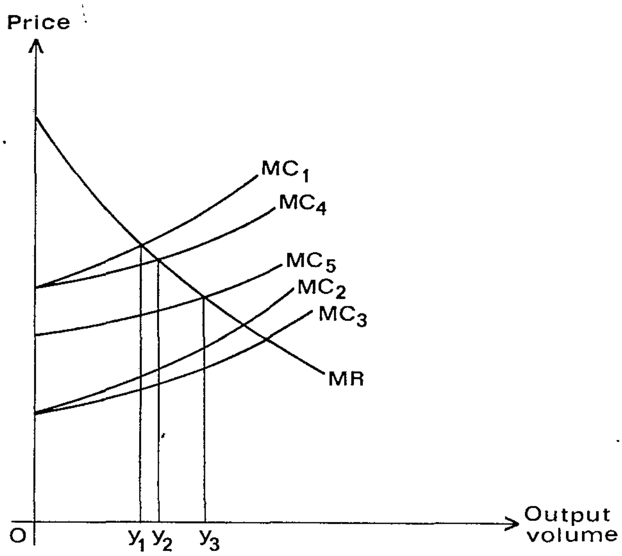


Figure IV.

ward shift that took place during the preceding period as a result of technological progress. Yet the new marginal cost curve will not coincide with MC_1: the new marginal cost curve will be MC_4, which is identical in shape with MC_3. Now the profit maximizing firm will increase its output to Y_2. The firm's profit will then be $A_3B_2C_2D_1$, which is larger than its rate of profit at the beginning of the preceding period, $A_1B_1C_1D_1$. This is so since, according to the marginalistic theory resting on the profit maximization hypothesis, $A_3B_2C_2D_1$ is larger than $EB_1C_1D_1$, which, in turn, is larger than $A_1B_1C_1D$.

Provided that the firm is able to forecast both technological progress and the concomitant change in costs correctly, it will increase its capital stock precisely to the extent that its productive capacity will be fully utilized when the output produced by the firm equals Y_2; thus, the profit $A_3B_2C_2D_1$ is interpretable as a normal profit. In that case the intermediate stage represented by the marginal cost curves MC_2 and MC_3 must only be regarded as an auxiliary expedient serving the purposes of illustration.

One of the basic assumptions underlying the model was that the labour force grows at a constant rate from one period to the next. When the effects of technological progress were analyzed, the growth in the labour force was accounted for, in that it was considered to lead to a lag in the reaction of wages to the increase in productivity. Even if the labour force had been fully employed when output equalled Y_1 (see Figure III), an additional assumption is necessary to ensure full employment at the new equilibrium level. Figure IV represents the same situation as Figure III, except that the demand curve D has been omitted for the sake of simplicity.

Before the analysis is carried further by means of Figure IV, there is reason to state that, according to equations (2:1), (1:2) and (7:2), full employment prevails if, and only if,

$$(11) \qquad Y_t = \varphi_n \, \tau^t \, \nu^t \bar{L}_o \, .$$

The parameter τ was discussed in connection with the analysis of technological development; thus, only the parameter ν needs to be considered here.

The firm obviously takes account of the opportunities provided by the increase in the labour force only on condition that this increase is also reflected in the entrepreneur's cost calculations. To put the same in another way, it must be assumed that the changes in the supply of labour are reflected in the movement of wages. We will suppose, therefore, that the increases in wages, even after they have reacted with a lag to the increase in productivity, may remain smaller than the increase in productivity. In Figure IV this assumption is reflected by the rise of the marginal cost curve from the position MC_3 to the position MC_5, rather than to the position MC_4, which it assumed in Figure III. Thus, an output level equal to Y_3 would ensure full employment. As a result of technological progress and the increase in the labour force, output would grow from Y_1 to Y_3, rather than to Y_2. This does not, of course, exclude the possibility of the output level Y_2 being compatible with full employment. The additional assumption is introduced merely to »ensure» the marginal cost curve sufficient mobility in response to changes in the labour force.

The decisive part played by expectations in investment decisions has already been emphasized in several contexts, because, among other things, investment is a time-consuming process. Yet, when the development of an economy is viewed from the cost side, it is not equally evident that the factors affecting the decisions are expectations. The diagrammatic analysis would appear to suggest, rather, that investment is influenced by past and present technological progress and by the changes that have taken place and are taking place in the labour force, insofar as these enable the firm to reduce its costs and, thus, provide it with an opportunity to enlarge its productive capacity. The following analysis will be based on this hypothesis.

To return once more to Figure IV, it can be argued that the larger the difference between the existing capital stock and the optimal capital stock — that is, between Y_1 and Y_3 — the more there is room for capital stock expansion.[13] This hypothesis can be given an analytical form, to obtain a

13. The shape of the demand curve of course contributes to determine how much room there is for expansion.

growth opportunity function

$$(12) \qquad k_t = \frac{\overline{\overline{Y}}_t}{\overline{Y}_t} - 1.$$

In principle, there is obviously nothing to prevent the capital stock from becoming excessively large, compared with the supply of labour. ($\overline{Y}$ may exceed $\overline{\overline{Y}}$.) When this is the case, the entrepreneurs may wish to reduce their capital stock; but this can only take place through wear and tear. A situation of this kind may well arise in a »mature» economy. Here too, however, the sphere of application of the present model will be confined to a 'growth-oriented' economy. In a previous context a condition concerning the nature of expectations was formulated to describe such an economy. At this point, the presence of growth opportunities is defined as another, objective characteristic of a 'growth-oriented' economy; it is postulated, in other words, that[14]

$$(12{:}1) \qquad k_t > 0\,.$$

Consideration of the demand and supply factors reflected in investment decisions has thus been completed. It should be noted, in particular, that the analysis involved the postulate of *price and wage flexibility*, even though the changes in prices and wages do not enter into the equations actually constituting the model. In this sense the present model falls, regarding its structure, somewhere between Keynesian and neo-classical theory.

When the analyses that led to the expectation and growth opportunity functions are combined to obtain an *investment function*, use is no longer made of the 'representative firm' model; instead, impulses affecting firms' investment in general — and coming from two directions: the demand side and the supply side — will be spoken of. The effects of these impulses are supposed to be additive.

The function given in equation (12) was termed a 'growth opportunity function'. This may give the impression that the growth opportunity function is not wholly independent of the expectation function. This is not, of course, the case, as was evident from the analysis in terms of a 'representative firm'. On the other hand, the weights of these two factors as determinants of investment are likely to depend on how growth-minded the entrepreneurs are.

14. The present concept of growth opportunities is narrower in scope than the 'stock of investment opportunities' concept introduced by R. A. Gordon. See his article »Investment Behavior and Business Cycles», *The Review of Economics and Statistics*, February 1955.

If the entrepreneurs are growth-minded, they are likely to make use of any opportunity furnished by technological progress and the supply of labour to expand their productive activity. If, by contrast, they are cautious, their attention is focused mainly on the movement of demand.

The investment function is obtained through weighting together equations (9) and (12):

$$(13) \qquad I_t = [(1 - \mu)\, g_t + \mu\, k_t]\, K_t \,.$$

Here the parameter μ (which is positive and less than or equal to unity) reflects the entrepreneurs' growth-mindedness. Setting $(1 - \mu)\, \bar{\alpha} = \alpha$ and $(1 - \mu)\, \bar{\beta} = \beta$, the investment function can be rewritten, by equation (5:1), as

$$(13{:}1) \qquad I_t = \frac{\alpha}{\varphi_k}\, Y_t - (\beta + \mu)\, K_t + \frac{\mu}{\varphi_k}\, \bar{\bar{Y}}_t .$$

According to (13:1), an increase in total output (Y_t), as well as technological progress and an increase in the labour force $(\bar{\bar{Y}}_t)$, reinforce the inducement to invest; whereas an increase in the capital stock (K_t) reduces the inducement to invest. The derivation of the investment function has shed light on the factors which may have a bearing on investment decisions. Thus the above analysis may also be helpful in evaluating the relative significance of the explanatory variables involved in the investment function.

5. STRUCTURAL ELEMENTS OF THE MODEL: THE CONSUMPTION FUNCTION

The structural elements of the model considered so far have related exclusively to the firms and their activity. Households constitute the other of the two sectors covered by the model: the national income, originating in the firms sector, is disposed of by the households sector. In accordance with Keynes's original theory of consumption, consumption is here assumed to be a function of national income alone, or

$$(14) \qquad C_t = \lambda\, Y_{t-1} + \gamma \,,$$

where λ = the marginal propensity to consume and γ = the minimum level of consumption.

Studies of the reaction of consumption to changes in income would have suggested the introduction of a distributed lag into equation (14). The unit period was assumed to be rather long, however, and thus it seems justifiable to assume that a single period is sufficient for the reaction to take place.

6. CONSTRUCTING THE MODEL

The main interdependences and definitions presented in the preceding sections will be collected below to give a total picture of the model. The equations concerned will be re-numbered for the purpose, indicating, however, their original numbers as well.

Initially, the most important definitional equations will be listed. The definition of national income (total output) has not yet been introduced. It is

$$\text{(I)} \qquad Y_t = I_t + C_t .$$

The capital stock is defined with the aid of net investment as follows (equation (8:2)):

$$\text{(II)} \qquad K_t = I_{t-1} + K_{t-1} .$$

The equations related to production include the production function (equation (1:2)),

$$\text{(III)} \qquad Y_t = \varphi_n N_t,$$

the supply of labour equation (2:1),

$$\text{(IV)} \qquad L_t = (\tau\nu)^t \bar{L}_o ,$$

the equation introducing the concept of the full-employment ceiling (equation (3:1)),

$$\text{(V)} \qquad \bar{\bar{Y}}_t = \varphi_n L_t ,$$

and the equation introducing the concept of the capital-stock ceiling (equation (5:1)),

$$\text{(VI)} \qquad \bar{Y}_t = \varphi_k K_t .$$

From equations (V) and (VI) the following restrictions regarding total output are derivable:

$$\text{(VII)} \qquad \bar{Y}_t \geqq Y_t \lesseqgtr \bar{\bar{Y}}_t .$$

Provided (VII) is a double equation, a *full equilibrium* prevails (equation (7)).

The investment function (equation (13:1)) reads

$$\text{(VIII)} \qquad I_t = \frac{\alpha}{\varphi_k} Y_t - (\beta + \mu) K_t + \frac{\mu}{\varphi_k} \bar{\bar{Y}}_t$$

When this function was derived, two restrictions additional to those given in (VII) were introduced (equations (9:2) and (12:1)):

$$\text{(IX)} \qquad Y_t \geqq \frac{\bar{\beta}}{\bar{\alpha}} \bar{Y}_t$$

and

$$\text{(X)} \qquad \bar{\bar{Y}}_t \geqq \bar{Y}_t \,.$$

The consumption function (equation (14)) is of the customary type,

$$\text{(XI)} \qquad C_t = \lambda Y_{t-1} + \gamma \,.$$

As regards the two variables — total output (Y_t) and the capital stock (K_t) — that are of principal interest here, the model is capable of further elaboration. Substituting from (VIII) and (XI) into (I), we have

$$\text{(XII)} \qquad Y_t = \frac{\lambda}{1 - \frac{\alpha}{\varphi_k}} Y_{t-1} - \frac{\beta + \mu}{1 - \frac{\alpha}{\varphi_k}} K_t + \frac{\frac{\mu}{\varphi_k}}{1 - \frac{\alpha}{\varphi_k}} \bar{\bar{Y}}_t + \frac{\gamma}{1 - \frac{\alpha}{\varphi_k}}$$

and substituting from (VIII) into (II),

$$\text{(XIV)} \qquad K_t = \frac{\alpha}{\varphi_k} Y_{t-1} + (1 - \beta - \mu) K_{t-1} + \frac{\mu}{\varphi_k} \bar{\bar{Y}}_{t-1} \,.$$

Finally, substituting (XIV) into (XII) and taking into account that $\bar{\bar{Y}}_t = \tau\nu \, \bar{\bar{Y}}_{t-1}$,

$$\text{(XIII)} \qquad Y_t = \frac{\lambda - (\beta + \mu) \frac{\alpha}{\varphi_k}}{1 - \frac{\alpha}{\varphi_k}} Y_{t-1} - \frac{(\beta + \mu)(1 - \beta - \mu)}{1 - \frac{\alpha}{\varphi_k}} K_{t-1} +$$

$$\left(1 - \frac{\beta + \mu}{\tau \nu}\right) \frac{\frac{\mu}{\varphi_k}}{1 - \frac{\alpha}{\varphi_k}} \bar{\bar{Y}}_t + \frac{\gamma}{1 - \frac{\alpha}{\varphi_k}} \,.$$

To simplify equations (XIII) and (XIV) the following notations are introduced:

$$a = \frac{\lambda - (\beta + \mu)\dfrac{\alpha}{\varphi_k}}{1 - \dfrac{\alpha}{\varphi_k}} \qquad G = \tau\nu$$

$$b = -\frac{(\beta + \mu)(1 - \beta - \mu)}{1 - \dfrac{\alpha}{\varphi_k}} \qquad \bar{\bar{Y}}_t = G^t \bar{\bar{Y}}_o$$

$$c = \frac{\alpha}{\varphi_k} \qquad n = \frac{\mu}{\varphi_k \tau\nu} \bar{\bar{Y}}$$

$$d = 1 - \beta - \mu \qquad T = \frac{\gamma}{1 - \dfrac{\alpha}{\varphi_k}}$$

$$m = \left(1 - \frac{\beta + \mu}{\tau\nu}\right) \frac{\dfrac{\mu}{\varphi_k}}{1 - \dfrac{\alpha}{\varphi_k}} \bar{\bar{Y}}_o$$

Equations (XIII) and (XIV) can now be rewritten as

(XIII:1) $\quad Y_t = aY_{t-1} + bK_{t-1} + mG^t + T$

and

(XIV:1) $\quad K_t = cY_{t-1} + dK_{t-1} + nG^t$.

They form a system of two first-order linear difference equations, the dynamic properties of which will be subjected to mathematical analysis in the following.

7. MATHEMATICAL SOLUTION OF THE MODEL

The solution will be found in two steps: first, the homogeneous part of the system will be dealt with and only then the entire system.

In matrix notation, the homogeneous part of the system consisting of equations (XIII:1) and (XIV:1) reads

$$\begin{pmatrix} Y_t \\ K_t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ K_{t-1} \end{pmatrix}. \tag{15}$$

To find a solution means that an attempt is made to find the t'th power of the coefficient matrix that, in conjunction with the initial values for Y and K, is capable of indicating the values of Y and K in any period by means of the following system of equations[15]

$$(16) \qquad \begin{pmatrix} Y_t \\ K_t \end{pmatrix} = A^t \begin{pmatrix} Y_o \\ K_o \end{pmatrix}.$$

The method of solution is based on Cayley-Hamilton's' theorem, according to which any square matrix satisfies its characteristic equation.[16]

The characteristic equation of the coefficient matrix A of (15) is

$$(17) \qquad \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - \lambda (a + d) + ad - bc = 0,$$

the roots of which are the characteristic roots of the coefficient matrix. These roots are

$$\lambda_{1,2} = \frac{a + d}{2} \pm \sqrt{\frac{(a - d)^2}{4} + bc}.$$

Assuming that $\lambda_1 \neq \lambda_2$, the system of simultaneous equations

$$(18) \qquad \begin{aligned} \lambda_1^t &= s + u \lambda_1 \\ \lambda_2^t &= s + u \lambda_2 \end{aligned}$$

is solved for the constants s and u to get

$$u = \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1} \qquad \text{and} \qquad s = \lambda_1^t - \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1} \lambda_1.$$

Applying the Cayley-Hamilton theorem,[17] the matrix A^t can be directly evaluated through substituting these constants into

$$(19) \qquad A^t = sI + uA,$$

where I = the unit matrix. Then

15. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

16. See Samuel Goldberg Introduction to Difference Equations, New York 1958, p. 229.

17. See Goldberg op. cit., p. 230

$$(20) \qquad A^t = \begin{pmatrix} \lambda_1^t + \dfrac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}(a - \lambda_1) & b\,\dfrac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1} \\ c\,\dfrac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1} & \lambda_1^t + \dfrac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}(d - \lambda_1) \end{pmatrix}$$

The solution of (15) is obtained by substituting (20) into (16) and taking account of the values of λ_1 and λ_2 as determined by equations (18).

The unreduced system is, in matrix notation,

$$(21) \qquad \begin{pmatrix} Y_t \\ K_t \end{pmatrix} = A \begin{pmatrix} Y_{t-1} \\ K_{t-1} \end{pmatrix} + V^t ,$$

where[18]

$$V^t = \begin{pmatrix} mG^t + T \\ nG^t \end{pmatrix}.$$

The solution of the reduced homogeneous system will be utilized in solving the unreduced system. Substituting 1, 2, . . ., r for t in (21),

$$\begin{pmatrix} Y_1 \\ K_1 \end{pmatrix} = A \begin{pmatrix} Y_o \\ K_o \end{pmatrix} + V$$

$$\begin{pmatrix} Y_r \\ K_r \end{pmatrix} = A^r \begin{pmatrix} Y_o \\ K_o \end{pmatrix} + A^{r-1} V + A^{r-2} V^2 + \ldots + AV^{r-1} + V^r .$$

Thus, in general,

$$(22) \qquad \begin{pmatrix} Y_t \\ K_t \end{pmatrix} = A^t \begin{pmatrix} Y_o \\ K_o \end{pmatrix} + (A^{t-1}, A^{t-2}, \ldots, A, I) \begin{pmatrix} V \\ V^2 \\ \vdots \\ V^t \end{pmatrix}.$$

Here

$$V^t = \begin{pmatrix} m \\ n \end{pmatrix} G^t + \begin{pmatrix} T \\ 0 \end{pmatrix}$$

Thus, the second term on the right in (22) can be written as

$$\sum_{r=o}^{t-1} A^r V^{t-r} = \left[\sum_{o}^{t-1} A^r G^{t-r} \right] \begin{pmatrix} m \\ n \end{pmatrix} + \left[\sum_{o}^{t-1} A^r \right] \begin{pmatrix} T \\ 0 \end{pmatrix}$$

$$= G^t \left[\sum_{o}^{t-1} A^r \frac{1}{G^r} \right] \begin{pmatrix} m \\ n \end{pmatrix} + \left[\sum_{o}^{t-1} A^r \right] \begin{pmatrix} T \\ 0 \end{pmatrix}.$$

18. It should be noted that G and T are coefficients.

Since

$$\left[\sum_{o}^{t-1} \frac{A^r}{G^r}\right] \times \left[I - \frac{A}{G}\right] = I - \frac{A^t}{G^t}$$

and provided that

$$\left| I - A \right| \neq 0 ,$$

we have

$$\sum_{o}^{t-1} \frac{A^r}{G^r} = \left(I - \frac{A^t}{G^t}\right)\left(I - \frac{A}{G}\right)^{-1} .$$

In consequence,

$$(23) \qquad \sum_{r=o}^{t-1} A^r\, V^{t-r} = G^t \left(I - \frac{A^t}{G^t}\right)\left(I - \frac{A}{G}\right)^{-1} \begin{pmatrix} m \\ n \end{pmatrix} + \left(I - A^t\right)\left(I - A\right)^{-1} \begin{pmatrix} T \\ 0 \end{pmatrix} .$$

Substituting (23) into (22), the solution of the unreduced system can be written

$$(24) \qquad \begin{pmatrix} Y_t \\ K_t \end{pmatrix} = A^t \begin{pmatrix} Y_o \\ K_o \end{pmatrix} + G^t \left(I - \frac{A^t}{G^t}\right)\left(I - \frac{A}{G}\right)^{-1} \begin{pmatrix} m \\ n \end{pmatrix} + \left(I - A^t\right)\left(I - A\right)^{-1} \begin{pmatrix} T \\ 0 \end{pmatrix} .$$

The terms on the right are, in expanded form, as follows:

$$A^t \begin{pmatrix} Y_o \\ K_o \end{pmatrix} = \begin{pmatrix} \lambda_1^t + \dfrac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}(a - \lambda_1) & \dfrac{b\,(\lambda_2^t - \lambda_1^t)}{\lambda_2 - \lambda_1} \\ \dfrac{c\,(\lambda_2^t - \lambda_1^t)}{\lambda_2 - \lambda_1} & \lambda_1^t + \dfrac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}(d - \lambda_1) \end{pmatrix} \begin{pmatrix} Y_o \\ K_o \end{pmatrix}$$

and introducing the following notations

$$H = \left[1 - \left[\lambda_1^t + \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}(a - \lambda_1)\right]\frac{1}{G^t}\right]\left[m\left(1 - \frac{d}{G}\right) + \frac{bn}{G}\right] + $$

$$+ \frac{(-b)\,(\lambda_2^t - \lambda_1^t)}{(\lambda_2 - \lambda_1)\, G^t}\left[\frac{cm}{G} + n\left(1 - \frac{a}{G}\right)\right] .$$

and

$$J = \frac{(-c)\,(\lambda_2^t - \lambda_1^t)}{(\lambda_2 - \lambda_1)\,G^t}\left[m\left(1 - \frac{d}{G}\right) + \frac{bn}{G}\right] + \left[1 - \left[\lambda_1^t + \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}\,(d - \lambda_1)\right]\frac{1}{G^t}\right]\left[\frac{cm}{G} + n\left(1 - \frac{a}{G}\right)\right]$$

and

$$M = \left[1 - \left[\lambda_1^t + \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}\,(a - \lambda_1)\right]\right] T\,(1 - d) + \frac{(-b)\,(\lambda_2^t - \lambda_1^t)}{\lambda_2 - \lambda_1}\,cT$$

$$V = \frac{(-c)\,(\lambda_2^t - \lambda_1^t)}{\lambda_2 - \lambda_1}\,T(1 - d) + \left[1 - \left[\lambda_1^t + \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}\,(d - \lambda_1)\right]\right] cT.$$

The second and the third term on the right can be written as

$$\left(I - \frac{A^t}{G^t}\right)\left(1 - \frac{A}{G}\right)^{-1}\begin{pmatrix} m \\ n \end{pmatrix} = \frac{1}{\begin{vmatrix} 1 - \frac{a}{G} & -\frac{b}{G} \\ -\frac{c}{G} & 1 - \frac{d}{G} \end{vmatrix}}\begin{pmatrix} H \\ J \end{pmatrix}$$

$$\left(I - A^t\right)\left(I - A\right)^{-1}\begin{pmatrix} T \\ 0 \end{pmatrix} = \frac{1}{\begin{vmatrix} 1 - a & -b \\ -c & 1 - d \end{vmatrix}}\begin{pmatrix} M \\ V \end{pmatrix}$$

The solution of the system for total output is

$$(25) \qquad Y_t = \left[\lambda_1^t + \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}\,(a - \lambda_1)\right] Y_o + \frac{b\,(\lambda_2^t - \lambda_1^t)}{\lambda_2 - \lambda_1}\,K_o +$$

$$\frac{G^t}{\begin{vmatrix} 1 - \frac{a}{G} & -\frac{b}{G} \\ -\frac{c}{G} & 1 - \frac{d}{G} \end{vmatrix}}\left\{\left[1 - \left[\lambda_1^t + \frac{\lambda_2^t - \lambda_1^t}{\lambda_2 - \lambda_1}\,(a - \lambda_1)\right]\frac{1}{G^t}\right] \times\right.$$

$$\left[m\left(1-\frac{d}{G}\right)+\frac{bn}{G}\right]+\frac{(-b)\ (\lambda_2^t-\lambda_1^t)}{(\lambda_2-\lambda_1)\ G^t}\left[\frac{cm}{G}+n\left(1-\frac{a}{G}\right)\right]\Bigg\}+$$

$$\frac{1}{\begin{vmatrix} 1-a & -b \\ -c & 1-d \end{vmatrix}}\left\{\left[1-\left[\lambda_1^t+\frac{\lambda_2^t-\lambda_1^t}{\lambda_2-\lambda_1}\ (a-\lambda_1)\right]\right]T(1-d)+\right.$$

$$\left.\frac{(-b)\ (\lambda_2^t-\lambda_1^t)}{\lambda_2-\lambda_1}\ cT\right\}$$

and, for the capital stock,

$$K_t=\frac{c\,(\lambda_2^t-\lambda_1^t)}{\lambda_2-\lambda_1}\ Y_o+\left[\lambda_1^t+\frac{\lambda_2^t-\lambda_1^t}{\lambda_2-\lambda_1}\ (d-\lambda_1)\right]K_o+ \tag{26}$$

$$\frac{G^t}{\begin{vmatrix} 1-\frac{a}{G} & -\frac{b}{G} \\ -\frac{c}{G} & 1-\frac{d}{G} \end{vmatrix}}\left\{\frac{(-c)\,(\lambda_2^t-\lambda_1^t)}{(\lambda_2-\lambda_1)\,G^t}\left[m\left(1-\frac{d}{G}\right)+\frac{bn}{G}\right]+\right.$$

$$\left.\left[1-\left[\lambda_1^t+\frac{\lambda_2^t-\lambda_1^t}{\lambda_2-\lambda_1}\ (d-\lambda_1)\right]\frac{1}{G^t}\right]\left[\frac{cm}{G}+n\left(1-\frac{a}{G}\right)\right]\right\}+$$

$$\frac{1}{\begin{vmatrix} 1-a & -b \\ -c & 1-d \end{vmatrix}}\left\{\frac{(-c)\ (\lambda_2^t-\lambda_1^t)}{\lambda_2-\lambda_1}\ T(1-d)+\right.$$

$$\left.\left[1-\left[\lambda_1^t+\frac{\lambda_2^t-\lambda_1^t}{\lambda_2-\lambda_1}\ (d-\lambda_1)\right]\right]cT\right\}.$$

II. EQUILIBRIUM GROWTH

The present chapter considers the structure of the model developed in the preceding chapter from the point of view of growth.

1. ON THE PROBLEMS OF THE THEORY OF FULL EQUILIBRIUM

As early as the late 1930s R. F. HARROD discussed, on the basis of KEYNES's General Theory, certain problems of economic growth, which in the 1950s and 1960s came to occupy a central position in theoretical investigation.[1] As the aspect from which Harrod (and Domar) approached the problems of growth is important for the present study, a concise description of their model is in order.

The model includes one behavioural equation — the saving function — according to which saving bears a constant proportion (s) to total income. Another basic equation of the model is a fixed-coefficient production function, relating the rate of change of the capital stock (i) via the capital coefficient $\left(\frac{1}{v}\right)$ to the rate of change of total income (y). A further assumption is that the population is growing at a given rate (n) as a result of non-economic factors.[2]

In consequence of these assumptions the »warranted» rate of growth in income cannot exceed the »natural» rate of growth determined by the growth of the population. In Harrod's terminology, the growth of total income is in »equilibrium» (g) when

$$I = S$$

and

$$\frac{1}{v} \cdot I = \Delta Y$$

and, thus,

1. R. F. HARROD »An Essay in Dynamic Theory», *Economic Journal*, March 1939. See also R. F. HARROD Towards a Dynamic Economics, London 1948; E. D. DOMAR »Capital Expansion, Rate of Growth and Employment», *Econometrica*, April 1946.

2. Technological progress can be incorporated into the model in the same way as the growth of the population.

$$g = \frac{\Delta Y}{Y} = \frac{I}{v \cdot Y} = \frac{S}{v}.$$

Full equilibrium or, in Harrod's terminology, full employment steady growth is attained when

$$n = g.$$

If $g < n$ and the coefficients of the model are regarded as given, the model does not include any flexible factor making it possible for the »warranted» rate to rise so as to equal the full equilibrium rate.

The Harrod—Domar system is structurally a short-run model: the »warranted» rate of growth (g) only expresses the condition of the equality of the supply and demand for goods in terms of the rates of change. This seemingly slight conceptual novelty in trade cycle theory was, however, an important step towards the theory of growth, and the problem of steady growth equilibrium has without doubt served as a source of stimulus for growth theory.

In a recent survey of the theory of growth[3] F. H. HAHN and R. C. O. MATTHEWS gave an excellent account of the varied solutions suggested for the above Harrod—Domar problem. Thus, a very concise description of the principal points of the theories concerned suffices here.

Solution of the Harrod—Domar problem presupposes, of course, that some parameter or parameters of the model (n, s and v) are considered as variables.

The Cambridge economists have devoted particular attention to the *propensity to save.*[4] In studies representing this approach, income distribution has been assumed to be flexible; provided that the capitalists' and the wage-earners' propensities to save differ, the distribution of income is capable of serving as an equilibrating factor, altering the economy's propensity to save in a direction consistent with full equilibrium.[5]

The problem of full equilibrium has in several studies also been approached from a neo-classical angle. In the studies attention has principally been focused on the *capital coefficient* as the equilibrating variable. SOLOW's and MEADE's contributions have undoubtedly been very significant here.[6]

3. F. H. HAHN and R. C. O. MATTHEWS »The Theory of Economic Growth: A Survey», *The Economic Journal,* December 1964.

4. See, e.g., N. KALDOR & J. A. MIRRLEES »A New Model of Economic Growth», *The Review of Economic Studies,* Vol. XXIX; J. ROBINSON The Accumulation of Capital, London 1956.

5. For details, see HAHN & MATTHEWS op cit., pp. 793—801 and 811—812.

6. R. M. SOLOW »A Contribution to the Theory of Economic Growth», *Quarterly Journal of Economics,* February 1956; J. E. MEADE A Neo-Classical Theory of Economic Growth, London 1961.

Solow's theory — which will here serve as an example of neo-classical theory — starts from the state of full employment. Now, if the capital coefficient is too small (from the standpoint of full employment steady growth), the »warranted» rate of growth *(g)* exceeds the growth rate compatible with full equilibrium. On the other hand, this may be interpreted as saying that the propensity to save is too large. As a result of excessive saving the capital stock increases at a faster rate than the population; and this, in turn, leads to a rise in the capital coefficient. The rise in the capital coefficient continues until full equilibrium is regained.[7]

The models where the growth of population is assumed to depend on economic factors are also of interest from the standpoint of the present study.[8] In such models it is consequently not the »warranted» growth rate *(g)* that adjusts; and in a Harrod-type of model this in fact means the possibility of prolonged underemployment equilibrium, for the rate of growth of the population can only change very slowly.

As regards the principles underlying *the present model*, it should be stated that the capital coefficient, the propensity to save and the growth rate of the population (and the rate of technological progress) are assumed to be given. *The flexible component of the model is the investment function;*[9] *thus, it is the investment function that determines whether growth follows the full equilibrium path or not.*

2. ON THE NATURE OF »EQUILIBRIUM»

Compatibility of an equilibrium analysis in terms of the basic model in the present study,

$$\text{(XIII:1)} \qquad Y_t = aY_{t-1} + bK_{t-1} + mG^t + T$$

$$\text{(XIV:1)} \qquad K_t = cY_{t-1} + dK_{t-1} + nG^t,$$

with the set of problems introduced in the preceding section presupposes that consumption bears a constant ratio to output (i.e., that $\gamma = 0$ in equation (XI)):

7. For neo-classical literature in general, see HAHN & MATTHEWS op. cit., pp. 787—793 and 809.

8. See, e.g., T. HAAVELMO A Study in the Theory of Economic Evolution, Amsterdam 1954, and the literature referred to in HAHN & MATTHEWS op.cit., pp. 801—804.

9. Some of the studies in the other groups described above also involve explicit investment functions. (This applies to the KALDOR—MIRRLEES model, for example.) As a rule, however, the centre of emphasis lies in the other structural elements of the models.

(XI:1) $C_t = \lambda\, Y_{t-1}$.

Hence, the last term on the right in equation (XIII:1) vanishes, since

$$T = \frac{\gamma}{1 - \dfrac{a}{\varphi_k}} .$$

The definition of *full equilibrium* was

(VII) $Y_t = \bar{Y}_t = \bar{\bar{Y}}_t$

The assumptions concerning the conditions of production (equations (III), (V) and (VII)) imply that, in full equilibrium, all the productive resources of the economy are fully employed and that output is the maximum producible from the existing resources.

Despite its being unrealistic, this equilibrium concept is, apart from being a useful analytical tool, serviceable in economic policy. From the standpoint of economic policy, it can be regarded as an efficiency norm that guides — or that should be permitted to guide — all economic policy measures. In the following, however, full equilibrium will be exclusively considered as a theoretical concept. The term itself suggests that the concept is not of much significance in the world of reality: what is concerned is a state of affairs that may be attainable in the long-run only.

Our intention is now to analyze the content of this equilibrium concept and the preconditions for this kind of equilibrium in the light of the present model (equations (XIII:1) and (XIV:1)). In the latter sense the concept becomes a *theoretical norm*.

3. THE EQUILIBRIUM SOLUTION

The present analysis presupposes consideration of a process over time. The solution set of the model has to be chosen as the point of departure.[10]

(24)
$$\begin{pmatrix} Y_t \\ K_t \end{pmatrix} = A^t \begin{pmatrix} Y_o \\ K_o \end{pmatrix} + G^t \left(I - \frac{A^t}{G^t}\right) \left(I - \frac{A}{G}\right)^{-1} \begin{pmatrix} m \\ n \end{pmatrix} + \left(I - A^t\right) \left(I - A\right)^{-1} \begin{pmatrix} T \\ 0 \end{pmatrix}$$

10. T is retained in the system, though it was assumed to equal zero. The reason for this will become apparent later.

Setting $A^t = 0$, an *equilibrium solution* (in the mathematical sense) is obtained:

$$(27) \qquad \begin{pmatrix} Y_t \\ K_t \end{pmatrix} = G^t \left(I - \frac{A}{G}\right)^{-1} \begin{pmatrix} m \\ n \end{pmatrix} + \left(I - A\right)^{-1} \begin{pmatrix} T \\ 0 \end{pmatrix}$$

or

$$(27{:}1) \qquad \begin{pmatrix} Y_t \\ K_t \end{pmatrix} = \frac{G^t}{\begin{vmatrix} 1 - \frac{a}{G} & -\frac{b}{G} \\ -\frac{c}{G} & 1 - \frac{d}{G} \end{vmatrix}} \begin{pmatrix} m\left(1 - \frac{d}{G}\right) + n\frac{b}{G} \\ m\frac{c}{G} + n\left(1 - \frac{a}{G}\right) \end{pmatrix} +$$

$$\frac{1}{\begin{vmatrix} 1 - a & -b \\ -c & 1 - d \end{vmatrix}} \begin{pmatrix} T(1-d) \\ Tc \end{pmatrix}.$$

Since

$$\begin{vmatrix} 1 - \frac{a}{G} & -\frac{b}{G} \\ -\frac{c}{G} & 1 - \frac{d}{G} \end{vmatrix} = \left(1 - \frac{a}{G}\right)\left(1 - \frac{d}{G}\right) - \frac{bc}{G^2}$$

and

$$\begin{vmatrix} 1 - a & -b \\ -c & 1 - d \end{vmatrix} = (1-a)\,(1-d) - bc,$$

we have

$$(28) \qquad Y_t = G^t \frac{m\left(1 - \frac{d}{G}\right) + n\frac{b}{G}}{\left(1 - \frac{a}{G}\right) 1 - \left(\frac{d}{G}\right) - \frac{bc}{G^2}} + \frac{T(1-d)}{(1-a)(1-d) - bc}$$

and

$$(29) \qquad K_t = G^t \frac{m\frac{c}{G} + n\left(1 - \frac{a}{G}\right)}{\left(1 - \frac{a}{G}\right)\left(1 - \frac{d}{G}\right) - \frac{bc}{G^2}} + \frac{Tc}{(1-a)\ (1-d) - bc}\,.$$

According to the definition given in equation (VII), total output has to be compatible with the capital stock; and, moreover, compatible with the labour force and technological progress. In other words[11]

$$(30) \qquad G^t \frac{m\left(1 - \frac{d}{G}\right) + n\frac{b}{G}}{\left(1 - \frac{a}{G}\right)\left(1 - \frac{d}{G}\right) - \frac{bc}{G^2}} + \frac{T\,(1-d)}{(1-a)\,(1-d) - bc} =$$

$$\varphi_k\, G^t \frac{m\frac{c}{G} + n\left(1 - \frac{a}{G}\right)}{\left(1 - \frac{a}{G}\right)\left(1 - \frac{d}{G}\right) - \frac{bc}{G^2}} + \frac{\varphi_k\, T}{(1-a)\,(1-d) - bc}$$

$$= G^t\, \bar{\bar{Y}}_o\,.$$

Obviously, equilibrium is attainable only on condition that the consumption function is proportional. Thus, this proportionality assumption is also necessary for reasons other than those mentioned in Section 2. When $T = 0$, the equilibrium conditions that the parameters have to satisfy can be written

$$(30{:}1) \qquad \frac{m\left(1 - \frac{d}{G}\right) + n\frac{b}{G}}{\left(1 - \frac{a}{G}\right)\left(1 - \frac{d}{G}\right) - \frac{bc}{G^2}} = \frac{\varphi_k\left[m\frac{c}{G} + n\left(1 - \frac{a}{G}\right)\right]}{\left(1 - \frac{a}{G}\right)\left(1 - \frac{d}{G}\right) - \frac{bc}{G^2}} = \bar{\bar{Y}}_o\,.$$

From the equations in (30:1) it is difficult to determine the combinations of the parameters that ensure the possibility of equilibrium: it should be recalled that each of the parameters involved in these equations is a combination of the original parameters of the model.

11. Equation (29) is multiplied by φ_k.

4. THE EQUILIBRIUM CONDITIONS FOR THE PARAMETERS

To derive the conditions of equilibrium that the original parameters must satisfy, we return to the definitional equation (I). Substituting $\bar{\bar{Y}}_t$ for Y_t,

$$(31) \qquad \bar{\bar{Y}}_t = I_t + C_t.$$

Let us then consider I_t in equilibrium. The investment function is

$$(13) \qquad I_t = [(1-\mu)\, g_t + \mu\, k_t]\, K_t.$$

In equilibrium, g and k are, by equations (9:3) and (12),

$$g = \bar{\alpha} - \bar{\beta}$$
$$k = 0,$$

and, thus,

$$(13{:}2) \qquad I_t = (1-\mu)\,(\bar{\alpha} - \bar{\beta})\, K_t.$$

On the other hand, in equilibrium,

$$\frac{\Delta K_t}{K_t} = \frac{\bar{\bar{Y}}_{t+1} - \bar{\bar{Y}}_t}{\bar{\bar{Y}}_t}$$
$$= \frac{[(\tau\nu)^{t+1} - (\tau\nu)^t]\,\varphi_n L_o}{(\tau\nu)^t\,\varphi_n L_o}$$
$$= \tau\nu - 1.$$

Recalling that $\Delta K_t = I_t$ and substituting from equation (13:2), the first equilibrium condition to be satisfied by the parameters can be written[12]

$$(32) \qquad (1-\mu)\,(\bar{\alpha} - \bar{\beta}) = \tau\nu - 1.$$

In addition to this condition concerning the parameters of the investment function it is necessary to formulate a condition that the propensity to consume and the capital coefficient have to fulfil. To this end, the investment function (13:2) and the consumption function (XI:1) are substituted into (31) to obtain

$$(31{:}1) \qquad \bar{\bar{Y}}_t = (1-\mu)\,(\bar{\alpha} - \bar{\beta})\, K_t + \lambda\, Y_{t-1}.$$

By virtue of (32) and the definition of $\bar{\bar{Y}}_t$, equation (31:1) can be written

12. $(1-\mu)\,(\bar{\alpha} - \bar{\beta}) = \alpha - \beta$.

$$(31{:}2) \qquad (\tau\nu)^t \,\overline{\overline{Y}}_o = (\tau\nu - 1)\, \frac{(\tau\nu)^t \,\overline{\overline{Y}}_o}{\varphi_k} + \lambda(\tau\nu)^{t-1}\, \overline{\overline{Y}}_o .$$

On simplification, this becomes

$$(33) \qquad \lambda = \left(1 - \frac{\tau\nu - 1}{\varphi_k}\right) \tau\nu ,$$

which is the equilibrium condition that the parameters of the production function and consumption function have to satisfy.

Making use of equations (32) and (33) the numerators and denominators in the equilibrium conditions in (30:1) can be written as follows:

$$(34{:}1) \qquad m\left(1 - \frac{d}{G}\right) + n\,\frac{b}{G} = \frac{\mu\,(a - \beta)}{\tau\nu\,(\varphi_k - a)} \cdot \overline{\overline{Y}}_o ,$$

$$(34{:}2) \qquad \varphi_k \left[m\,\frac{c}{G} + n \cdot \left(1 - \frac{a}{G}\right)\right] = \frac{\mu\,(a - \beta)}{\tau\nu\,(\varphi_k - a)} \cdot \overline{\overline{Y}}_o$$

and

$$(34{:}3) \qquad \left(1 - \frac{a}{G}\right)\left(1 - \frac{d}{G}\right) - \frac{bc}{G^2} = \frac{\mu\,(a - \beta)}{\tau\nu\,(\varphi_k - a)} .$$

Thus, when the values of the parameters satisfy the conditions in (32) and (33), the equilibrium equation in (30:1) is also satisfied. In consequence, (32) and (33) are a set of sufficient *equilibrium conditions*.

5. THE STABILITY CONDITIONS FOR THE PARAMETERS

Employing the equilibrium conditions derived in the preceding section, it is possible to specify the parameters of the model in such a way that an appropriate choice of the initial values of total output and capital stock causes the system to move according to the full equilibrium path. But this does not yet ensure that this equilibrium path represents a »true» or stable equilibrium; i.e., this does not ensure that, if the values of total output and capital stock depart from the equilibrium values, their subsequent course will be toward the equilibrium path.

For the equilibrium to be stable, the parameters must have values ensuring that the roots, λ_1 and λ_2 of the characteristic equation

$$(17) \qquad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

are both less than unity.[13]

According to the general test of stability,[14] the following inequality must hold:

$$(35) \qquad \begin{vmatrix} 1 & ad-bc \\ ad-bc & 1 \end{vmatrix} > 0$$

Expanding this we have

$$(35:1) \qquad -1 < ad - bc < 1.$$

The same test also requires that

$$(36) \qquad \begin{vmatrix} 1 & 0 & ad-bc & -a-d \\ -a-d & 1 & 0 & ad-bc \\ ad-bc & 0 & 1 & -a-d \\ -a-d & ad-bc & 0 & 1 \end{vmatrix} > 0$$

or that

$$(36:1) \qquad (1 + ad - bc)^2 > (a + d)^2.$$

Let us consider what requirements are imposed by these stability conditions upon the original parameters of the model. Substituting the original parameters into (35:1) and simplifying, the *first stability condition* becomes

$$(35:2) \qquad -1 < \frac{\lambda\,(1-\beta-\mu)}{1-\dfrac{\alpha}{\varphi_k}} < 1.$$

Likewise, substituting into (36:1) and recalling that $1 - \lambda > 0$ and $\beta + \mu > 0$, the *second stability condition* becomes

$$(36:2) \qquad \beta + \mu < \frac{2\left(1-\dfrac{\alpha}{\varphi_k}+\lambda\right)}{1+\lambda}.$$

Since $\beta + \mu > 0$, it should be noticed that, by inequality (36:2),

$$(36:3) \qquad \frac{\alpha}{\varphi_k} < 1 + \lambda.$$

13. These roots must not be confused with the propensity to consume, denoted by λ.

14. See, e.g., William J. Baumol Economic Dynamics, 2nd Edition, New York 1959, pp. 246—248.

6. THE EQUILIBRIUM AND STABILITY CONDITIONS

The analysis performed in Sections 5 and 6 showed that full equilibrium, in the sense that

$$\text{(VII)} \qquad Y_t = \bar{Y}_t = \bar{\bar{Y}}_t \,,$$

is attainable provided that

$$\text{(32)} \qquad \alpha - \beta = \tau\nu - 1$$

$$\text{(33)} \qquad \lambda = \left(1 - \frac{\tau\nu - 1}{\varphi_k}\right) \tau\nu \,,$$

$$\text{(35:2)} \qquad -1 < \frac{\lambda\,(1 - \beta - \mu)}{1 - \dfrac{\alpha}{\varphi_k}} < 1$$

and

$$\text{(36:2)} \qquad \beta + \mu < \frac{2\left(1 - \dfrac{\alpha}{\varphi_k} + \lambda\right)}{1 + \lambda}$$

hold, the last inequality having the implication

$$\text{(36:3)} \qquad \frac{\alpha}{\varphi_k} < 1 + \lambda \,.$$

7. THE EQUILIBRIUM CONDITIONS: A NUMERICAL ANALYSIS

To elucidate the nature and implications of the equilibrium conditions in (32) and (33), the parameters were given certain numerical values, chosen with regard to the assumption that the length of the unit period was one year. Table 1 illustrates the interdependence of the parameters involved in (33).

Inspection of the table reveals, among others, the following facts.

1. In stationary equilibrium $(\tau\nu = 1.000)$ the propensity to consume (λ) equals unity and is independent of the capital-output ratio $\left(\dfrac{1}{\varphi_k}\right)$.
2. If the capital-output ratio is large, acceleration in the growth of the popu-

Table 1.

τ	ν	$\tau\nu$	λ			
			$\varphi_k = 0.5$	$\varphi_k = 0.4$	$\varphi_k = 0.3$	$\varphi_k = 0.2$
1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.005	1.005	0.99495	0.99244	0.98825	0.979875
1.000	1.01	1.010	0.9898	0.98475	0.97633	0.9595
1.005	1.01	1.01505	0.9845	0.97686	0.96413	0.93867
1.01	1.01	1.0201	0.97909	0.96884	0.95175	0.91758
1.02	1.01	1.0302	0.96798	0.95242	0.92649	0.87464
1.02	1.015	1.0353	0.96221	0.94393	0.91348	0.85257
1.03	1.01	1.0403	0.95645	0.93549	0.90055	0.83068
1.03	1.015	1.04545	0.95042	0.92666	0.88706	0.80787
1.04	1.01	1.0504	0.94452	0.91805	0.87393	0.7857
1.04	1.015	1.0556	0.93822	0.90887	0.85996	0.76214
1.05	1.01	1.0605	0.93218	0.90010	0.84663	0.73970
1.05	1.015	1.06575	0.92560	0.89057	0.83217	0.71538

lation (ν) and technological progress (τ) presuppose a larger decline in the propensity to consume than is necessary when the capital-output ratio is small. For example, if $\tau\nu$ increases from 1.0302 to 1.0403, the propensity to consume must decrease from 0.96798 to 0.95645 if the capital-output ratio is 2 $(\varphi_k = 0.5)$; but from 0.87464 to 0.83068 if the capital-output ratio is 5 $(\varphi_k = 0.2)$.

The consequences of the second equilibrium condition are illustrated in Table 2 $(\bar{g} = \bar{\alpha} - \bar{\beta})$. The parameter μ describes the entrepreneurs' general mode of reaction to demand prospects on the one hand, and to the oppor-

Table 2.

$\tau\nu$	μ			
	$\bar{g} = 0.03$	$\bar{g} = 0.04$	$\bar{g} = 0.05$	$\bar{g} = 0.06$
1.000	1.000	1.000	1.000	1.000
1.005	0.833	0.875	0.900	0.917
1.010	0.667	0.750	0.800	0.833
1.01505	0.4983	0.624	0.699	0.749
1.0201	0.33	0.498	0.598	0.665
1.0302	—	0.245	0.396	0.497
1.0353	—	0.1175	0.294	0.412
1.0403	—	—	0.194	0.328
1.04545	—	—	0.091	0.243
1.0504	—	—	—	0.16
1.0556	—	—	—	0.073

tunities provided by declining costs on the other. The larger the value of μ, the greater the role played by the development of costs in investment decisions.

Inspection of the table reveals the following:

1. The larger the expected long-term growth rate of demand at a given value of τv, the larger must be the proportion of the entrepreneurs who base their investment decisions upon the development of cost.

2. If the expected long-term rate of growth in demand equals the rate at which the full-employment ceiling is rising, it is necessary for equilibrium to have $\mu = 0$, since $(1-\mu)\ \bar{g} = \tau v - 1$. Provided the system remains in equilibrium for a sufficient length of time, an adjustment of expectations may be possible. In this event the investment function reduces to one typical of a customary acceleration model; for, in such a case, we have from equation

$$(13{:}2) \qquad \frac{I_t}{K_t} = \frac{\Delta \bar{\bar{Y}}_t}{\bar{\bar{Y}}_t}.$$

3. Equilibrium is rather difficult to maintain, as the equilibrium condition requires that the psychological expectations and the physical preconditions of growth are interrelated in a given, specific manner.

8. THE STABILITY CONDITIONS: A NUMERICAL ANALYSIS

To be able to perform a numerical analysis of the stability conditions

$$(35{:}2) \qquad -1 < \frac{\lambda\ (1-\beta-\mu)}{1-\dfrac{\alpha}{\varphi_k}} < 1$$

and

$$(36{:}2) \qquad \beta+\mu < 2\,\frac{1-\dfrac{\alpha}{\varphi_k}+\lambda}{1+\lambda}$$

one must specify the values of $\bar{\alpha}$ and $\bar{\beta}$ and not only the value of their difference. From the expectation function

$$(9) \qquad g_t = \bar{\alpha}\,\frac{Y_t}{\bar{\bar{Y}}_t} - \beta$$

the expected long-term growth in demand was

(9:3) $$\bar{g} = \bar{\alpha} - \bar{\beta}.$$

Eliminating $\bar{\beta}$, we have

(37:1) $$\bar{\alpha} - \bar{g} = \bar{\alpha} \frac{Y_t}{\bar{Y}_t} - g_t ,$$

and, further,

(37:2) $$\bar{\alpha} = \frac{\bar{g} - g_t}{1 - \frac{Y_t}{\bar{Y}_t}} .$$

Setting $g_t = 0$ and denoting the corresponding value of $\frac{Y_t}{\bar{Y}_t}$ by $\bar{y}$, i.e., $\bar{y}$ is the degree of capacity utilization at which the entrepreneurs have no longer any growth expectations, $\bar{\alpha}$ can be evaluated from

(37:3) $$\bar{\alpha} = \frac{\bar{g}}{1 - \bar{y}} .$$

According to this equation, the value of $\bar{\alpha}$ reflects not only the entrepreneurs' growth expectations but also the sensitivity of these expectations to the existence of excess capacity. The larger the excess capacity, in relative terms, that is necessary for the disappearance of the entrepreneurs' growth expectations, the higher is the value of $\bar{\alpha}$.

A number of alternative values of $\bar{\alpha}$ are presented in Table 3.

Table 3.

$\bar{y}$ \ $\bar{g}$	0.03	0.04	0.05	0.06
0.6	0.075	0.1	0.125	0.15
0.75	0.12	0.16	0.20	0.24
0.9	0.3	0.4	0.5	0.6

Inspection of the table immediately reveals that the value of $\bar{\alpha}$ is far more sensitive to changes in $\bar{y}$ than to changes in $\bar{g}$.

When the value of $\bar{\alpha}$ is known, the value of $\bar{\beta}$ can be determined from equation (9:3).[15]

15. The α and β involved in the equilibrium conditions were defined in terms of $\bar{\alpha}$ and $\bar{\beta}$ as follows: $\alpha = (1 - \mu)\, \bar{\alpha}$ and $\beta = (1 - \mu)\, \bar{\beta}$.

Table 4.

Row No.	$\tau\nu$	φ_k	λ	$\bar{g}$	μ	$\bar{a}$ 0.6	$\bar{a}$ 0.75	First stability condition[16]	Second stability condition[16]
1.	1.000	0.5	1.000	0.03	1.000	0.075	—	0	satisfied
2.	1.005	0.5	0.9945	0.03	0.8333	0.075	—	satisfied	,,
3.	1.005	0.5	0.9945	0.03	0.8333	—	0.12	,,	,,
4.	1.005	0.3	0.9883	0.03	0.8333	0.075	—	,,	,,
5.	1.005	0.3	0.9883	0.04	0.875	0.1	—	,,	,,
6.	1.0201	0.5	0.9791	0.03	0.33	0.075	—	,,	,,
7.	1.0201	0.5	0.9791	0.03	0.33	—	0.12	,,	,,
8.	1.0201	0.3	0.9518	0.03	0.33	0.075	—	,,	,,
9.	1.0201	0.3	0.9518	0.03	0.33	—	0.12	,,	,,
10.	1.0201	0.3	0.9518	0.04	0.4975	0.1	—	,,	,,
11.	1.0201	0.3	0.9518	0.04	0.4975	—	0.16	,,	,,
12.	1.0201	0.2	0.9176	0.03	0.33	0.075	—	,,	,,
13.	1.0201	0.2	0.9176	0.03	0.33	—	0.12	,,	,,
14.	1.0201	0.2	0.9176	0.04	0.4975	0.1	—	,,	,,
15.	1.0201	0.2	0.9176	0.04	0.4975	—	0.16	,,	,,
16.	1.0302	0.5	0.9680	0.04	0.245	0.1	—	,,	,,
17.	1.0302	0.5	0.9680	0.04	0.245	—	0.16	,,	,,
18.	1.0302	0.3	0.9265	0.04	0.245	0.1	—	,,	,,
19.	1.0302	0.3	0.9265	0.04	0.245	—	0.16	not satisfied	,,
20.	1.0302	0.3	0.9265	0.05	0.396	0.125	—	satisfied	,,
21.	1.0302	0.3	0.9265	0.05	0.396	—	0.2	,,	,,
22.	1.0302	0.2	0.8746	0.04	0.245	0.1	—	,,	,,
23.	1.0302	0.2	0.8746	0.04	0.245	—	0.16	not satisfied	,,
24.	1.0302	0.2	0.8746	0.05	0.396	0.125	—	satisfied	,,
25.	1.0302	0.2	0.8746	0.05	0.396	—	0.2	not satisfied	,,
26.	1.0455	0.5	0.9504	0.05	0.091	0.125	—	,,	,,
27.	1.0455	0.5	0.9504	0.05	0.091	—	0.2	,,	,,
28.	1.0455	0.3	0.8871	0.05	0.091	0.125	—	,,	,,
29.	1.0455	0.3	0.8871	0.05	0.091	—	0.2	,,	,,
30.	1.0455	0.3	0.8871	0.06	0.2425	0.15	—	satisfied	,,
31.	1.0455	0.3	0.8871	0.06	0.2425	—	0.24	not satisfied	,,
32.	1.0455	0.2	0.8079	0.05	0.091	0.125	—	,,	,,
33.	1.0455	0.2	0.8079	0.05	0.091	—	0.2	,,	,,
34.	1.0455	0.2	0.8079	0.06	0.2425	0.15	—	,,	,,
35.	1.0455	0.2	0.8079	0.06	0.2425	—	0.2	,,	,,
36.	1.0455	0.2	0.8079	0.07	0.3507	0.175	—	,,	,,
37.	1.0455	0.2	0.8079	0.07	0.3507	—	0.28	,,	,,
38.	1.0455	0.2	0.8079	0.08	0.4319	0.2	—	satisfied	,,
39.	1.0455	0.2	0.8079	0.08	0.4319	—	0.32	not satisfied	,,

16. The first stability condition reads $-1 < \dfrac{\lambda(1-\beta-\mu)}{1-\dfrac{a}{\varphi_k}} < 1$ and,

the second stability condition $\beta + \mu < \dfrac{2\left(1-\dfrac{a}{\varphi_k}+\lambda\right)}{1+\lambda}$

Equilibrium values of the parameters corresponding to a number of the values of the rate at which the full-employment ceiling rises are set out in Table 4, indicating also whether the values of the parameters satisfy the stability conditions.

The conclusions suggested by the table include the following:

1. If the value of the capital-output ratio $\left(\frac{1}{\varphi_k}\right)$ departs markedly from the percentage rate at which the full-employment ceiling is rising (i.e., from $100 \times (\tau\nu - 1)$), the equilibrium process is likely to be unstable. This is the case when $\tau\nu = 1.0302$ and $\varphi_k = 0.2$, and when $\tau\nu = 1.04545$ and $\varphi_k = 0.5$.
2. On the other hand, it is possible for the process to be stable when the capital-output ratio and the percentage rate at which the full-employment ceiling is rising are of the same order of magnitude. This is suggested by the cases where $\tau\nu = 1.0302$ and $\varphi_k = 0.3$ and where $\tau\nu = 1.04545$ and $\varphi_k = 0.3$.
3. Total output can only remain below the path of the full-employment ceiling, but it cannot possibly be above it; and thus it is natural that a sufficiently large value of the growth-in-demand expectations $(\bar{g})$ will enhance the possibility of a stable process. This is, for example, the case when $\tau\nu = 1.0302$, $\varphi_k = 0.2$ and $\bar{g} = 0.05$; and the same applies when $\tau\nu = 1.04545$, $\varphi_k = 0.2$ and $\bar{g} = 0.07$.
4. If the entrepreneurs' expectations are sensitive to a decrease in demand — which sensitivity is indicated by a high value of $\bar{\alpha}$ — the chances of stability are weakened. This state of affairs is exemplified by the rise of $\bar{\alpha}$ from 0.1 to 0.16 when $\tau\nu = 1.0302$ and $\varphi_k = 0.2$.
5. The higher the rate at which the full-employment ceiling is rising, the smaller are the chances of a stable equilibrium, unless the capital-output ratio »follows suit».

9. ON FULL CAPACITY UTILIZATION

A full equilibrium, in the sense defined here, was found to be an ideal state of affairs, difficult to attain. Full capacity utilization, on the other hand, which is reached when

$$\text{(VII:1)} \qquad Y_t = \bar{Y}_t$$

is satisfied, is a less restrictive equilibrium concept.

Provided the capital stock continues to be in full use from one period to the next, in the sense of equation (VII:1), supply and demand in the markets for goods are in equilibrium, and the expectations underlying the entrepreneurs' investment decisions continue to be realized. By contrast, an equilibrium in the labour market will only be attained, in principle, provided

$$(\text{VII:2}) \qquad Y_t = \bar{\bar{Y}}_t.$$

A more complete treatment of the labour market would of course imply that allowance should be made for frictional unemployment, amounting, say, to some 1 or 2 per cent. However, the line of approach followed here does not necessitate this.

10. THE PRECONDITIONS FOR FULL CAPITAL STOCK UTILIZATION

Let us now proceed to consider under what conditions the state of full capital stock utilization can persist over time. The analysis will be confined to cases where the parameters of the model satisfy the full equilibrium and stability conditions derived in the foregoing. Thus, the task is to analyze the time path of total output, postulating that the capital stock continues to be fully utilized $(Y_t = \bar{Y}_t)$ and assuming that total output, when it is below its equilibrium path $(\bar{\bar{Y}}_t)$, will approach this equilibrium path in the limit.

If $\gamma = 0$, (XIII) can be rewritten

$$(\text{XIII:2}) \qquad Y_t = \frac{\lambda - \left(\beta + \mu\right) \dfrac{\alpha}{\varphi_k}}{1 - \dfrac{\alpha}{\varphi_k}} Y_{t-1} - \frac{(\beta + \mu)(1 - \beta - \mu)}{1 - \dfrac{\alpha}{\varphi_k}} K_{t-1} + \left(1 - \frac{\beta + \mu}{\tau \nu}\right) \frac{\dfrac{\mu}{\varphi_k}}{1 - \dfrac{\alpha}{\varphi_k}} \bar{\bar{Y}}_t\,.$$

But the equilibrium concept now under consideration implies that $K_{t-1} = \dfrac{Y_{t-1}}{\varphi_k}$. Substituting this into (XIII:2),

$$(38{:}1)\qquad \Upsilon_t = \frac{\lambda - \left(\beta+\mu\right)\dfrac{\alpha}{\varphi_k}}{1-\dfrac{\alpha}{\varphi_k}}\,\Upsilon_{t-1} - \frac{(\beta+\mu)\,(1-\beta-\mu)}{\left(1-\dfrac{\alpha}{\varphi_k}\right)\varphi_k}\,\Upsilon_{t-1} +$$

$$\left(1-\frac{\beta+\mu}{\tau\nu}\right)\frac{\dfrac{\mu}{\varphi_k}}{1-\dfrac{\alpha}{\varphi_k}}\,\bar{\bar{\Upsilon}}_t\,.$$

Solving this equation for Υ_t,

$$(38{:}2)\qquad \Upsilon_t = \left[\frac{1}{\varphi_k-\alpha}\left(\lambda\varphi_k-(\beta+\mu)\,(1+\alpha-\beta-\mu)\right)\right]^t\Upsilon_o +$$

$$\frac{1}{\varphi_k-\alpha}\left(\frac{(\tau\nu-\beta-\mu)\,\mu}{\tau\nu}\,\bar{\bar{\Upsilon}}_t\right)$$

$$\left[\frac{1-\left[\dfrac{1}{\varphi_k-\alpha}\left(\lambda\varphi_k-(\beta+\mu)\,(1+\alpha-\beta-\mu)\right)\right]^t}{1-\dfrac{1}{\varphi_k-\alpha}\left(\lambda\,\varphi_k-(\beta+\mu)\,(1+\alpha-\beta-\mu)\right)}\right]$$

Next, in equation (XIV), $\dfrac{\bar{\Upsilon}_{t-1}}{\varphi_k}$ is substituted for K_{t-1}, and $\bar{\Upsilon}_t$ is written for Υ_t and $\dfrac{\bar{\bar{\Upsilon}}_t}{\tau\nu}$ for $\bar{\bar{\Upsilon}}_{t-1}$. When both sides of the equation thus obtained are multiplied by φ_k, the equation to be solved reads

$$(39{:}1)\qquad \bar{\Upsilon}_t = \alpha\,\bar{\Upsilon}_{t-1} + (1-\beta-\mu)\,\bar{\Upsilon}_{t-1} + \frac{\mu}{\tau\nu}\,\bar{\bar{\Upsilon}}_t\,.$$

Solving this equation for $\bar{\Upsilon}_t$,

$$(39{:}2)\qquad \bar{\Upsilon}_t = (1+\alpha-\beta-\mu)^t\,\bar{\Upsilon}_o + \frac{\mu}{\tau\nu}\,\bar{\bar{\Upsilon}}_t\left(\frac{1-(1+\alpha-\beta-\mu)^t}{\beta+\mu-\alpha}\right).$$

By the condition given in (VII:1) the right-hand sides of equations (38:2) and (39:2) must be equal. In consequence

$$(40{:}1)\qquad \frac{1}{\varphi_k-\alpha}\left(\lambda\varphi_k-(\beta+\mu)\,(1+\alpha-\beta-\mu)\right) = 1+\alpha-\beta-\mu$$

and

$$(40:2) \qquad \frac{1}{\varphi_k - \alpha} \, \frac{(\tau\nu - \beta - \mu)\,\mu}{\tau\nu} \, \overline{\overline{Y}}_t = \frac{\mu}{\tau\nu} \, \overline{\overline{Y}}_t\,,$$

for the coefficients of Y_o and $\overline{Y}_o$ must be equal, and the coefficients of $\overline{\overline{Y}}_t$ must be the same in both equations.

From (40:2) it follows that

$$(41) \qquad \varphi_k = 2\tau\nu - (1 + \mu).$$

According to this equation the value of the capital coefficient is a function of the rate of technological progress (τ), the growth rate of the population (ν) and the weights of the factors affecting the entrepreneurs' investment decisions (μ).

Finally, from (40:1) and (41), in combination with the equilibrium conditions given in (32) and (33),

$$(42) \qquad 2\,\tau\nu - (1 + \mu) = 2\,\tau\nu - (1 + \mu).$$

This signifies that the condition imposed by equation (41) upon the parameters is sufficient for the attainment of full capital stock utilization.

Thus, in order for total output to approach the full-employment ceiling $\overline{\overline{Y}}_t$ in the limit, at the same time as the capital stock continues to be fully utilized, the parameters of the model have to fulfil the following conditions:

$$(32) \qquad \alpha - \beta = \tau\nu - 1,$$

$$(33) \qquad \lambda = \left(1 - \frac{\tau\nu - 1}{\varphi_k}\right) \tau\nu,$$

$$(41) \qquad \varphi_k = 2\,\tau\nu - (1 + \mu),$$

$$(35:2) \qquad -1 < \frac{\lambda\,(1 - \beta - \mu)}{1 - \dfrac{\alpha}{\varphi_k}} < 1$$

and

$$(36:2) \qquad \beta + \mu < \frac{2\left(1 - \dfrac{\alpha}{\varphi_k} + \lambda\right)}{1 + \lambda}.$$

11. FULL CAPITAL STOCK UTILIZATION: THE SHAPE OF THE TIME PATH

The time path of the total output ensuring full capital stock utilization is non-oscillatory provided the parameters of the model satisfy the following conditions:

$$(43{:}1) \qquad a + d > 0,$$

stating that the sum of the roots of the solution must be positive;

$$(43{:}2) \qquad ad - bc > 0,$$

requiring that the product of the roots of the solution must be positive; and

$$(43{:}3) \qquad \frac{(a - d)^2}{4} + bc > 0\,,$$

ensuring that there are two distinct roots, neither of which is complex.

Let us first consider the values of the parameters in the light of the condition given in (43:1).

In terms of the original parameters of the model, this condition reads

$$(43{:}1) \quad a + d = \frac{\lambda - \left(\beta + \mu\right)\dfrac{\alpha}{\varphi_k}}{1 - \dfrac{\alpha}{\varphi_k}} + (1 - \beta - \mu) = 1 - \frac{\lambda - \beta - \mu}{1 - \dfrac{\alpha}{\varphi_k}} > 0$$

The conditions concerning the parameters that were derived previously do not allow the conclusion that $(a + d)$ is invariably positive. Therefore, those of the combinations of the parameter values for the stable processes in Table 4 which permit conclusions concerning the sign of $(a + d)$ are collected in Table 5.

The sum is negative in none of the cases. Thus it would appear that, at the economically meaningful values of the parameters, the condition given in (43:1) is fulfilled.

In terms of the original parameters, equation (43:2) can be rewritten as

$$(43{:}2) \qquad ad - bc = \left(\frac{\lambda - \left(\beta + \mu\right) \dfrac{\alpha}{\varphi_k}}{1 - \dfrac{\alpha}{\varphi_k}} \right) (1 - \beta - \mu) -$$

$$- \frac{(\beta + \mu)\ (1 - \beta - \mu)\ \alpha}{\left(1 - \dfrac{\alpha}{\varphi_k} \right) \varphi_k} > 0 \ .$$

What was said of the condition given in (43:1) also applies to this condition: it seems to be satisfied by all economically meaningful values of the parameters.

Table 5.

Row of Table 4	$\lambda - (\beta + \mu) \dfrac{\alpha}{\varphi_k}$	$1 - \beta - \mu$
2	0.96057	0.15167
3	0.96057	0.15167
4	0.95326	0.15917
5	0.95153	0.11750
6	0.94290	0.63985
7	0.91634	0.60970
8	0.89142	0.63985
9	0.84715	0.6097
10	0.86337	0.47235
11	0.80226	0.4422
12	0.82709	0.63985
13	0.76068	0.6097
14	0.78501	0.47235
15	0.69334	0.4422
16	0.92416	0.70970
17	0.88692	0.66440
18	0.81544	0.55870
20	0.81544	0.55870
21	0.73056	0.51340
22	0.76501	0.70970
24	0.90801	0.55870
30	0.76943	0.68932
38	0.52380	0.49995

The third condition becomes

$$(43:3) \qquad \frac{(a-d)^2}{4} + bc =$$

$$\frac{\left(\dfrac{\lambda - \left(\beta+\mu\right)\dfrac{a}{\varphi_k}}{1-\dfrac{a}{\varphi_k}} - (1-\beta-\mu)\right)^2}{4} - \frac{(\beta+\mu)\,(1-\beta-\mu)\,a}{\left(1-\dfrac{a}{\varphi_k}\right)\varphi_k} > 0\,.$$

The values of $\dfrac{(a-d)^2}{4} + bc$ corresponding to the stable processes represented in Table 4 are given in Table 6.

Table 6.

Row of Table 4	$\frac{(a-d)^2}{4} + bc$	Row of Table 4	$\frac{(a-d)^2}{4} + bc$
2	0.176266	13	— 0.05030
3	0.17481	14	— 0.00066
4	0.16871	15	— 0.03721
5	0.18707	16	— 0.00076
6	0.01595	17	— 0.00726
7	0.01253	18	— 0.02290
8	0.00006	20	— 0.01244
9	— 0.01216	21	— 0.04238
10	0.02958	24	— 0.06580
11	0.01655	30	— 0.05515
12	— 0.02332	38	— 0.20182

A majority of the stable processes failed to satisfy the third condition. This is not surprising in itself: what was concerned were parameter values satisfying the conditions of stable full equilibrium. Thus it is necessary to examine, separately in each particular case, whether the third condition is fulfilled by the parameters ensuring continuous full capital stock utilization.

Even where the time path of a process is non-oscillatory, the total output variable may approach the full-employment ceiling from above. This is not, however, compatible with the definition of a full-employment ceiling. If the sign of the second derivative of the solution changes in the course of the process, it is possible, in principle, to have such a time path incompatible with this definition. Differentiating (39:2) twice yields

$$(44) \quad \Upsilon''_t = (1 + \alpha - \beta - \mu)^t \, (\log\, (1 + \alpha - \beta - \mu))^2 \, \Upsilon_o + \frac{\mu}{\tau\nu\, (\mu - \alpha + \beta)} \Big[(\log\, (\tau\nu))^2 - (1 + \alpha - \beta - \mu)^t \, (\log\, (\tau\nu) + \log\, (1 + \alpha - \beta - \mu))^2 \Big] \, \bar{\bar{\Upsilon}}_t \, .$$

Let us substitute $t = 1$ into this expression. If it turns out that Υ''_1 can possibly be negative, the process will involve a point of inflexion; for the second derivative of the equilibrium path is $\bar{\bar{\Upsilon}}''_t > 0$.

The right-hand side of (44) is negative if

$$(44{:}1) \quad (1 + \alpha - \beta - \mu) \Big[\Big(\frac{\mu}{\mu - \alpha + \beta} \Big) (\log\, (\tau\nu) + \log\, (1 + \alpha - \beta - \mu)^2) \, \bar{\bar{\Upsilon}}_o - (\log\, (1 + \alpha - \beta - \mu))^2 \, \Upsilon_o \Big] > \frac{\mu}{\mu - \alpha + \beta} \, (\log\, \tau\nu)^2 \, \bar{\bar{\Upsilon}}_o \, .$$

If $\mu - \alpha + \beta > 0$, the inequality can hold provided either $(\bar{\bar{\Upsilon}} - \Upsilon_o)$ or $(\bar{\alpha} - \bar{\beta})$ is sufficiently large.[17]

12. FULL CAPITAL STOCK UTILIZATION: A NUMERICAL ANALYSIS

To clarify the nature of the processes ensuring full capital stock utilization, the movements of total output and capital stock were explored by employing sets of parameter values satisfying the conditions given by (32), (33), (41), (35:2), (36:2), (43:1), (43:2) and (43:3). The numerical trials were carried out on an electronic computer.[18]

The above conditions confine the values of the parameters to rather narrow limits. In the trials the values of the parameters were the following:[19]

17. In the latter case $(\alpha - \beta)$ increases at the cost of μ.

18. The IBM 1440 program was made by Mr RAIMO HEISKANEN employing the Fortran II language.

19. Another series of trials was made using the values $\tau = 1.03$ and $\nu = 1.01$, but the results were qualitatively similar. In this series the parameters had the following values:

τ	ν	$\bar{\alpha} - \bar{\beta}$	$\bar{\alpha}$	μ	φ_k	λ
1.03	1.01	0.05	0.125	0.194	0.8866	0.99301
1.03	1.01	0.06	0.15	0.32833	0.6866	0.97924
1.03	1.01	0.07	0.175	0.42429	0.4866	0.95414

	τ	ν	$\bar{a}-\bar{\beta}$	a	μ	φ_k	λ
A_1	1.02	1.01	0.04	0.1	0.245	0.8154	0.9920
A_2	,,	,,	,,	0.16	,,	,,	,,
B_1	,,	,,	0.05	0.125	0.396	0.6644	0.9834
B_2	,,	,,	,,	0.2	,,	,,	,,
C_1	,,	,,	0.06	0.15	0.4967	0.5637	0.9750
C_2	,,	,,	,,	0.24	,,	,,	,,

The point of departure was the »plausible» assumption that the rise in the full-employment ceiling is of the order of 3 per cent a year. Thus it was only possible to given alternative values to the entrepreneurs' growth expectations $(\bar{a} - \bar{\beta})$ and the sensitivity-to-demand coefficient $\bar{a}$; two values of the latter were used: in one case the expected growth rate was assumed to be zero when the degree of capital stock utilization was 60 per cent (these trials are denoted by the subscript 1), and, in the other case, when it was 75 per cent (subscript 2). The following questions were considered:

1. How is the course of the process affected by the initial value of total output (relative to the initial value of the full-employment ceiling (= 100))?
2. How are the changes in the expectations regarding demand reflected in the process?
3. What is the bearing of the changes on the sensitivity coefficient?

Table 7 A.

Time period	Y_t A_1	A_2	B_1	B_2	C_1	C_2	$\bar{\bar{Y}}_t = 1.0302^t \bar{\bar{Y}}_o$
1	98.86	98.83	99.85	99.85	100.35	100.35	103.02
2	102.64	102.58	104.12	104.12	104.70	104.70	106.13
3	106.37	106.30	108.06	108.06	108.57	108.57	109.33
4	110.09	110.00	111.83	111.83	112.22	112.22	112.63
5	113.82	113.73	115.53	115.53	115.81	115.81	116.04
6	117.58	117.49	119.23	119.22	119.42	119.42	119.54
7	121.40	121.31	122.96	122.95	123.08	123.08	123.15
8	125.28	125.20	126.75	126.75	126.82	126.83	126.87
9	129.24	129.17	130.64	130.63	130.67	130.67	130.70
10	133.30	133.23	134.62	134.61	134.62	134.63	134.65
11	137.45	137.39	138.70	138.69	138.69	138.70	138.71
12	141.70	141.66	142.91	142.89	142.88	142.89	142.90
13	146.08	146.04	147.23	147.22	147.20	147.20	147.22
14	150.57	150.54	151.69	151.67	151.64	151.65	151.67
15	155.19	155.17	156.27	156.26	156.22	156.23	156.25
16	159.94	159.93	161.00	160.98	160.94	160.95	160.96
17	164.82	164.84	165.86	165.84	165.80	165.81	165.83
18	169.85	169.88	170.87	170.85	170.81	170.81	170.83
19	175.03	175.07	176.03	176.01	175.97	175.97	175.99
20	180.36	180.42	181.35	181.33	181.28	181.29	181.31

The values of total output (Y) and the full-employment ceiling $(\bar{\bar{Y}}_t)$ for twenty successive periods, corresponding to an initial value of 95 for total output, are shown in Table 7 A. (Thus, the initial value of total output was assumed to be 5 per cent below the full-employment ceiling.) The values of Y_t and $\bar{\bar{Y}}_t$ corresponding to an initial value of 80 for the total output are set out in Table 7 B. The values of the capital stock are not indicated in either table, as the capital coefficient remains constant in each trial. It appears from Table 7 A, first of all, that an increase in the entrepreneurs' sensitivity to demand $(\bar{\alpha})$ affects the movement of total production only to a very minor extent. For example, during the 10th period in Trial A_1, $Y = 133.30$, while in Trial A_3 the corresponding value is $Y = 133.23$. In the other trials the changes in $\bar{\alpha}$ had virtually no effect on the course of total output.

Table 7 B.

Time period	Y A_1	Y B_1	Y C_1	$\bar{\bar{Y}}_t = 1.0302^t \bar{\bar{Y}}_0$
1	87.31	90.33	92.34	103.02
2	93.79	98.09	100.43	106.13
3	99.65	104.24	106.29	109.33
4	105.03	109.41	111.01	112.63
5	110.06	113.99	115.16	116.04
6	114.85	118.25	119.07	119.54
7	119.46	122.34	122.89	123.15
8	123.97	126.36	126.73	126.87
9	128.42	130.39	130.62	130.70
10	132.85	134.46	134.60	134.65
11	137.30	138.60	138.68	138.71
12	141.79	142.84	142.88	142.90
13	146.34	147.19	147.20	147.22
14	150.97	151.66	151.64	151.67
15	155.69	156.26	156.22	156.25
16	160.52	160.98	160.94	160.96
17	165.47	165.85	165.80	165.83
18	170.55	170.87	170.81	170.83
19	175.76	176.03	175.97	175.99
20	181.12	181.30	181.28	181.31

A rise in the expected rate of growth of demand (i.e., a shift from Trial *A* to Trial *B*, and, further, to Trial *C*) affects the time path of total output very clearly during the first periods. Thus it seems obvious that a rise in the

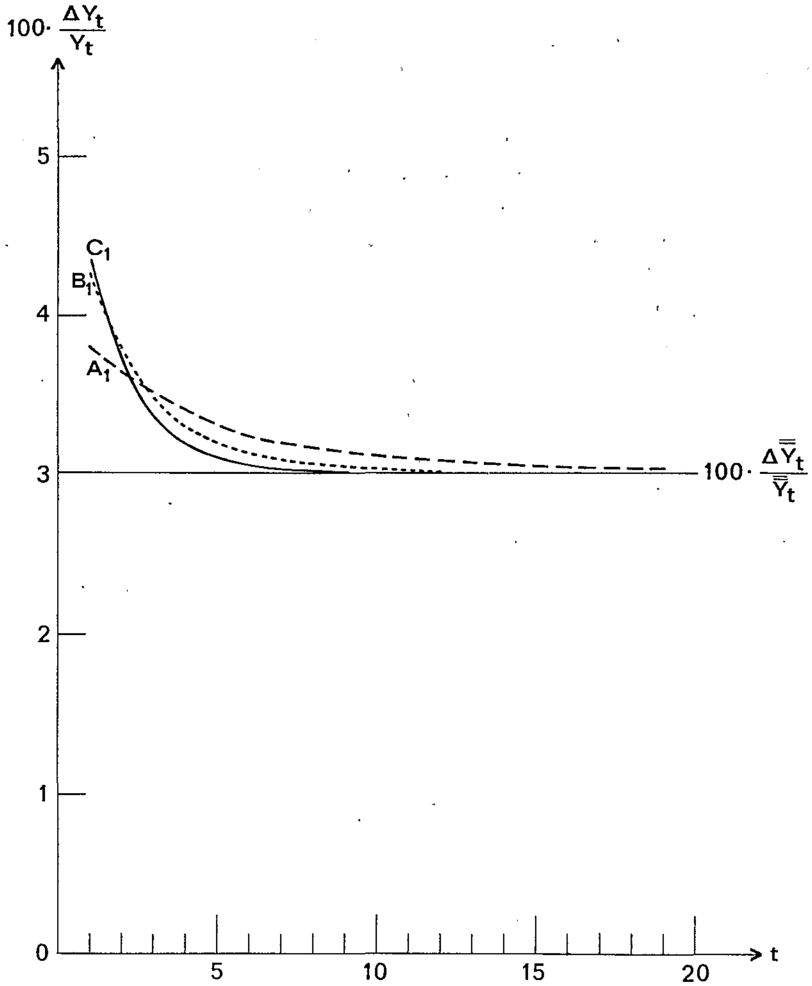


Figure V.

expected rate of growth of demand at first increases the growth rate of total output, so that this approaches the full-employment ceiling. Later on, however, the influence of the rise in the expected growth rate of demand diminishes rather sharply. The three curves in Figure V represent the *rates of change* of total output corresponding to three trials (A_1, B_1 and C_1) illustrated in Table 7 A. The curves elucidate the effect of expectations regarding growth upon the movement of total output. It should be pointed out, however, that the differences in total output due to differences in growth expectations

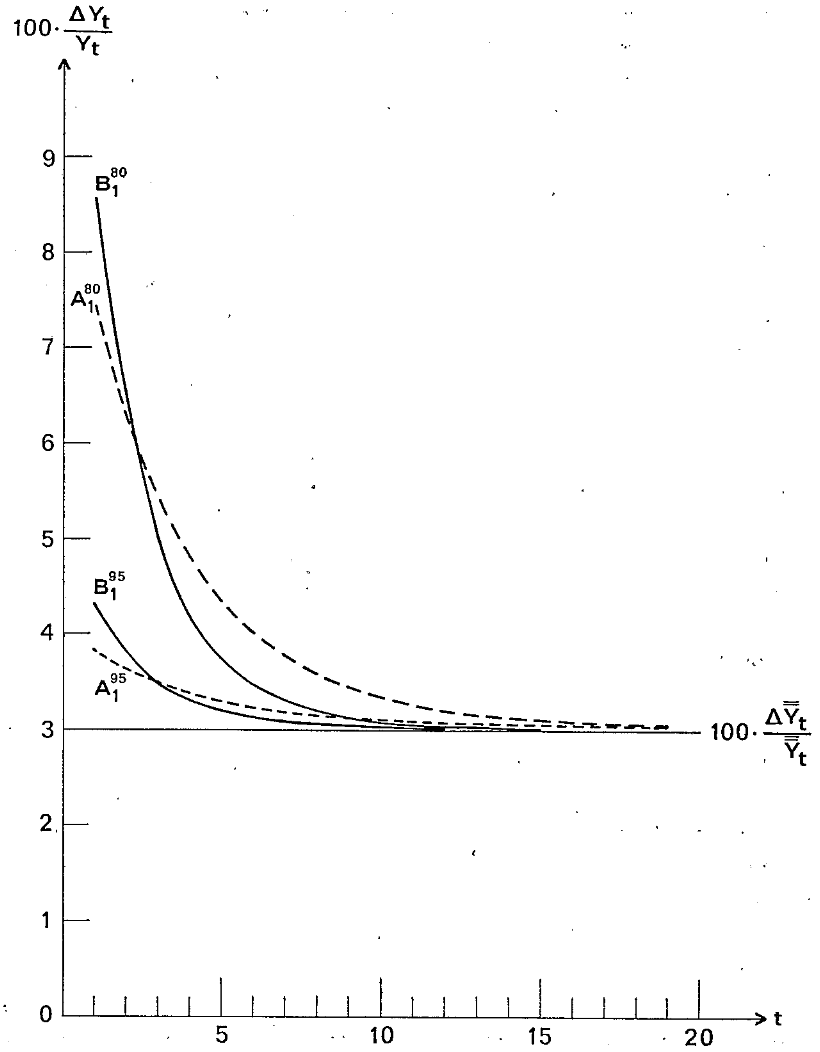


Figure VI.

are in the 20th period — which is the last one included in the tables — already negligible.

One is led to the following conclusion: According to the present model, the higher the expected rate of growth of demand, the more rapidly will total output approach the full-employment ceiling. This presupposes, of course, that the expected growth in demand exceeds the rate at which the full-employment ceiling is rising.

To discover the significance of the initial value of total output, the results of the trials corresponding to one another in Tables 7 A and 7 B have to be compared.

In the trials recorded in Table 7 B the initial value of total output $(Y_o = 80)$ was lower than in the trials recorded in Table 7 A $(Y_o = 95)$. It is seen that the lower the initial value — and the larger the excess capacity — the more rapid is the rate of increase of total output during the early phases of the process. In point of fact, if the total output starts from a low level, it will gradually exceed the level reached in a process that starts from a lower initial value. This is exemplified by the values of Y_t yielded by the two Trials A_1 for the 15th period.

To illustrate the matter, the rates of change of total output in Trial A_1 and Trial B_1 corresponding to the initial values $Y_o = 95$ and $Y_o = 80$ are represented in Figure VI. The influence of the initial values is clearly reflected by the curves. The curves in Figure V would appear to suggest that an increase in the growth rate due to the initial value is likely to decelerate more slowly than an increase due to a change in the expectations regarding the growth of demand. The same is also suggested by the figures describing the movement of total output.

Thus it is legitimate to conclude that the larger the deviation of total output from the full-employment ceiling, the more powerful is the impetus this deviation gives to the growth of total output. The background factor at work here is the opportunity for growth provided by the underemployment of the productive resources that the entrepreneurs take into account, according to the model, in their investment decisions. The smaller the deviation of total output from the full-employment ceiling, the smaller is the investment induced by this factor.

As is obvious from Figures V and VI, in each trial the course of the process during the first periods suggested the existence of a point of inflexion. This possibility was already considered in the preceding section. When the trial was continued until a point of inflexion was attained, it was necessary in some cases to include as many as 200 periods in the process. In all of the cases explored the point of inflexion was met when total output was very close to the full-employment ceiling.[20] After the point of inflexion was passed, for a time the growth rate of total output remained smaller than the rate

20. The small inaccuracies due to the rounding had also to be taken into account in the anlysis.

at which the full-employment ceiling was rising; before long, however, the growth of total output began to accelerate again and total output began to approach the full-employment ceiling. To provide an example, in Trial A_1 (where $Y_0 = 80$), the increase in total output was 3.0198 per cent at the point of inflexion, attained in the 39th period, or 0.0002 percentage points below the rate of increase in the full-employment ceiling.

13. THE RESULTS OF THE ANALYSIS

The results of the foregoing analysis can be generalized as follows:

If the parameters of the model satisfy the conditions derived, the total output ensuring full capital stock utilization $(\bar{Y}_t)$ approaches the full-employment ceiling $(\bar{\bar{Y}}_t)$ along a time path whose second derivative is negative. After a point of inflexion, the gap between the full-employment ceiling and total output may widen for a few periods. Soon, however, total output begins

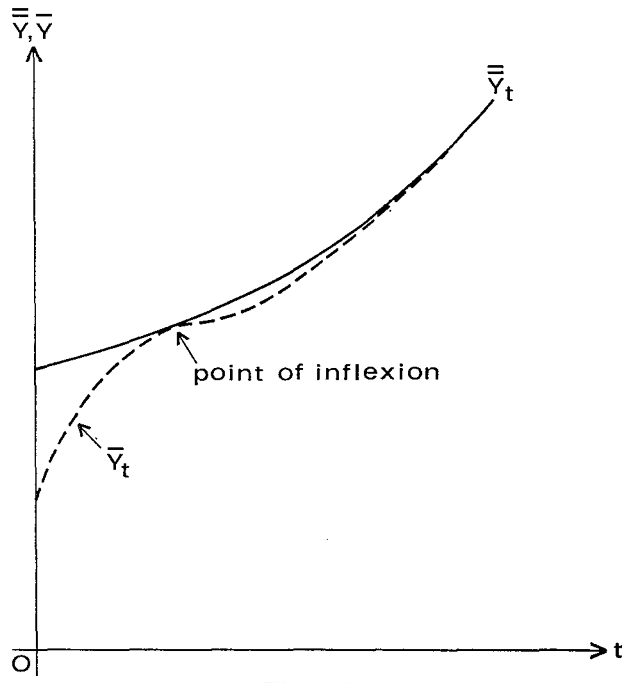


Figure VII.

to approach the full-employment ceiling, i.e. full equilibrium, again. These conclusions are illustrated in Figure VII.

Through the model constructed in the present study it was possible to demonstrate that the Harrod—Domar problem can be solved by means of a flexible investment function, although it also turned out that, in order for a solution to exist, the parameters had to satisfy a number of conditions restricting their range of variation considerably.

III. ON CYCLICAL FLUCTUATIONS

In the preceding chapter the nature of 'equilibrium' was analyzed in detail from the point of view of the present macro-dynamic model. In the present chapter the results of the analysis will be used in an attempt to approach the trade cycle phenomenon.

1. A THEORY OF THE TRADE CYCLE

Taking full equilibrium as a theoretical norm and the analysis carried out so far as a point of departure, the model will be utilized now to discover factors that may give rise to cyclical fluctuations in economic activity.[1]

The theoretical norm chosen for the analysis implies that the occurrence of cyclical fluctuations must be regarded, in a sense, as a *departure from equilibrium* in the economy. It follows from this norm, moreover, that cyclical fluctuations in total output take place *below the full equilibrium*.[2] The equilibrium is assumed to be stable.

This assumption is based on considerations exogenous to the model. In analyzing cyclical fluctuations in a modern mixed-type economy, the fact cannot be disregarded that stabilization targets are a distinctive characteristic of government activity — not even where the government sector itself is left outside the analysis. It is indeed obvious that the firms' and households' awareness of the goals of the economic policy pursued is apt to mold their behaviour in such a way that economic activity will tend toward equilib-

1. See above p. 14.

2. Until very recently it has been customary in the theory of the trade cycle to consider the cyclical fluctuation as taking place above or around a stationary or evolutionary equilibrium. See, e.g., J. A. SCHUMPETER Business Cycles, Vol. I, New York 1939, Chapter IV; J. R. HICKS A Contribution to the Theory of the Trade Cycle, Oxford 1950, Chapters V and VI; PAUL A. SAMUELSON »Interaction between Multiplier Analysis and the Principle of Acceleration», *The Review of Economic Studies* XXI (2). It should be mentioned that in the late 1930s models were presented by certain authors where cyclical fluctuations were due to shifts in the short-run equilibrium: M. KALECKI Essays in the Theory of Economic Fluctuations, London 1939, Chapter 6; NICHOLAS KALDOR »A Model of the Trade Cycle», *Economic Journal*, March 1940. JAMES S. DUESENBERRY's business cycle theory from 1958 is one of those representing the view adopted in the present study; see especially op. cit., Chapter 9.

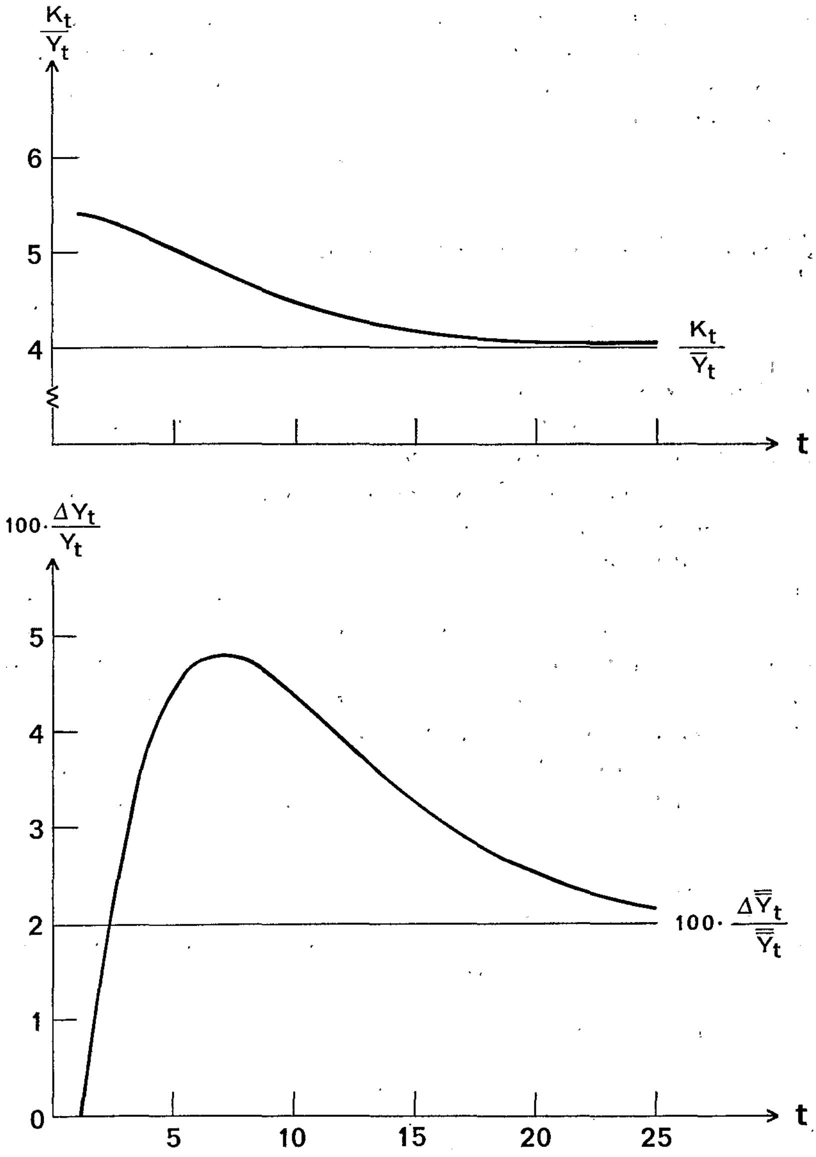

$\frac{K_t}{Y_t}$
6
5
4
$\frac{K_t}{\bar{Y}_t}$
t
$100 \cdot \frac{\Delta Y_t}{Y_t}$
5
4
3
2
1
0
$100 \cdot \frac{\Delta \bar{\bar{Y}}_t}{\bar{\bar{Y}}_t}$
5
10
15
20
25
t

Figure VIII.

rium. It is this assumption of a certain sort of economic policy at work in the background that justifies the analytical standpoint taken here.

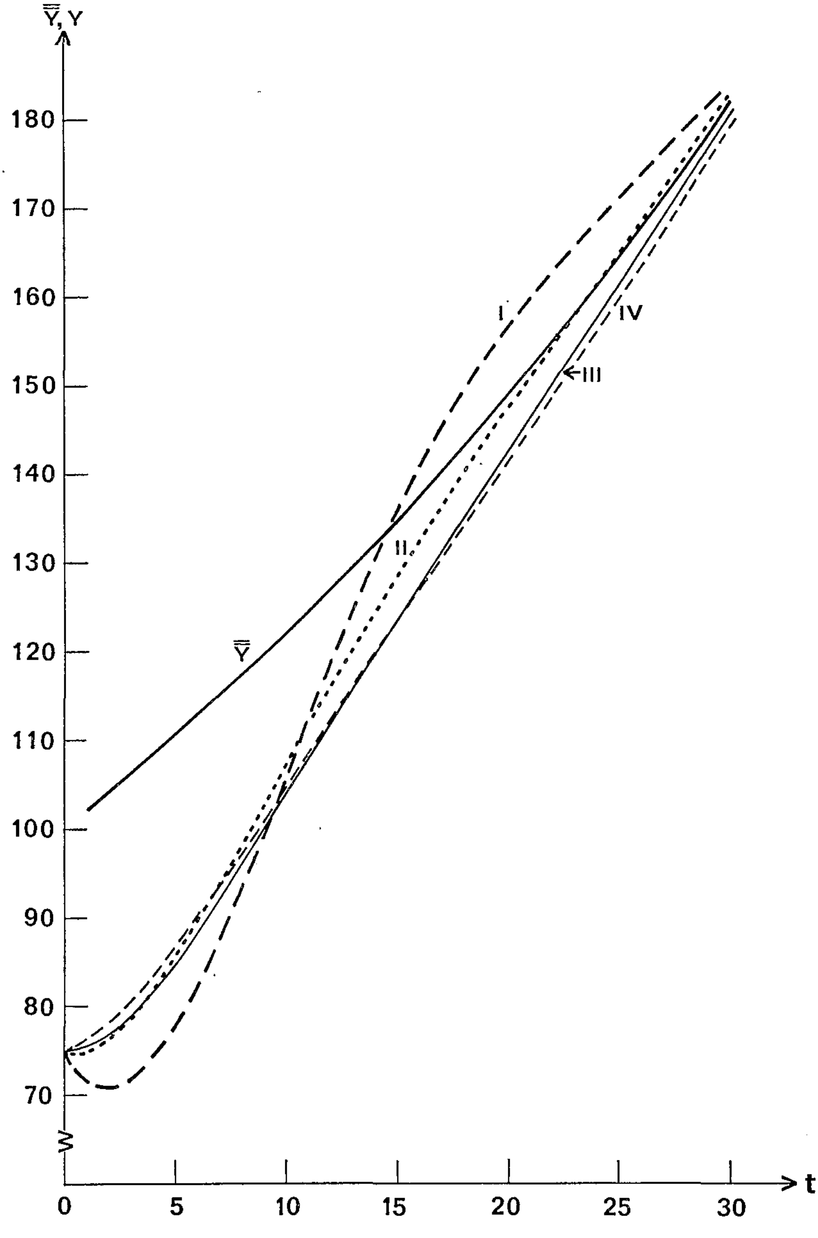
Examination of the shapes of the time-paths of certain stable processes tending toward full equilibrium in Tables 5 and 6[3] revealed that some of them were oscillating. Yet the oscillating time paths are problematic, in that total output, when it exceeds the full-employment ceiling, no longer satisfies the boundary conditions of the model. Thus, the oscillating time paths are not fit in themselves for a description of the trade cycle. The factors making for cyclical fluctuations must consist of other dynamic properties of the model.

To identify these properties, the stable processes presented in Table 4[4] were investigated by means of an electronic computer. It was supposed that total output had, for exogeneous reasons, fallen to a level where the entrepreneurs, as a result of increased uncertainty, no longer expected demand to grow $(g = 0)$.[5] The capital stock was assumed to be optimal with respect to the full-employment ceiling $(\varphi_k K_o = \bar{Y}_o)$. These assumptions imply that the fall of total output from the equilibrium level had been quite rapid.

The results of the trials showed that the oscillating processes attained and exceeded the full-employment ceiling rather soon, while the non-oscillating ones approached from below the full-employment ceiling and, thus, the equilibrium more slowly. Another feature of the processes was this: excess capacity disappeared simultaneously as (or one period before) total output reached the full-employment ceiling. In other words, the capital-stock ceiling does not in practice hinder total output from increasing until it hits the full-employment ceiling. The graph of the (oscillating) rate of growth of total output and that of the capital-output ratio in one of the trials are represented in Figure VIII. When total output attains the full-employment ceiling during the 25th period, the relative rate of increase in total output exceeds the relative rate of rise of the full-employment ceiling (2.01), whereas the capital-output ratio has just declined to a level presupposed by full capital stock utilization (4).

As regards the beginning of the processes, it should be noted that, owing to the investment potential inherent in the growth opportunities, total output will start to grow, despite the non-existent growth-in-demand expectations.[6]

3. See pp. 55 and 56.
4. See p. 49.
5. See equation (9:2) on p. 20.
6. In R. A. Gordon's trade cycle theory the investment potential has a similar stimulating effect on the cyclical upswing. See his article on »Investment Behavior and Business Cycles», *The Review of Economics and Statistics*, February 1955.



Y̿, Y
180
170
160
150
140
130
120
110
100
90
80
70
0
5
10
15
20
25
30
t
Y̿
I
II
III
IV

Figure IX.

The processes whose parameters were the following were investigated in detail.

	w	φ_k	λ	$\bar{g}$	μ	$\bar{a}$ 0.75	$\bar{a}$ 0.6	$\bar{a}$ 0.5
I	1.0201	0.25	0.93808	0.03	0.33	0.12	—	—
II	1.0201	0.25	0.93808	0.03	0.33	—	0.075	—
III	1.0201	0.3	0.95175	0.03	0.33	—	0.075	—
IV	1.0201	0.3	0.95175	0.03	0.33	—	—	0.06

Two of the processes, I and II, are oscillating, while the other two, III and IV, are non-oscillating. The processes themselves are represented in Figure IX and their rates of change in Figure X.

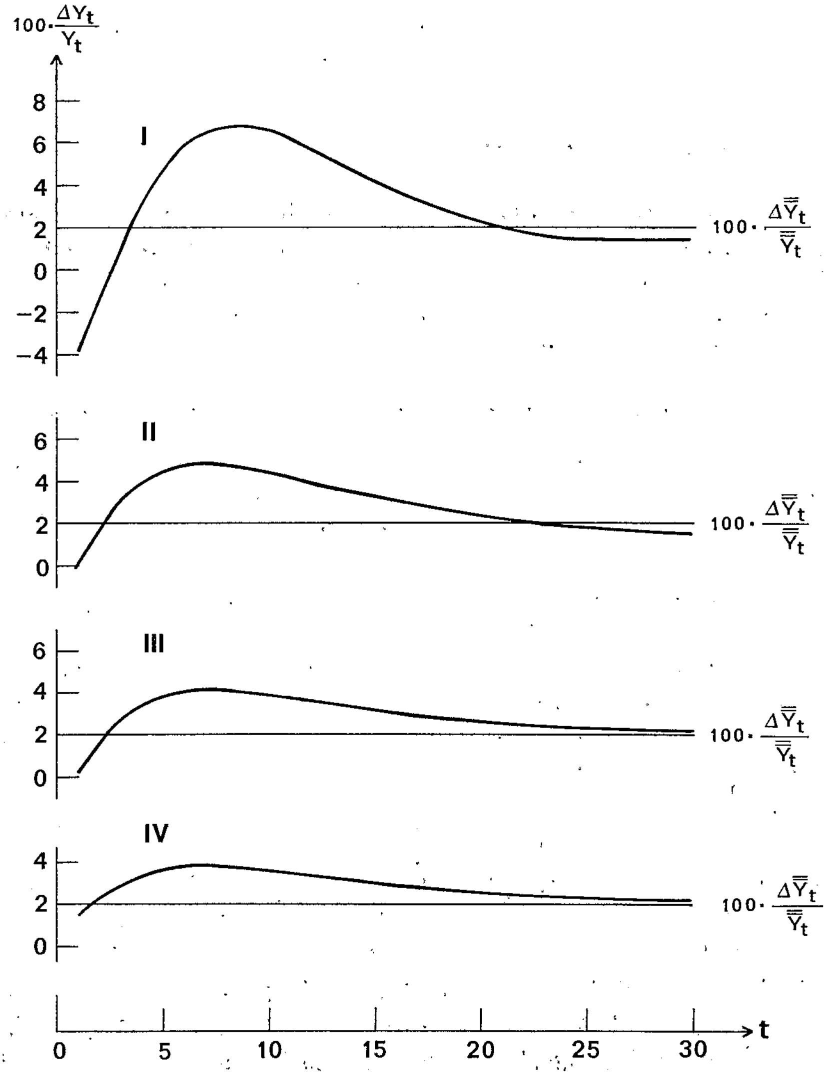
Process I differs from process II in the value of $\bar{a}$. When the sensitivity of expectations to cyclical changes, as measured in terms of $\bar{a}$, declines from 0.75 to 0.6, the process assumes a decidedly more damped shape; and, in consequence, total output hits the full-employment ceiling later. Process III, in turn, differs from process II in regard to the capital-output ratio $\left(\frac{1}{\varphi_k}\right)$. Reduction of this parameter from 4 to 3.3 has changed the process from an oscillating into a non-oscillating one, and has delayed the attainment of the full-employment ceiling. And process IV is more sensitive than process III — $\bar{a}$ has declined from 0.6 to 0.5 — and, hence, the time path is more damped. It should be pointed out that, during the early phase, process III rather than process IV is more damped.

To be able to construct an explanation of the trade cycle by employing these elements, one has to interpret the basic model flexibly. Judging from the processes just described, this implies that the technological capital-output ratio $\left(\frac{1}{\varphi_k}\right)$ and the psychological sensitivity coefficient (a) must be permitted to vary to a certain degree.

The basic underlying idea is as follows: owing to the structure of the economy, total output approaches full equilibrium steadily, but the variability of the coefficients causes fluctuations in economic activity.

The fluctuation in the capital-output ratio is supposed to be due to the fact that the growth rates of the capital-intensive sectors (where the capital-output ratio is high) and the labour-intensive sectors (where the capital-output ratio is low) may diverge at times; thus, of course, the capital-output ratio for the entire economy will vary.

The changes in the sensitivity coefficients are supposed to reflect the asymmetry of entrepreneurs' expectations when economic activity rises and declines: during an upswing the entrepreneurs' uncertainty regarding the



$100 \cdot \frac{\Delta Y_t}{Y_t}$
I
II
III
IV
8
6
4
2
0
−2
−4
$100 \cdot \frac{\Delta \overline{\overline{Y}}_t}{\overline{\overline{Y}}_t}$
0
5
10
15
20
25
30
t

Figure X.

future does not increase as rapidly as it decreases during a downswing. In Figure II,[7] the expectation function g_2 might be one typical of a downswing, and a function whose graph is paralled with g_1 and passes through the point at which g_2 intersects the vertical axis might be an expectation function for periods of upswing. This would again be replaced by the function g_1 once the full-employment ceiling had been reached.

The course of development toward full equilibrium could thus be described as follows.

There will be a revival of economic activity when, as a result of technological progress and the growth of the population, investment potential is present. Before an upswing starts, the capital stock must adjust itself during the downswing to these primary growth factors, or, from equation (12) and condition (12:1)[8]

$$\bar{\bar{Y}}_t \doteq \bar{Y}_t .$$

When the revival commences, total output is assumed to follow a non-oscillating time path (such as, say, the one of process III). A »disturbance» in this upwards movement may take place when the entrepreneurs' expectations have been reinforced to such an extent, and the coefficient $\bar{a}$ has perhaps risen, that firms particularly in the capital-intensive fields begin to undertake time-consuming investments. This implies that, in the investment function

$$\text{(VIII)} \qquad I_t = \frac{a}{\varphi_k} Y_t - (\beta + \mu) K_t + \frac{\mu}{\varphi_k} \bar{\bar{Y}}_t$$

the capital-output ratio $\left(\frac{1}{\varphi_k}\right)$ increases. An investment boom is thus started. The growth rate of total output rises (so as to correspond, for example, to process I).

Though the degree of capital stock utilization will now rise at a rapid rate, it may not be the case that the entrepreneurs' expectations will become increasingly favourable at precisely the same rate; for during the later phases of the upswing the uncertainty regarding the future may not diminish as rapidly as in the initial stages of the investment boom: the parameter a declines. This reduces the inducement to invest (the decline in $\bar{a}$ obviously affects the investment function more through a than through β, with the result that the growth of total output will decelerate; i.e., the time path assumes a shape comparable to that of process II).

7. See p. 21.
8. See p. 26.

At the stage when the greater part of the investments of the capital-intensive industries are maturing, the capital-output ratio in the investment function falls. It is justifiable to assume that the development has continued smoothly up to this point. Yet, the change that now takes place in the capital-output ratio is rather sharp — exactly how sharp it is will depend on the strength of the investment boom; and thus the course of development can hardly continue without at least a slight drop to a lower level of activity. How large the fall is depends on how much »excess» capacity has been created during the investment boom. Here, excess capacity is interpreted as measured in terms of the degree of capital stock utilization $\left(\frac{Y_t}{\bar{Y}_t}\right)$.

As was stated above, the preconditions for a revival exist when technological progress and population growth have resulted in the emergence of investment potential. But a revival can only take place after a time interval during which the growth rate of the capital stock has been lower than the rate of rise in the full-employment ceiling (τv). Then a new growth process begins, and its properties may be similar to those of the preceding one.

But if the rise starts from a level differing only slightly from the full-equilibrium level, the possibilities offered by the technological advances and population growth may not be sufficient to give rise to an investment boom.[9] Even if the growth of total output continued without »jumps», the assumption concerning expectations is such that a retardation of the process is possible (in the course of time, process III may turn into process IV).

When total output reaches the full-equilibrium level, either through approaching it indefinitely or through »hitting» it, full employment of all resources may continue for some time. But today's economic policy, characterized at the beginning of this section, also seeks to stabilize the value of money, which generally tends to be endangered under conditions of full resource utilization. The government can then be expected to undertake economic policy measures which may cause the volume of total output to decline somewhat from the full-employment level. If the decision-makers expect, on the basis of past experience, that an economic policy of this kind will be pursued, this may tend to depress their growth expectations $(\bar{g} = \bar{\alpha} - \bar{\beta})$. Thus, total output may also depart from the full-equilibrium level for reasons inherent in the economic system itself. Only a change in economic policy, which is likely to follow sooner or later, may be capable of raising expectations to their former level. It should

9. See p. 26.

be noted that this last step in the explanation of the trade cycle presupposed that one more parameter of the model — $\bar{g}$ — was permitted to change at times.

* * *

Flexible application of the basic macro-dynamic model led in the foregoing to an explanation of the trade cycle. This explanation rested on the view that factors — exogenous from the point of view of the model, it is true — are at work within the economy that at times cause total output to fall short of the value corresponding to the full-equilibrium level. Cyclical fluctuations occur when »upwards jumps» due to investments take place in the growth process tending toward equilibrium. During a cyclical downswing, however, total output is not likely to return to its former level at the beginning of the preceding upswing, as the investment potential generated by technological progress and population growth is liable to cause a new upswing before that level has been attained.

2. SOME ALTERNATIVE STARTING POINTS

The trade cycle theory set forth in the preceding section was based on the concept of full equilibrium. Another two theoretical norms on which trade cycle analysis might be based will be discussed briefly in the present section. No additional theories resting on these norms will, however, be put forward here.

Full capital-stock utilization might furnish a serviceable starting point for a trade cycle theory; in symbols,

$$\text{(VII:1)} \qquad Y_t = \bar{Y}_t .$$

The conditions of this type of »balanced growth» can be spelled out through following a line of approach similar to that used in the preceding chapter. Let us first consider the proportional case $(\gamma = 0)$.

The conditions of full equilibrium are not necessarily satisfied here. Let us substitute $\frac{Y_t}{\varphi_k}$ for K_t in equations (XIII) and (XIV) and write $\bar{\bar{Y}}_{t-1} = \frac{\bar{\bar{Y}}_t}{\tau\nu}$

The growth will be »balanced» if the solution equations (38:2) and (39:2) of these two equations are identical.[10] This requirement implies that

10. See p. 52.

(45:1) $\varphi_k = \tau\nu + \alpha - \beta - \mu$.

and

(45:2) $$\lambda = \frac{1 + \alpha - \beta - \mu}{\tau\nu + \alpha - \beta - \mu}\, \tau\nu :$$

These two conditions can also be expressed through a single equation:

(45:3) $$\lambda = \frac{1 + \alpha - \beta - \mu}{\varphi_k} \cdot \tau\nu .$$

Here there is one equilibrium condition less than in the case of full equilibrium (equations (32), (33) and (41)).

From equation (45:2) and assuming that the propensity to consume (λ) is less than unity we have the following inequality:

(46:1) $$\frac{1 + (\alpha - \beta - \mu)}{\tau\nu + (\alpha - \beta - \mu)} \cdot \tau\nu < 1 .$$

In consequence,

(46:2) $(\alpha - \beta - \mu)\, \tau\nu < \alpha - \beta - \mu ,$

and, thus,

(46:3) $\alpha - \beta - \mu < 0,$

if $\tau\nu > 1$.

Provided that the condition given in (45:3) is satisfied, the time path of »balanced» growth can be examined with the aid of the equation

(39:2) $$\bar{Y}_t = (1 + \alpha - \beta - \mu)^t\, Y_o + \frac{\mu}{\tau\nu}\, \bar{\bar{Y}}_t \left(\frac{1 - (1 + \alpha - \beta - \mu)^t}{\beta + \mu - \alpha} \right).$$

By (46:3) it can be concluded that, as t increases indefinitely,

(47:1) $$\bar{Y}_\infty = \frac{\mu}{\tau\nu\, (\beta + \mu - \alpha)}\, \bar{\bar{Y}}_\infty .$$

From this equation it is seen that

(46:2) $\bar{Y}_\infty \gtreqless \bar{\bar{Y}}_\infty$

depending on whether

$$\frac{\mu}{\tau\nu\, (\beta + \mu - \alpha)} \gtreqless 1$$

or

$$(46{:}3) \qquad \frac{\mu}{\beta + \mu - \alpha} \gtreqless \tau\nu\,.$$

An economy may be characterized as *strongly growth-oriented* if

$$\frac{\mu}{\beta + \mu - \alpha} > \tau\nu$$

and *weakly growth-oriented* if

$$\frac{\mu}{\beta + \mu - \alpha} < \tau\nu\,.$$

Regarding these conditions particular attention must be drawn to the fact that only the parameters of the investment function appear to be relevant for an economy's power of growth.

As $(1 - \mu)\,\bar{g} = \alpha - \beta$, the power-of-growth coefficient can be written

$$\frac{\mu}{\mu\,(1 - \mu)\,\bar{g}}$$

which indicates how a rise in the expectations regarding the increase in demand increases the growth power. When the coefficient is written as

$$\frac{\mu}{\mu\,(1 + \bar{g}) - \bar{g}}$$

it is seen that a rise in μ will diminish the growth power.[11]

The stability conditions being the same as previously, the conditions of »balanced» growth are

$$(44{:}3) \qquad \lambda = \frac{1 + \alpha - \beta - \mu}{\varphi_k} \cdot \tau\nu$$

$$(35{:}2) \qquad -1 < \frac{\lambda\,(1 - \beta - \mu)}{1 - \dfrac{\alpha}{\varphi_k}} < 1$$

and

$$(36{:}2) \qquad \beta + \mu < \frac{2\left(1 - \dfrac{\alpha}{\varphi_k} + \lambda\right)}{1 + \lambda}.$$

11. It should be emphasized that $\mu\,(1 + \bar{g}) - \bar{g} > 0$ by the condition given in (46:3).

Let us finally return to the basic model in its original form, dropping the assumption of a proportional consumption function. (That is, permitting the case where $\gamma \neq 0$). From equations (28) and (29)[12] we have for a definition of »balanced» growth

$$(48) \qquad G^t \frac{m\left(1-\frac{d}{G}\right)+n\frac{b}{G}}{\left(1-\frac{a}{G}\right)\left(1-\frac{d}{G}\right)-\frac{bc}{G^2}} + \frac{T(1-d)}{(1-a)(1-d)-bc} =$$

$$\varphi_k G^t \frac{m\frac{c}{G}+n\left(1-\frac{a}{G}\right)}{\left(1-\frac{a}{G}\right)\left(1-\frac{d}{G}\right)-\frac{bc}{G^2}} + \frac{T\varphi_k c}{(1-a)(1-d)-bc}.$$

The equilibrium condition is fulfilled if

$$(48:1) \qquad m\left(1-\frac{d}{G}\right)+n\frac{b}{G} = \varphi_k\left[m\frac{c}{G}+n\left(1-\frac{a}{G}\right)\right]$$

and

$$(48:2) \qquad 1-d = \varphi_k c.$$

Substituting the original parameters of the model into (48:2), the *first equilibrium condition* is obtained directly:

$$(48:3) \qquad \alpha = \beta + \mu.$$

Making use of the condition in (48:3), the *second equilibrium condition* is obtained from equation (48:1) in terms of the original parameters:

$$(48:4) \qquad \varphi_k = \frac{\tau\nu(\tau\nu-1)}{\tau\nu-\lambda}.$$

The stability conditions are the same as before, i.e., those given in (35:2) and (36:2).

From equations (30) and (48) it appears that, in the last analysis, the power of growth of the economy depends on whether

$$(48) \qquad \frac{m\left(1-\frac{d}{G}\right)+n\frac{b}{G}}{\left(1-\frac{a}{G}\right)\left(1-\frac{d}{G}\right)-\frac{bc}{G^2}} \gtreqless \Upsilon_o$$

12. See p. 40.

or, employing the original parameters and making use of the equilibrium conditions in (48:3) and (48:4), whether

$$\frac{\mu}{(\tau\nu - 1)(\tau\nu - \lambda)} \gtreqless 1. \tag{49}$$

* * *

It is obvious that these starting points can, in principle, be developed to obtain an explanation of the trade cycle. The purely technical difficulties of model building are, however, considerably greater than those encountered in the case dealt with in the preceding section.

CONCLUDING REMARKS

The investment function developed for the model, according to which entrepreneurs' expectations, sensitive to cyclical fluctuations in economic activity, in combination with technological progress and population growth, affect their investment decisions, made it possible to solve the so-called Harrod—Domar problem. In this solution, related to full capital-stock utilization, the movement of national income involved cyclical fluctuations in embryo. The trade cycle analysis proper only suggested the occurrence of fluctuations sharper than those implied by the solution of the Harrod—Domar problem.

In regard to the methodological aspects of the study, it should be pointed out that the tentative numerical analysis made possible by electronic computers proved very helpful for the theoretical analysis.

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