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Determinacy and E-stability with Interest Rate Rules at the Zero Lower Bound*

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Abstract

We evaluate and compare alternative monetary policy rules, namely average inflation targeting, price level targeting, and traditional inflation targeting rules, in a standard New Keynesian model that features recurring, transient zero lower bound regimes. We use determinacy and expectational stability (E-stability) of equilibrium as the criteria for stabilization policy. We find that price level targeting policy, including nominal income targeting as a special case, most effectively promotes determinacy and E-stability among the policy frameworks, whereas standard inflation targeting rules are prone to indeterminacy. Average inflation targeting can induce determinacy and E-stability effectively, provided the averaging window is sufficiently long.

Keywords: Zero Lower Bound; Markov-Switching; Expectations; Price level targeting; Average inflation targeting; Nominal income targeting;

JEL classification: E31; E47; E52; E58

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1 Introduction

1.1 Motivation and Main Findings

The zero lower bound (ZLB) on interest rates has become a pervasive constraint on monetary policy in several advanced economies. In some countries, notably Japan and the U.S., the ZLB is a recurring phenomenon, and some expect the ZLB to bind with increasing frequency in the future.[1] The ZLB on interest rates or, more generally, interest rate pegs, can destabilize expectations, and it should worry policymakers. If interest rates become constrained by the ZLB under a given interest rate rule, then the rule might induce multiple equilibria ("indeterminacy"). Some of the multiple equilibria subject the economy to extraneous, beliefs-driven volatility, and therefore they might be viewed as undesirable. In addition, it is important to understand whether agents can learn a particular equilibrium when they are assumed not to have rational expectations à la Evans and Honkapohja (2001). Under adaptive learning, the ZLB and interest rate pegs more broadly can lead to dynamically unstable inflation and inflation expectations (e.g., "deflationary spirals").[2] As a result, they are both widely associated with the non-existence of an expectationally stable ("E-stable" or "learnable") equilibrium, that is, a dynamically stable rational expectations equilibrium (REE) that could emerge from an econometric learning process involving imperfectly informed agents. When expectational stability (E-stability) conditions are not satisfied, policymakers may fail to anchor expectations under learning–even if agents' initial expectations are close to policymakers' target equilibrium.

Concerns about the recurrence of ZLB events have generated interest in alternative policy frameworks such as average inflation targeting and price level targeting that may mitigate the problems associated with the ZLB. For example, the Federal Reserve adopted an average inflation targeting framework in August, 2020, after conducting its own policy strategy

[1]Many policymakers and economists have suggested that the ZLB could bind with increasing frequency in the U.S. and other economies (e.g., see Bernanke et al. (2019)). Similarly, the New York Fed's Survey of Primary Dealers for the period of 2016 to 2019 indicate that market participants placed substantial probability on the prospect of a second ZLB event in the U.S. economy.

[2]See Howitt (1992), Evans and Honkapohja (2001), Evans and Honkapohja (2003) and Evans and McGough (2018) for evidence of instability under learning with interest pegs. Evans et al. (2008), Benhabib et al. (2014), and Honkapohja and Mitra (2020), among others, document similar instabilities in nonlinear New Keynesian models with a binding ZLB.

review. Both average inflation targeting and price level targeting promise inflation in excess of the inflation target following a period of below-target inflation. In principle, expectations of higher future inflation can elevate and help anchor inflation expectations when interest rates are at the ZLB.

The main contribution of this paper is to evaluate alternative policy rules, including average inflation targeting and price level targeting rules, when interest rates are frequently constrained by the ZLB. We ask whether these alternative policy rules help to rule out multiple equilibria when the ZLB is a recurring phenomenon. In addition, we examine whether agents can learn the equilibrium of the model under the alternative policy rules using the criterion of E-stability. More generally, policymakers should consider determinacy and E-stability desiderata for monetary policy when evaluating alternatives to the inflation targeting status quo.

We conduct our analysis using a standard New Keynesian model with recurring, transient ZLB events. Similar to Bianchi and Melosi (2017), our model features a persistent, possibly recurring two-state demand shock that follows an exogenous Markov process, and a central bank that sets interest rates with the ZLB binding when demand is low (i.e., following a contractionary demand shock). Agents are uncertain about the future path of the demand and monetary policy state, but form expectations that account for the possibility of ongoing regime changes. In this framework, recurring ZLB regimes affect the stability of expectations. Thus, we investigate whether alternative monetary policy rules can preclude multiple equilibria under rational expectations, and deflationary spirals under learning, *given* the expected duration and frequency of ZLB events.[3] Specifically, we compare three simple interest rate rules that describe three related policy strategies: (i) inflation targeting (a Taylor-type rule); (ii) average inflation targeting; and (iii) price level targeting (a Wicksellian rule). Note that nominal income targeting is a special case of the price level targeting rules when the policy coefficients for price level and output are the same.

Our findings have important implications for stabilization policy in the current low in-

[3]Standard adaptive learning models can make strong, counterfactual predictions about the possibility of deflationary spirals at the ZLB. In these frameworks, agents have linear forecasting models and do not anticipate eventual escape from the ZLB. In contrast, the learning agents in our model form expectations that account for the recurrence of ZLB and active monetary policy regimes. Expectations of future active policy mitigate the potential for deflationary spirals.

terest rate environment. We find that the model under a simple inflation targeting rule is prone to indeterminacy, although under some empirically relevant assumptions about the expected ZLB duration, simple Taylor rules can promote an E-stable REE. In contrast, the model under price level targeting is almost certain to admit a unique, E-stable equilibrium, as long as agents put an arbitrarily small, positive probability on the prospect of exiting the ZLB regime. The average inflation targeting rule can also promote determinacy and E-stability quite effectively, provided that the measure of average inflation is sufficiently backward looking. For all these rules, we provide real-time learning simulations that demonstrate the absence of deflationary spirals and the convergence of learning agents' expectations to rational expectations when the E-stability criterion is satisfied. These findings are also applicable to a general setting of interest rate pegs.

After a brief literature review, the paper proceeds as follows. Section 2 describes our model and the policy rules under consideration. Section 3 considers a version of the model with flexible prices, thus providing intuitive reasoning for our numerical results. Sections 4 and 5 describe our numerical results for determinacy and E-stability, and related robustness concerns. Section 6 concludes.

1.2 Literature Review

A number of papers have documented indeterminacy and E-instability of REE in standard models with exogenous (or pegged) nominal interest rates. It is well-known that passive monetary policy rules, including interest pegs, permit "local indeterminacy" of the target steady state, i.e., the existence of multiple stable solution paths that converge to the steady state with positive interest rates (see Woodford (2003)).[4] Howitt (1992), Evans and Honkapohja (2001), and Evans and McGough (2018) demonstrate E-instability of REE under interest rate pegs when agents are learning; and Evans et al. (2008) document E-instability of the low inflation steady state with the binding ZLB, and the possibility of deflationary spirals

[4] See also Aruoba and Schorfheide (2013), Aruoba et al. (2018), Cochrane (2017), Armenter (2018) Christiano et al. (2018), Bilbiie (2019), Holden (2020), and Mertens and Ravn (2014), for more on the multiplicity of equilibria at the ZLB. We focus on "local determinacy" of the target steady state with inflation equal to the inflation target, whereas Benhabib et al. (2001) studies "global indeterminacy", i.e., existence of two steady states: the target steady state and a low inflation steady state.

under learning at the ZLB. In most of the above-mentioned papers, the monetary policy regime is expected to last forever, but a determinate equilibrium may exist in models with an inflation targeting policymaker and persistent, transitory passive monetary regimes, as shown by Cho (2016) and Barthélemy and Marx (2019), and earlier considered by Davig and Leeper (2007).[5] Similarly, Mertens and Ravn (2014) and Christiano et al. (2018) show when expectations are stable under adaptive learning in economies that are subject to a *one-time* transient ZLB regime; and McClung (2020) shows that Markov-switching models with an inflation targeting central bank and recurring interest rate peg regimes can admit E-stable REE if interest peg regimes are not expected to last too long.[6] Therefore, the ability of policymakers to manage expectations subject to interest peg regimes such as ZLB events depends crucially on the expected duration and frequency of the interest peg regime. This paper builds on this literature by examining the stabilization properties of alternative monetary policy strategies *given* the expected ZLB duration and frequency.

Consequently, this paper also contributes to a broad literature on alternative policy frameworks in low interest rate environments. Early works, including Svensson (2003), Eggertsson and Woodford (2003), and Auerbach and Obstfeld (2005), argue that price level targeting or policies that engineer temporary overshooting of the inflation target are approximately optimal strategies during liquidity traps.[7] Kiley and Roberts (2017) and Bernanke et al. (2019) study the stabilization properties of lower-for-longer strategies, and Nakata and Schmidt (2019), Nakata and Schmidt (2020), and Bianchi et al. (2020) focus on pathologies of and policy strategies for recurring ZLB events. Other recent works focus on average inflation targeting, including Mertens and Williams (2019), Budianto et al. (2020), and Amano et al. (2020).

[5] A number of papers show how to restore determinacy under persistent interest rate pegs, including Acharya and Dogra (2020), Bilbiie (2019), Gabaix (2020), Diba and Loisel (forth.), and Roulleau-Pasdeloup (2020).

[6] McClung (2020) and this paper focus on REE. Self-confirming restricted perceptions equilibria of models with regime-switching Taylor rules are studied by Airaudo and Hajdini (forth.) and Ozden and Wouters (2020) .

[7] Woodford (2003) and Giannoni (2014) explore optimal monetary policy and price level targeting when interest rates are not constrained by the ZLB. Similarly, Nessén and Vestin (2005) and Eo and Lie (2020) examine the welfare implications of average inflation targeting as a makeup policy. Policymakers have drawn considerable attention to these alternative policy frameworks and related academic findings (e.g., see Evans (2012), Williams (2017), Bernanke (2017)).

A strand of the literature studies the stabilization properties of these alternative policy frameworks when agents are capable of learning. Our study is most closely related to that of Honkapohja and Mitra (2020), who compare price level targeting with inflation targeting in a liquidity trap with adaptive learning agents. They find that employing price level targeting can guide the economy out of the ZLB, even if agents initially put little weight on information about the price level target path when forecasting inflation. In contrast, we study an environment with recurring ZLB events, and we find that deflationary spirals are absent under price level targeting if learning agents put a small probability on exiting the low demand state.[8]

A few recent works specifically examine price level targeting and determinacy at the ZLB. Armenter (2018) documents multiple stable solution paths in a model with an optimal price level targeting policy, including solutions for which the ZLB binds indefinitely. Thus, price level targeting is no panacea. However, we can rule out analogous outcomes in our model by assuming occasional active monetary regimes. Our results are more in line with those of Holden (2020), who finds that price level targeting in a model with an occasionally binding ZLB constraint ensures a unique perfect foresight path that converges toward the intended steady state. In contrast to Holden (2020), we abstract from occasionally binding constraints, and instead focus on a framework with stochastic, exogenous regime changes in order to study the stability of expectations *given* the frequency and duration of ZLB events. Our approach also considers both sunspot and fundamental equilibria, a broader class of interest rate rules, and in some cases, adaptive learning agents. Thus, our work and that of Holden (2020) are complementary.

2 Model and Equilibrium Concepts

Here we present a standard New Keynesian model with a discrete-valued preference shock that induces the ZLB for our analysis. This section also introduces important concepts related to equilibrium and stability of equilibrium, as well as the different models of expectations formation considered in the remainder of this paper.

[8] See Mele et al. (2020) for a study on price level targeting and learning when the central bank is rational.

2.1 Model Description

The model economy consists of a representative household, a continuum of monopolistically competitive firms that face price-adjustment costs, and a monetary authority that adjusts the (short-term) nominal interest rate in response to economic conditions. The government balances its budget in each period using lump-sum taxes.

Households. The representative household maximizes its expected lifetime utility

$$E_0 \left[\sum_{t \geq 0} \beta^t exp(\xi_t) \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{h_t^{1+\eta}}{1+\eta} \right) \right], \tag{1}$$

subject to

$$P_t C_t + B_t = W_t h_t + R_{t-1} B_{t-1} + P_t D_t - T_t, \tag{2}$$

where P_t is the price level, C_t is consumption, W_t is the nominal wage, h_t is hours worked, T_t is lump-sum taxes, R_t is the nominal interest rate, B_t is nominal government debt, and D_t is real dividends from the economy's firms. Similar to Bianchi and Melosi (2017), the preference shock $\xi_t = \overline{\xi_{s_t}} + \epsilon_{d,t}$ is the sum of a continuous i.i.d. variable, $\epsilon_{d,t}$, with mean-zero, and a discrete-valued shock, $\overline{\xi_{s_t}}$, with two realized shocks of $\overline{\xi_0}$ and $\overline{\xi_1}$. The regime variable $s_t \in \{0, 1\}$ follows a first-order Markov process according to the transition matrix

$$P = \begin{pmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{pmatrix} \tag{3}$$

where $p_{ij} = Pr(s_{t+1} = j | s_t = i)$ for $i, j = 0, 1$. The values of $\overline{\xi_0}$, $\overline{\xi_1}$, p_{00}, and p_{11} are set such that the unconditional mean of ξ_t is zero where $\overline{\xi_0} < 0 < \overline{\xi_1}$. When the low value for the preference shock $\overline{\xi_0}$ is realized, inflation and aggregate demand fall, which affects the monetary policy stance, leading the nominal rate to hit the ZLB.[9]

Firms. Firm j uses labor as its only input in production. The production function is

[9]Our approach to motivating the ZLB using discrete-valued demand shocks is similar to that of Bianchi and Melosi (2017), and is also related to the approaches of Eggertsson and Woodford (2003), Bilbiie (2019), and Nakata and Schmidt (2019), among others.

given by

$$Y_t(j) = h_t(j), \tag{4}$$

where $h_t(j)$ denotes labor demand allocated to firm j. Firm j operates in a monopolistically competitive market and faces the following downward-sloping demand curve

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_{s,t}} Y_t, \tag{5}$$

where Y_t is aggregate output and $\epsilon_{s,t}$ is the elasticity of demand for each intermediate good, which is a random i.i.d. variable with mean ϵ_s. Firm j chooses $P_t(j)$ to maximize the discounted present value of expected real profits, where real profits at t are given by

$$D_t = \frac{P_t(j)Y_t(j)}{P_t} - \frac{W_t h_t(j)}{P_t} - AC_t(j) \tag{6}$$

where $AC_t(j)$ is a price-adjustment cost term in the tradition of Rotemberg (1982)

$$AC_t(j) = \frac{\gamma}{2}\left(\frac{P_t(j)}{\Pi P_{t-1}(j)} - 1\right)^2 Y_t \tag{7}$$

and Π is the gross inflation rate in the steady state, which is consistent with the central bank's inflation target.

Aggregate Resource Constraint. All intermediate goods producing firms make identical choices so that the market clearing condition is given by

$$C_t = Y_t - \frac{\gamma}{2}\left(\frac{P_t}{\Pi P_{t-1}} - 1\right)^2 Y_t. \tag{8}$$

For the stability analysis, we derive a log-linear approximation of the model around the deterministic intended steady state (i.e., the steady state consistent with the central bank's policy objective) in the presence of a recurring ZLB, as in Bianchi and Melosi (2017). Under rational expectations, the log-linearized equations that determine the inflation gap, π, and the output gap, y, are given by

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa y_t + v_{s,t} && (9) \\ y_t &= E_t y_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + u_t + v_{d,t} && (10) \end{aligned}$$

where $u_t = \sigma^{-1}(\xi_t - E_t \xi_{t+1})$, $v_{s,t} \sim \mathcal{N}(0, \sigma_s^2)$ and $v_{d,t} \sim \mathcal{N}(0, \sigma_d^2)$ are proportional to $\epsilon_{s,t}$ and $\epsilon_{d,t}$, respectively, and i_t is determined in conjunction with a policy rule for the nominal interest rate. All variables are expressed in terms of percentage deviations from the steady state. Because some of the policy rules under consideration here respond directly to the price level rather than inflation, it is convenient to rewrite (9) and (10) by using $\pi_t = p_t - p_{t-1}$ and $E_t \pi_{t+1} = E_t p_{t+1} - p_t$ as follows:[10]

$$\begin{aligned} p_t &= \frac{\beta}{1+\beta} E_t p_{t+1} + \frac{\kappa}{1+\beta} y_t + \frac{1}{1+\beta} p_{t-1} + \frac{1}{1+\beta} v_{s,t} && (11) \\ y_t &= E_t y_{t+1} - \sigma^{-1}(i_t - E_t p_{t+1} + p_t) + u_t + v_{d,t}. && (12) \end{aligned}$$

Monetary Authority. The monetary policymaker tacitly conducts monetary policy using simple interest rate rules, i_t^*, which will be presented later, subject to a lower bound constraint,

$$i_t = \max\{i_t^*, -\bar{i}\}$$

where i_t is the nominal policy rate and $-\bar{i} < 0$.[11] We want to study how expectations behave given the recurrence of the ZLB, and so we simply assume $i_t = -\bar{i}$ in the low demand, low inflation state ($s_t = 0$), and $i_t = i_t^*$ otherwise. Under this simplifying assumption we need not worry about the values of $\bar{i}$ and $\overline{\xi_{s_t}}$, which turn out to be utterly irrelevant in our determinacy and E-stability analysis.[12] In turn, this allows us to focus on the consequences of agents'

[10] We implicitly define p_t as the log deviation of the price level from some constant price level target, p^*. We can allow for a time-varying price level target without altering our main results.

[11] Here, i_t is a percentage deviation from the steady state. Assuming a zero inflation target, $\bar{i} = 1/\beta - 1$, gives the ZLB constraint.

[12] Our model's determinacy and E-stability conditions do not depend on the calibration of the model's regime-switching intercept term (i.e., $\bar{i}$ and $\overline{\xi_{s_t}}$). The value of the intercept matters for the existence of equilibria such that $i_t^* < -\bar{i}$ if and only if $s_t = 0$ (e.g. see Ascari and Mavroeidis (2021) for a general treatment of related existence and multiplicity issues in models with occasionally binding constraints). However, existence considerations are beyond the scope of this study, which focuses on the stability of expectations *given the frequency and persistence of ZLB events*. See Appendices A.3-A.4 for more details.

beliefs about the persistence and frequency of the low inflation, low interest rate regime (i.e., agents' beliefs about p_{00} and p_{11}) for determinacy and E-stability. Thus, we have a tractable framework for analyzing the stability of expectations *given* recurring ZLB regimes.[13]

We consider three monetary policy rules, which are expressed below in terms of log-linearized variables (after incorporating the above assumptions) and the regime indicator s_t:

1. **Inflation targeting (Taylor rule):**

$$i_t = s_t(\phi_\pi \pi_t + \phi_y y_t) - (1 - s_t)\bar{i} \tag{13}$$

2. **Price level targeting:**

$$i_t = s_t(\phi_p p_t + \phi_y y_t) - (1 - s_t)\bar{i} \tag{14}$$

   Note that the price level targeting rule coincides with a simple nominal income targeting rule when $\phi_y = \phi_p$. Thus, we use (14) to study closely related price level targeting and nominal income targeting strategies.

3. **Average inflation targeting:**

$$i_t = s_t(\phi_{\bar{\pi}} \bar{\pi}_{t,t-m+1} + \phi_y y_t) - (1 - s_t)\bar{i} \tag{15}$$

   where $\bar{\pi}_{t,t-m+1} = \frac{1}{m}\sum_{j=0}^{m-1} \pi_{t-j}$. We consider two values of $\phi_{\bar{\pi}}$: (i) $\phi_{\bar{\pi}} = \phi_\pi$, such that the central bank targets a simple average of inflation with a target window m; and (ii) $\phi_{\bar{\pi}} = \phi_\pi m$, such that the central bank targets the unweighted sum of the m most recent inflation observations. Both interpretations assume that the policy rate depends on a long history of inflation data. However, as shown later, the second formulation is helpful for comparing outcomes under inflation targeting, average inflation targeting, and price level targeting.

[13] Tractability comes at the expense of ignoring the possibility that $i_t^* < -\bar{i}$ when $s_t = 1$ *given arbitrary values of* $\bar{i}$ and $\overline{\xi_{s_t}}$. However, in our numerical analysis of E-stable equilibria, we typically find $\overline{\xi_{s_t}}$ such that $i_t^* < \bar{i}$ if and only if $s_t = 0$.

As presented previously, s_t is the two-state exogenous Markov process driving $\overline{\xi_{s_t}}$ according to (3). Because i_t is the percentage deviation of the nominal interest rate from its steady state, $\bar{i}$, the nominal interest rate is equal to zero when $s_t = 0$.

For each policy rule we consider, we can express our model in the following general form

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t) + D(s_t)v_t \tag{16}$$

where x_t collects endogenous variables such as p_t, y_t, and i_t and their lagged variables depending on the model, $C(s_t)$ is a function of u_t and $\bar{i}$, and $v_t = (v_{s,t},\ v_{d,t})'$.

2.2 Rational Expectations Equilibrium

For our rational expectations analysis, we assume that agents possess complete, homogeneous information of the economy and form true mathematical expectations, $E_t x_{t+1}$, conditional on complete time-t information.[14] A rational expectations solution is any stochastic process $\{x_t\}$ that solves the model (16) under the above-mentioned assumptions. In general, there can be two types of solutions: (i) minimal state variable (MSV) solutions, which express x_t as a function of fundamental predetermined variables, x_{t-1}, $C(s_t)$, s_t, and no other variables; and (ii) non-fundamental (sunspot) solutions, which express x_t as a function of x_{t-1}, $C(s_t)$, s_t, and extraneous variables that do not appear in (16). An REE is a mean-square stable rational expectations solution.[15] *If* a unique REE of (16) exists, then it assumes the MSV form:

$$x_t = \Omega(s_t)x_{t-1} + Q(s_t)v_t + \Gamma(s_t) \tag{17}$$

[14]Note that agents do not know s_{t+j} for any $j \geq 1$ at time t.

[15] Mean-square stability is the most widely used stability concept in the Markov-switching dynamic stochastic general equilibrium (DSGE) literature. Intuitively, a stochastic process is mean-square stable if it has finite first and second moments. Interested readers are referred to Farmer et al. (2009), Cho (2016), and Cho (2021) for more on mean-square stability. Note that Barthélemy and Marx (2019) provide conditions for the uniqueness of a boundedly stable REE.

where

$$\Omega(s_t) = \left(I - A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} \Omega(s_{t+1}) \right)^{-1} B(s_t) \tag{18}$$

$$Q(s_t) = \left(I - A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} \Omega(s_{t+1}) \right)^{-1} D(s_t) \tag{19}$$

$$\Gamma(s_t) = \left(I - A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} \Omega(s_{t+1}) \right)^{-1} \left(C(s_t) + A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} \Gamma(s_{t+1}) \right). \tag{20}$$

Numerous methods have been developed to obtain solutions of the form (17); here we use the forward method of Cho (2016).[16] After obtaining an MSV solution (17), we assess the uniqueness of the equilibrium using the determinacy conditions in Cho (2016) and Cho (2021), which are tractable conditions that depend only on $A(s_t), B(s_t), \Omega(s_t)$, p_{00}, and p_{11}. These determinacy conditions, when satisfied, ensure that the MSV solution (17) is the unique REE of the model (16). Thus, if a given MSV solution (17) satisfies the conditions in Cho (2021), then all non-fundamental solutions of (16) and all other MSV solutions of (16) are mean-square *unstable*. If the determinacy conditions fail, and the MSV solution we obtain is mean-square stable, then we have indeterminacy.[17] In order to apply the solution approach and determinacy conditions of Cho (2016) and Cho (2021) to a model of the form (16), which contains a regime-switching intercept term, we need to make slight modifications to the model (see Appendix A.3 for more details).

2.3 Adaptive Learning Framework

As an alternative to rational expectations, we consider the adaptive learning approach in the spirit of Evans and Honkapohja (2001). This approach relies on more realistic assumptions about information availability than does rational expectations. For example, adaptive learning agents are not assumed to know the full structure of the economy when forming expectations of aggregate variables that matter for their decisions related to optimal pric-

[16] Alternative solution techniques for Markov-switching DSGE models are developed by Foerster et al. (2016), Maih (2015), and Farmer et al. (2011), among others.

[17] We always obtain at least one mean-square stable solution for any calibration of the model we consider in this study. Therefore, we can always determine whether our model is determinate or indeterminate.

ing for firms, and optimal savings, labor and consumption for households. These learning agents form expectations using a subjective forecasting model for the aggregate variables, often referred to as a "perceived law of motion" (PLM) for the aggregate variables, which they estimate in real time (e.g., using the recursive least squares method). In each period, households and firms make decisions contingent on these forecasts. After the markets clear, the aggregate implications of these decisions are summarized by the Euler equation and the Phillips curve, given the learning agents' inflation and output forecasts.[18]

A primary focus of this study is to identify policy rules (13)–(15) that select an equilibrium that is "E-stable" or "stable under (adaptive) learning." Intuitively, adaptive learning agents' PLM (and therefore the *actual* learning equilibrium law of motion) may converge to the REE law of motion in real time if the E-stability conditions are satisfied. Importantly, if the learning equilibrium converges to the (mean-square stable) REE then inflation is mean-square stable in the learning equilibrium. Thus, a deflationary spiral (i.e., a situation such that $E_0\pi_t \to -\infty$ as $t \to \infty$) does not occur under learning if agents learn an E-stable REE. More generally, E-unstable REE cannot be the outcome of an adaptive learning process, and therefore E-instability is a warning signal that inflation expectations can become severely de-anchored. Consequently, we should avoid policies that do not promote E-stability.

To derive the E-stability conditions (i.e., the conditions under which an REE is locally E-stable),[19] we first establish the information set, $\mathcal{I}_t$, available to adaptive learning agents when forming expectations at time t. We assume that agents have "contemporaneous information": $(P, x_t, s_t, v_t) \in \mathcal{I}_t$.[20] We also assume that learning agents recursively estimate the coefficients, $(a(s_t), b(s_t), c(s_t))$, of the following PLM:

$$x_t = a(s_t) + b(s_t)x_{t-1} + c(s_t)v_t + \tilde{\epsilon}_t, \tag{21}$$

[18] In keeping with much of the adaptive learning literature, we assume that the Euler equation and the Phillips curve describe the evolution of equilibrium output and inflation given adaptive learning forecasts. However, a large body of literature studies an alternative approach in which agents' decisions, and hence economic outcomes, depend on their long-horizon expectations. See Preston (2005), Eusepi and Preston (2011), Evans et al. (2013), and Bullard and Eusepi (2014) for more on infinite-horizon learning.

[19] We stress that in models with lagged endogenous variables, the E-stability conditions are "local" conditions in the sense that E-stability only predicts the convergence of adaptive learning beliefs to REE beliefs if the initial beliefs are in some neighborhood of the REE beliefs.

[20] Later we discuss implications of excluding x_t from $\mathcal{I}_t$.

where $\tilde{\epsilon}_t$ is the perceived i.i.d. noises; $x_t = (\pi_t, y_t)'$ if the model is given by (9), (10) and (13) for inflation targeting; $x_t = (p_t, y_t)'$ if the model is given by (11), (12) and (14) for price level targeting; and $x_t = (\pi_t, \ldots, \pi_{t-m+2}, y_t)'$ if the model is given by (9), (10) and (15) for average inflation targeting. It is important to recognize a couple of facts about the agents' PLM and our model of adaptive learning. First, (21) has the same functional form as the MSV solution (17). That is, we assume agents use a correctly-specified econometric model, but do not know the parameter values. Second, because agents know s_t, they do not need sophisticated Markov-switching VAR techniques to estimate the coefficients of (21); they need only estimate two linear models in real time (one for each regime) using standard techniques such as the least squares method.[21]

Under the assumption of contemporaneous information, agents form expectations in real time as follows:[22]

$$\hat{E}_t x_{t+1} = \sum_{s_{t+1}} p_{s_t s_{t+1}} \left\{ a(s_{t+1})_{t-1} + b(s_{t+1})_{t-1} x_t \right\}$$

where $a(s_t)_{t-1}$, $b(s_t)_{t-1}$ and $c(s_t)_{t-1}$ denote agents' estimates of $a(s_t)$, $b(s_t)$ and $c(s_t)$, respectively, using all information available at the end of $t-1$. In what follows, we suppress the $t-1$ subscripts in the agents' PLM and let $(a(s_t), b(s_t), c(s_t))$ represent $(a(s_t)_{t-1}, b(s_t)_{t-1}, c(s_t)_{t-1})$. If agents make decisions contingent on these forecasts (i.e., if we substitute $\hat{E}_t x_{t+1}$ into (16)), then the equilibrium at time t is given by

$$\begin{aligned} x_t &= \left(I - A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1}) \right)^{-1} (B(s_t) x_{t-1} + D(s_t) v_t) \\ &+ \left(I - A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1}) \right)^{-1} \left(C(s_t) + A(s_t) \sum_{s_{t+1}} p_{s_t s_{t+1}} a(s_{t+1}) \right). \end{aligned} \tag{22}$$

After observing x_t, agents update their estimates of $a(s_t)$, $b(s_t)$ and $c(s_t)$ using standard techniques such as least squares, holding fixed their beliefs about $a(j)$, $b(j)$, $c(j)$ where $j \neq s_t$. From (22), it is apparent that the agents' beliefs, $a(s_t)$, $b(s_t)$, and $c(s_t)$, are self-

[21] Recursive approaches to estimating (21) are briefly discussed in section 5.2 and also McClung (2020).

[22] Here, and throughout the paper, we use $\hat{E}_t$ to denote (potentially) non-rational expectations formed under adaptive learning. E_t denote rational expectations.

confirming only if

$$b(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1})\right)^{-1} B(s_t) \tag{23}$$

$$c(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1})\right)^{-1} D(s_t) \tag{24}$$

$$a(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} b(s_{t+1})\right)^{-1} \left(C(s_t) + A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}} a(s_{t+1})\right). \tag{25}$$

Note that these conditions are identical to (18)-(20). Therefore, learning agents' beliefs only converge to self-confirming values under adaptive learning with correctly-specified PLM (21) if the agents learn the coefficients of an REE (17). Formally, we say that adaptive learning agents learn the MSV solution (17) if $(a(s_t), b(s_t), c(s_t)) \to (\Gamma(s_t), \Omega(s_t), Q(s_t))$ as $t \to \infty$. Proposition 1 of McClung (2020) derives the E-stability conditions under which agents may learn an MSV solution (17) by estimating (21) in real time and making forecasts contingent on these estimates using contemporaneous information. When the E-stability conditions fail (i.e. "E-instability" obtains), agents will not learn the MSV solution. For convenience, the E-stability conditions are presented in Appendix A.4.

2.4 The Relationship between Determinacy and E-stability

McClung (2020) also shows that determinacy implies the E-stability of the unique mean-square stable equilibrium when agents have contemporaneous information.[23] Therefore, if the model (16) is determinate, then the unique mean-square stable REE (17) is E-stable. However, the converse is not true; E-stability may select an equilibrium that the determinacy criterion would not select. Therefore, the regions of the model parameter space that generate E-stable MSV solutions are larger than those of the parameter space that generate a unique mean-square stable REE. In fact, McClung (2020) shows that the E-stability conditions for models of the form (16) can be significantly weaker than the determinacy conditions.

[23]See Propositions 1 and 2 of McClung (2020) for details. Branch et al. (2013) also studies uniqueness and E-stability of (regime-dependent) REE in a class of purely forward-looking regime-switching models.

3 Analysis with a Simplified Model

Before exploring a fully fledged New Keynesian model, we first study (in)determinacy and E-(in)stability in a simplified version of the model (11)-(12). In the simplified model, as in Davig and Leeper (2007), prices are assumed to be fully flexible (i.e., as $\kappa \to \infty$ such that $y_t = 0$ for all t). However, we study a variety of targeting rules at the ZLB, while Davig and Leeper (2007) examine the equilibrium properties of recurring active and passive monetary regimes in the context of a Taylor-type rule. With flexible prices (and also assuming $v_t = 0$ for exposition), the model reduces to the Fisherian model:

$$i_t \quad = \quad E_t p_{t+1} - p_t + \sigma u_t. \tag{26}$$

To characterize the equilibrium dynamics of inflation, we need only pair the Fisher equation (26) with one of the interest rate rules from (13)–(15). Analytical determinacy results are available, which help develop our predictions and intuition for the numerical analysis presented in Sections 4 and 5.

First, consider inflation targeting (13) with the Fisher equation (26), yielding the following Markov-switching expectational difference equation for inflation:

$$\phi_\pi s_t \pi_t = E_t \pi_{t+1} + (1 - s_t)\,\bar{i} + \sigma u_t \tag{27}$$

where $\phi_\pi > 0$ and $y_t = 0$ for all t is imposed in (13) with flexible prices.

Proposition 1 considers determinacy and E-stability under inflation targeting in the simplified model with recurrent ZLB events.

Proposition 1 *Consider a simple model of inflation (27) that combines the Fisher equation (26) and inflation targeting (13), and assume $\phi_\pi \geq 0$. Then, (27) is indeterminate and the MSV solution is E-unstable for any finite value of $\phi_\pi > 0$.*

Proof: See Appendix A.1. ■

Proposition 1 suggests that inflation targeting is always ineffective as stabilization policy in our simplified model with recurring ZLB events, regardless of their frequency and duration.[24]

It is instructive to consider an alternative interest rate rule, and here we the consider price level targeting rule (14). If we substitute (14) into (26), then we arrive at the following Markov-switching expectational difference equation for the price level, p_t:

$$p_t \quad = \quad (1+\phi_p s_t)^{-1} E_t p_{t+1} + (1+\phi_p s_t)^{-1} \left((1-s_t)\,\bar{i} + \sigma u_t \right). \tag{28}$$

Proposition 2 shows that we have determinacy and E-stability under price level targeting, provided that interest rates can respond to prices some of the time, even if policy is only unconstrained on an extremely infrequent basis.

Proposition 2 *Consider a simple model of inflation (28) that combines the Fisher equation* (26) *and price level targeting* (14)*, and assume $\phi_p \geq 0$ and $p_{00} + p_{11} > 1$. Then, (28) is determinate and the unique REE is E-stable if and only if $\phi_p > 0$ and $p_{00} < 1$.*

Proof: See Appendix A.2. ■

According to Proposition 2, all that is required for determinacy under price level targeting is that the ZLB regime must be transitory ($p_{00} < 1$) and monetary policy must be expected to respond to the price level following exit from the ZLB regime ($\phi_p > 0$).[25] Importantly, there are no restrictions on p_{11}, apart from $p_{00} + p_{11} > 1$. Thus, price level targeting in a simplified model leads to determinacy even if the ZLB can be recurring (i.e., $p_{11} < 1$) and occuring more frequently than the unconstrained monetary regime (i.e., $p_{11} < p_{00}$).

Average inflation targeting rules of the form (15) with $\phi_{\bar{\pi}} = \phi_\pi m$ are an intermediate case between inflation targeting and price level targeting. For example, if we set $m = 1$, then average inflation targeting (15) collapses to inflation targeting (13). For $m \geq 1$, we can

[24] In the absence of ZLB events (i.e., $s_t = 1$ for all t), and assuming $\phi_\pi \geq 0$, the simplified model with inflation targeting is a determinate model of inflation if and only if the Taylor Principle, $\phi_\pi > 1$, is satisfied. If the model is indeterminate, such that multiple rational expectations equilibria exist, there will always be a unique MSV solution. The MSV solution of the simplified model is unique owing to the lack of lagged endogenous variables in the model. Therefore, it is apparent that a zero interest rate policy or, more generally, interest rate pegs (i.e., $\phi_\pi = 0$), does not promote determinacy or E-stability in the simple Fisherian model (26) with inflation targeting.

[25] In the absence of ZLB events (i.e., $s_t = 1$ for all s_t), and assuming $\phi_p \geq 0$, the simplified model with price level targeting is a determinate model if and only if $\phi_p > 0$ is satisfied. Woodford (2003), Giannoni (2014), and Honkapohja and Mitra (2020) show this result in more general models. Furthermore, the unique MSV solution is E-stable if $\phi_p > 0$. Hence, a permanent interest rate peg (i.e., $\phi_p = 0$) is on the boundary of the determinacy and E-stability region of the simple price level targeting model's parameter space.

rewrite the average inflation targeting rule (15) with $\phi_{\bar{\pi}} = \phi_\pi m$ in terms of the price level, p_t (assuming $\phi_y = \bar{i} = 0$ for the sake of exposition):

$$\begin{aligned} i_t &= \phi_{\bar{\pi}} s_t \bar{\pi}_{t,t-m+1} \\ &= \phi_\pi m s_t \bar{\pi}_{t,t-m+1} \\ &= \phi_\pi s_t \left\{ (p_t - p_{t-1}) + (p_{t-1} - p_{t-2}) + \ldots + (p_{t-m+2} - p_{t-m+1}) \right\} \\ &= \phi_\pi s_t \left(p_t - p_{t-m+1} \right). \end{aligned}$$

In the limit $m \to \infty$, we have:

$$i_t = \phi_\pi s_t \left(p_t - p_0 \right)$$

where p_0 is some arbitrary initial condition. If we normalize $p_0 = 0$, then the average inflation targeting rule with $\phi_{\bar{\pi}} = \phi_\pi m$ clearly becomes the price level targeting rule when $m \to \infty$. Thus, we expect an increase in m to yield determinacy results that are closer to the prediction of Proposition 2 for price level targeting than they are to that of Proposition 1 for inflation targeting. Section 4 considers a fully fledged New Keynesian model and shows that this is generally the case.

4 Rational Expectations and Determinacy

This section considers the New Keynesian model described by (9) and (10), and examines the determinacy properties for each policy rule. For each policy rule, we consider a benchmark calibration that is well within the range of calibrations studied in the literature: $\beta = 0.9975$, $\kappa = 0.05$, $\sigma = 2$, $\phi_\pi = 2$, and $\phi_y = 0.5/4$ for inflation targeting (13) and average inflation targeting (15), and $\phi_p = 0.25$ and $\phi_y = 0.5/4$ for (14), loosely following Williams (2010). Robustness concerns related to the calibration are discussed throughout this section.[26] As mentioned in Section 2, there is no need to calibrate $\bar{i}$ or $\overline{\xi_{s_t}}$. The irrelevance of $\overline{\xi_{s_t}}$ underscores the fact that the persistence and frequency of passive monetary spells is what matters for

[26]Note that σ_d and σ_s are irrelevant for determinacy and E-stability analysis (e.g. see Cho (2021) and McClung (2020)).

the stability of beliefs under rational expectations or adaptive learning.

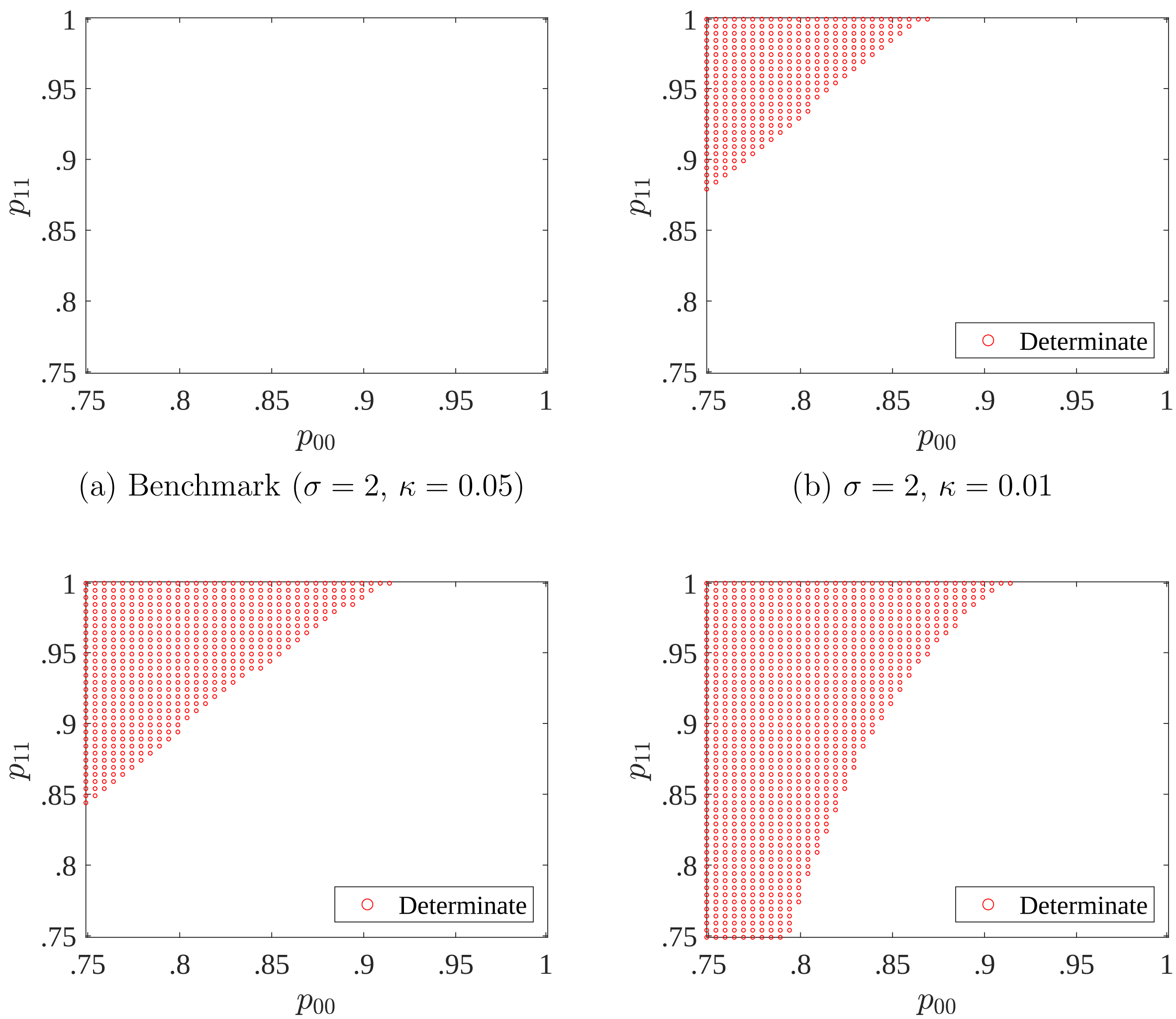
For each rule, we consider a range of transition probability calibrations that imply a transient ($p_{00} < 1$) and recurring ($p_{11} < 1$) ZLB regime. We use the average duration of each regime, k, which is given by $(1 - p_{kk})^{-1}$, for $k = 0, 1$, to identify reasonable values of (p_{00}, p_{11}). Recent works estimate the expected duration of the binding ZLB regime for the U.S. economy for the period of 2008 to 2015. For example, Swanson and Williams (2014) find that the Blue Chip expectation of the ZLB duration fluctuated between two and five quarters prior to the Fed's calendar-based forward guidance in 2011, when the expected duration increased to seven or more quarters and the median expected duration in the New York primary dealer survey increased to nine quarters.[27] Kulish et al. (2017) estimate the path of expected durations of the ZLB and obtain similar results, ranging from three to 12 quarters. In our model, $p_{00} = 0.75$ corresponds to an expected duration of four quarters; $p_{00} = 0.8$ corresponds to five quarters; $p_{00} = 0.9$ corresponds to 10 quarters; $p_{00} = 0.917$ corresponds to 12 quarters; and $p_{00} = 0.95$ corresponds to 20 quarters.

4.1 Inflation Targeting

The baseline model with inflation targeting is formed by (9), (10) and (13). We first examine the determinacy properties of the model as a function of the transition probabilities p_{00} and p_{11}, with all other parameters set at their benchmark values. Panel (a) of Figure 1 shows that the benchmark calibration model with recurring ZLB episodes and inflation targeting is indeterminate for all empirically plausible values of p_{00} and p_{11}. However, panels (b) and (c) of Figure 1 show that the determinacy region in the (p_{00}, p_{11})-space expands as the Phillips curve flattens (κ decreases) or as risk aversion increases (σ increases), holding all other parameters at the benchmark values. Intuitively, a lower κ or higher σ reduces the positive, expectations-destabilizing feedback from the expectations to the equilibrium outcomes which give rise to extraneous self-fulfilling fluctuations. A lower κ reduces the sensitivity of inflation to the output expectations because $\pi_t = \kappa \sum_{k \geq 0} \beta^k E_t y_{t+k}$; and a higher σ reduces the sensitivity of output to the real interest rate expectations because

[27] The forecast horizon in the Blue Chip consensus expectation of the ZLB duration is only six quarters. See Swanson and Williams (2014) for more.

Figure 1: Determinacy and Inflation Targeting (the Taylor rule)



(a) Benchmark ($\sigma = 2$, $\kappa = 0.05$)

(b) $\sigma = 2$, $\kappa = 0.01$

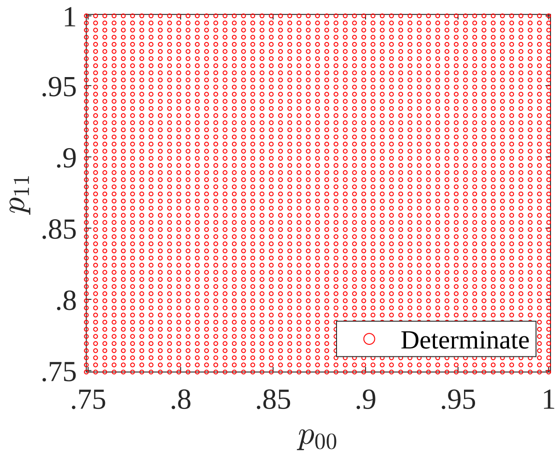
(c) $\sigma = 5$, $\kappa = 0.01$

(d) $\phi_\pi = 5$, $\sigma = 5$, $\kappa = 0.01$

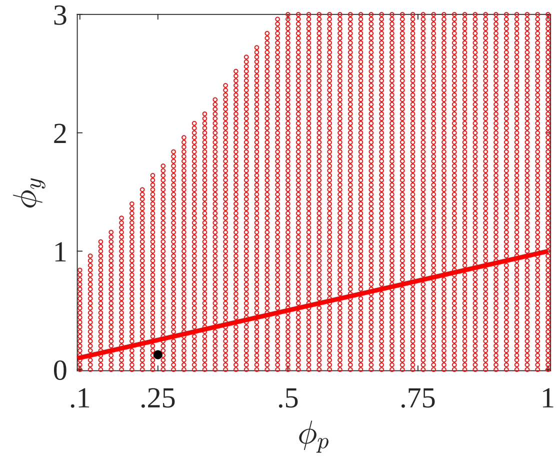
Note: The REE for inflation targeting rules is depicted with respect to p_{00} and p_{11} as follows. The red (circle) area denotes determinacy; and the white area denotes indeterminacy. Other model parameters are given by $\beta = 0.9975$, $\phi_\pi = 2$, and $\phi_y = 0.5/4$ throughout this exercise unless noted otherwise.

$y_t = -\sigma^{-1} \sum_{k \geq 0}(i_{t+k} - E_t \pi_{t+k+1})$. For brevity, Figure 1 does not depict the effects of reducing β. However, a lower β also reduces positive expectational feedback, thus enlarging the determinacy regions. Furthermore, aggressive active monetary policy (i.e., higher ϕ_π) can enlarge the determinacy regions when κ is sufficiently low and σ is sufficiently high (see panel (d) of Figure 1). Finally, panels (b)–(d) of Figure 1 show that the low frequency (a high value of p_{11}) and short duration (a low value of p_{00}) of ZLB events are key to ensuring

Figure 2: Determinacy and Price Level Targeting



(a) Benchmark ($\phi_p = 0.25$, $\phi_y = 0.5/4$)



(b) $p_{00} = 0.965$, $p_{11} = 0.975$

Note: The REE for a price level targeting rule is depicted with respect to p_{00} and p_{11} in panel (a) and with respect to ϕ_p and ϕ_y in panel (b) as follows. The red (circle) area denotes determinacy; the white area denotes indeterminacy. The black square in panel (b) depicts the benchmark calibration, and the red line is the set of points satisfying the nominal income targeting restriction $\phi_p = \phi_y$. The model parameters are set at the benchmark values throughout this exercise unless noted otherwise.

a unique REE.

We conclude that inflation targeting under a Taylor-type rule is prone to indeterminacy in a model subject to recurring ZLB events. However, this problem of indeterminacy is mitigated provided the Phillips curve is sufficiently flat, monetary policy is very active away from the ZLB, or agents are very risk averse.

4.2 Price level targeting

Now we consider determinacy under a price level targeting rule of the form (14). The result is stark, and consistent with the findings from the simple Fisherian model with price level targeting (i.e., Proposition 2): for sufficiently small ϕ_y, we have determinacy for all (p_{00}, p_{11}), provided $p_{00} < 1$. We have determinacy even if ZLB events are more frequent and persistent than are the unconstrained policy regimes (i.e., $p_{11} < p_{00}$). Figure 2 (a) illustrates the basic result for the benchmark calibration. Furthermore, in line with Proposition 2, we could instead set $\phi_p = 0.0001$ and $\phi_y = 0$ and also obtain the same determinacy region as depicted in Figure 2 (a). Finally, when $\phi_p = \phi_y$ such that the price level targeting rule (14) implements nominal GDP targeting policy, we again find the determinacy region depicted in Figure 2(a).[28] Therefore, nominal income targeting policy is a highly effective means of stabilizing expectations when the ZLB is expected to bind frequently.

Proposition 2 and the numerical results presented in this section indicate that policymakers can almost always manage rational expectations under price level targeting or nominal income targeting in an economy subject to ongoing ZLB events. However, there is a caveat: policymakers may destabilize the economy when responding too aggressively to output relative the price level in (14). Figure 2 (b) shows that large values of ϕ_y (relative to ϕ_p) may lead to indeterminacy for a calibration of p_{00} that matches the persistence of the U.S. ZLB episode of 2008 to 2015. This result suggests that central banks targeting the price level should respond only mildly to output.

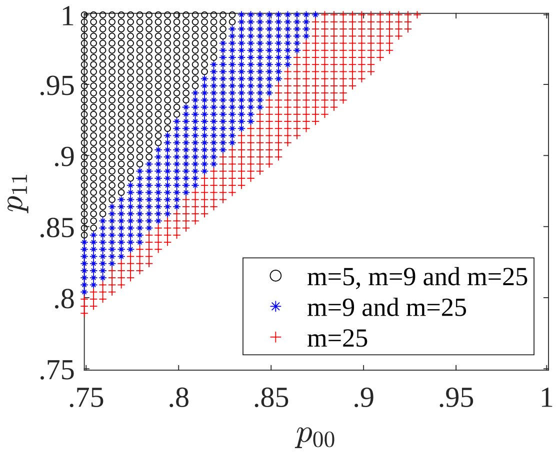
4.3 Average inflation targeting

In this section, we examine the model incorporating the average inflation targeting rule (15), the New Keynesian Phillips curve (9), and IS curve (10). We consider three measures of average inflation, indexed by $m = 5, 9, 25$, that vary in the degree of history-dependence. These values of m correspond to a target that averages over the past year, two years, and six years of inflation data, respectively. As mentioned in Section 2.1, we also consider two interpretations of the average inflation targeting rule. First, we set $\phi_{\bar{\pi}} = \phi_\pi$, such that the central bank targets a simple average of the most recent m quarters of inflation data. Second, we set $\phi_{\bar{\pi}} = \phi_\pi m$ and interpret the target as an unweighted sum of the most recent m quarters of inflation. In this policy framework, average inflation targeting with $\phi_{\bar{\pi}} = \phi_\pi m$ converges to (14) as $m \to \infty$, as argued in Section 3.

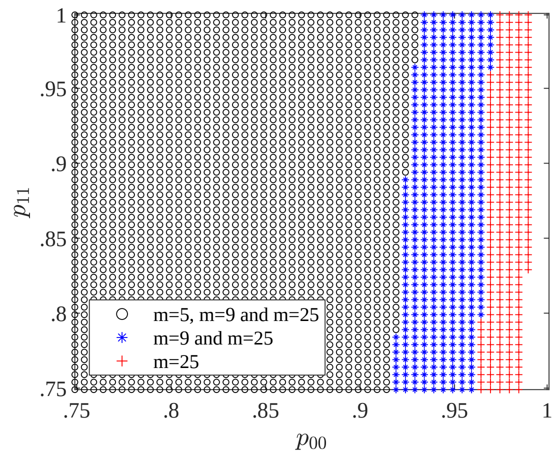
As shown in Figure 3, under both interpretations of the rule, a higher m expands the determinacy region in the (p_{00}, p_{11})-space. Thus, we have the smallest determinacy region for inflation targeting (13), which is equivalent to (15) with $m = 1$, and the largest determinacy region for the price level targeting rule, (14), which can be obtained from (15) as $m \to \infty$.

[28] We explored determinacy regions in the model with many values of $\phi_p \in [0.0001, 100]$ given small ψ_y and also $\phi_p = \phi_y \in [0.0001, 100]$. We invariably obtain the determinacy region depicted in Figure 2(a).

Figure 3: Determinacy and Average Inflation Targeting



(a) Simple Average Target ($\phi_{\bar{\pi}} = \phi_\pi$).



(b) Unweighted Target ($\phi_{\bar{\pi}} = \phi_\pi m$).

Note: The REE for average inflation targeting with various target windows of m is depicted with respect to p_{00} and p_{11}. The black (circle) region is the determinacy region for $m = 5$; the determinacy region for $m = 9$ consists of the black and blue (asterisk) regions; the determinacy region for $m = 25$ consists of the black, blue, and red (plus) regions; and the white region denotes indeterminacy. The model parameters are set at the benchmark values throughout this exercise unless noted otherwise.

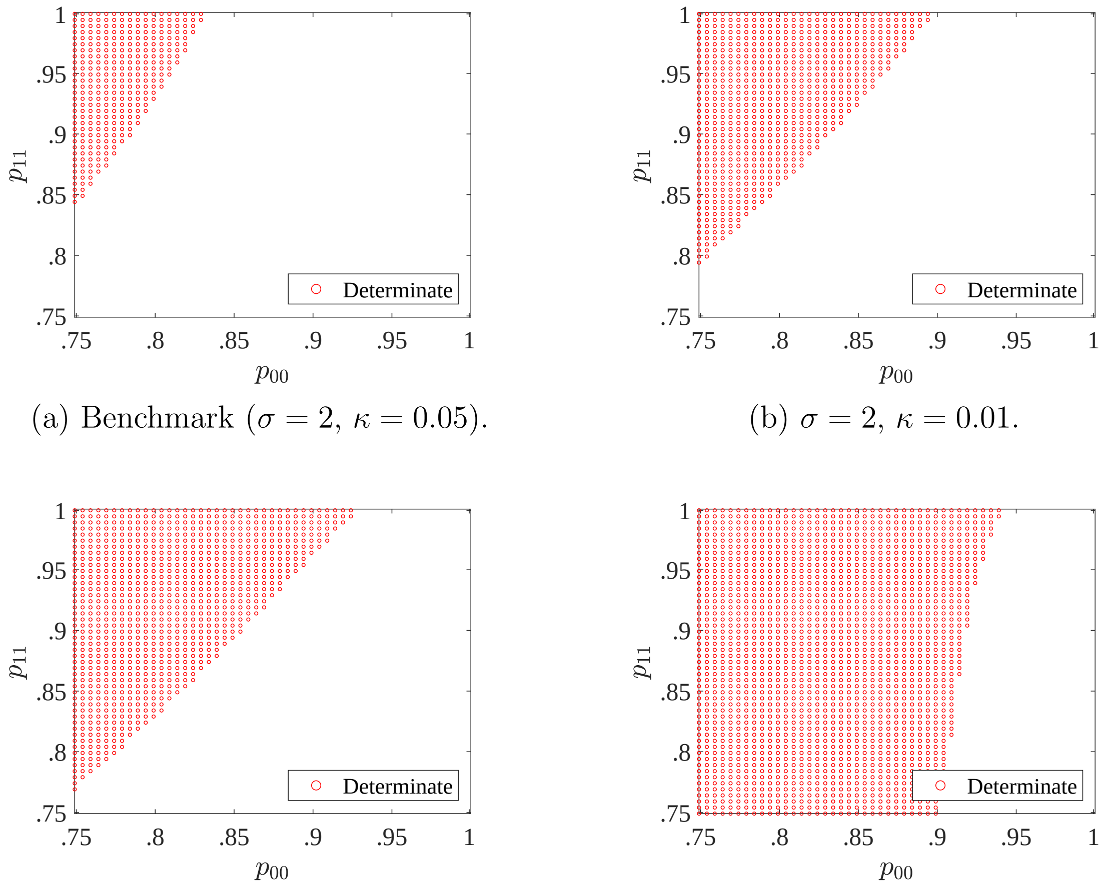
In addition, as expected, the determinacy regions under the unweighted sum interpretation ($\phi_{\bar{\pi}} = \phi_\pi m$) are strictly larger than those under the simple average inflation target interpretation ($\phi_{\bar{\pi}} = \phi_\pi$), because the former generates a more active response to inflation during unconstrained regimes.

Figure 4 demonstrates the effects of varying κ, σ, β, and ϕ_π under the simple average inflation target interpretation ($\phi_{\bar{\pi}} = \phi_\pi$) when $m = 5$. As we found in our determinacy analysis with inflation targeting, a lower κ, higher σ, lower β, and higher ϕ_π enlarge the determinacy regions under average inflation targeting. The same intuition as before applies to the average inflation targeting case.

5 Adaptive Learning and E-stability

This section considers the New Keynesian model of Section 2.1, and examines the E-stability properties for each policy rule. We also demonstrate useful applications of E-stability, and consider robustness issues.

Figure 4: Determinacy and Average Inflation Targeting under Alternative Parameterizations



(a) Benchmark ($\sigma = 2$, $\kappa = 0.05$).

(b) $\sigma = 2$, $\kappa = 0.01$.

(c) $\sigma = 5$, $\kappa = 0.01$.

(c) $\sigma = 5$, $\kappa = 0.01$, $\phi_{\bar{\pi}} = 5$.

Note: For various parameterizations, the REE for an average inflation targeting rule with $m = 5$ is depicted with respect to p_{00} and p_{11} as follows: red area (determinacy) and white area (indeterminacy). This figure assumes the simple average inflation targeting rule in (15) with $\phi_{\bar{\pi}} = \phi_{\pi}$. The model parameters are set at the benchmark values throughout this exercise unless noted otherwise.

5.1 (In)Determinacy and E-(in)stability

Our discussion in Section 2.4 predicts the first basic conclusion of our E-stability analysis under the model of adaptive learning: a determinate equilibrium is E-stable. However, while determinacy implies E-stability, the converse is not true. This implies that using E-stability as a criterion instead of determinacy/indeterminacy may alter the evaluation of alternative policy rules. Therefore, we can think of the E-stability criterion as a minimal requirement for stabilization policy. We examine this possibility using our fully fledged New

Keynesian model developed in Section 2.1. Figure 5 (a) shows that inflation targeting (13) is capable of generating a unique E-stable MSV solution, despite model indeterminacy over the entire parameter space of (p_{00}, p_{11}). Similarly, Figure 5 (b) shows that indeterminate models with the average inflation targeting rule (15) admit E-stable solutions. Finally, because price level targeting (14) generates determinacy regions that virtually exhaust the model parameter space (assuming ϕ_y is not too large), we find that E-stability regions also exhaust the policy parameter space. In short, while inflation targeting can be associated with E-stability for some parameter region that is associated with indeterminacy, and average inflation targeting has a larger parameter region of E-stability than that for determinacy, price level targeting always leads to determinacy *and* E-stability for the entire parameter space considered (assuming ϕ_y is not too large). Therefore, price level targeting is the most effective stabilization policy according to our determinacy and E-stability analysis.[29]

5.2 E-stability and Deflationary Spirals

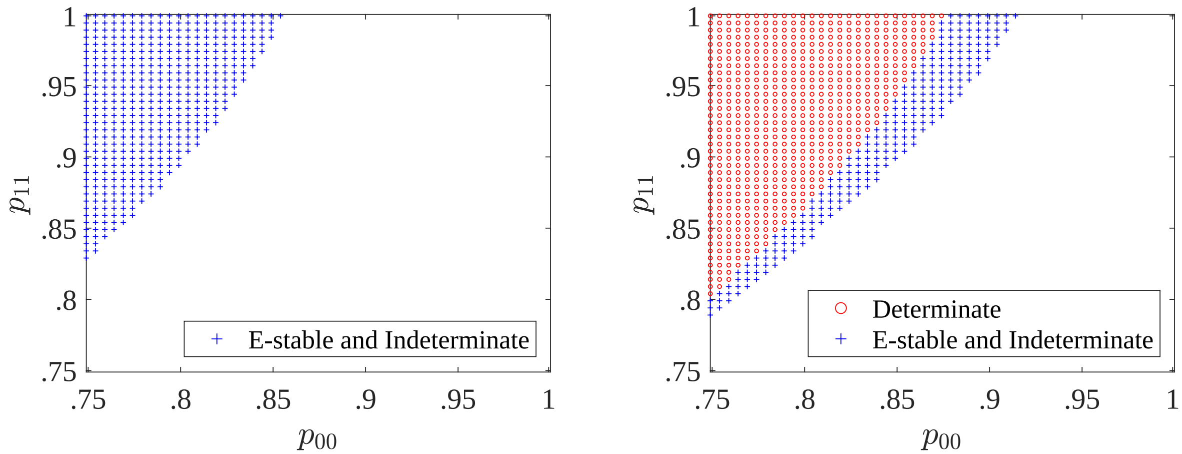
It has been well-established that deflationary spirals may occur under adaptive learning when interest rates are pegged at zero (e.g., see Evans et al. (2008)). This section illustrates how and why deflationary spirals are absent at the ZLB when agents are learning adaptively but are sufficiently optimistic about the possibility of escaping the current liquidity trap (i.e., they forecast regime changes, and p_{00} (p_{11}) is sufficiently low (high) to deliver an E-stable REE). For the purpose of this analysis, we define a deflationary spiral as

$$\lim_{t\to\infty} E_0\pi_t = -\infty.$$

The usual mathematical expectation operator E_0 denotes model-consistent expectations (which may not coincide with agents' expectations).

[29] We also considered the following lagged information assumption: $(P, x_{t-1}, s_t, v_t) \in \mathcal{I}_t$, but $x_t \notin \mathcal{I}_t$. Our numerical E-stability results for the model (9)-(10) and any one of the policy rules from (13)-(15) are the same under this alternative lagged information assumption as those under the contemporaneous information assumption. Also note that it is quite uncommon for indeterminate models to yield an E-stable REE in standard linearized models under assumptions analogous to those in Section 2. For example, the model given by (9), (10), and (13) *without regime switching* admits an E-stable (and dynamically stable) solution if and only if the model is determinate. See Bullard and Mitra (2002) and Evans and McGough (2005) for more details.

Figure 5: Determinacy and E-stability



(a) Inflation Targeting

(b) Avg. Inflation Targeting ($\phi_{\bar{\pi}} = \phi_{\pi}$)

Note: The REE for inflation targeting and average inflation targeting are depicted with respect to p_{00} and p_{11} in panels (a) and (b), respectively. The red (circle) area denotes determinacy and E-stability; the blue (plus) area denotes E-stability and indeterminacy; and the white area denotes E-instability and indeterminacy. Average inflation targeting has the target window of $m = 9$ for all the simulations.

5.2.1 E-stability and Convergence to REE

First, we demonstrate numerically that E-stability can predict convergence of the learning equilibrium law of motion to the mean-square stable REE law of motion. Formally, convergence obtains in real time if $(a(s_t)_t, b(s_t)_t, c(s_t)_t) \rightarrow (\Gamma(s_t), \Omega(s_t), Q(s_t))$ for $s_t = 0, 1$ as $t \rightarrow \infty$, given $(a(s_t)_0, b(s_t)_0, c(s_t)_0)$ in some suitable neighborhood of the REE. If the learning equilibrium law of motion converges to the REE law of motion in real time then deflationary spirals will not occur because mean-square stability ensures that $lim_{t\rightarrow\infty} E_0 \pi_t$ is finite (e.g., see Cho (2016) for details).

To demonstrate convergence in practice, we suppose agents update $\Phi(k)_t = (a(k)_t, b(k)_t, c(k)_t)'$ using the recursive estimator

$$\Phi(k)_t = \Phi(k)_{t-1} + \psi(k)_t R(k)_t^{-1} z_t (x_t - \Phi(k)'_{t-1} z_t)' \quad (29)$$

$$R(k)_t = R(k)_{t-1} + \psi(k)_t (z_t z'_t - R(k)_{t-1}) \quad (30)$$

where $z_t = (1 \ x_t' \ v_t')'$, $\psi(k)_t = 1/t_k^{\alpha}$ if $s_t = k$ and 0 otherwise, t_k is the number of periods such that $s_t = k$, $\alpha \in (0, 1]$ and $k = 0, 1$.[30] Intuitively, (29)-(30) is a recursive (weighted) least squares estimator of the two linear regime-dependent PLMs.

Under contemporaneous information,[31] time-t equilibrium is determined as follows.

Step 1 At the end of $t-1$, agents update $\Phi(k)_{t-1}$ using time-$t-1$ information and (29)-(30).

Step 2 At time-t, temporary equilibrium is given by substituting $\Phi(k)_{t-1}$ into (22).

We can repeat Steps 1 and 2 to solve for temporary equilibrium at $t+1$ and so on. Figure 6 illustrates convergence to the REE law of motion in cases where E-stability conditions are satisfied, and divergence in cases where the conditions are not satisfied. Each panel of the figure illustrates the maximum distance between agents' current estimate of any of the K coefficients of $\Phi(k)_t$ for $k = 0, 1$ (i.e., $b_{j,t}$ for $j = 1, \ldots, K$) and the true REE value of that coefficient (i.e., $\bar{b}_j$). Convergence occurs if $\max_j |b_{j,t} - \bar{b}_j| \to 0$ as t increases. In each panel we assume that initial beliefs, $\Phi(k)_0$, are different from the true REE beliefs, and then simulate each model 50 times. Further, we set $p_{11} = 0.975$ and consider two different values of p_{00} to generate both E-stable and E-unstable models. Particular attention is paid to $p_{00} = 0.965$, which ensures an expected ZLB duration equal to the duration of the 2008-2015 U.S. ZLB event.[32] The other value is given by $p_{00} = 0.99$, which leads to a longer expected duration of the ZLB than $p_{00} = 0.965$. All other parameters are set at the benchmark values as discussed in Section 4. Across the simulations, we observe convergence to REE when E-stability is obtained and $\Phi(k)_0$ is sufficiently close to the REE, and divergence when E-stability conditions are not satisfied.

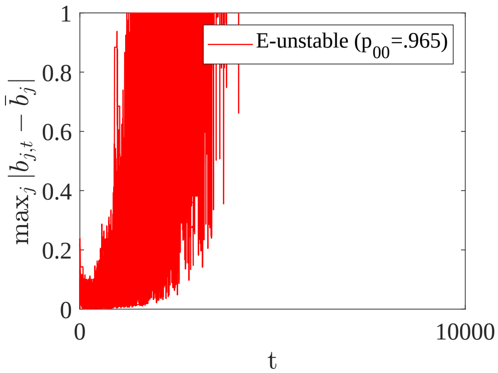
As depicted in Figure 6, the coefficients in the simulation with inflation targeting for both values of p_{00} diverge away from their REE coefficients, whereas those for price level targeting

[30] We also consider cases with constant gain (i.e. $\psi(k)_t = \psi \in (0, 1]$) as noted below.

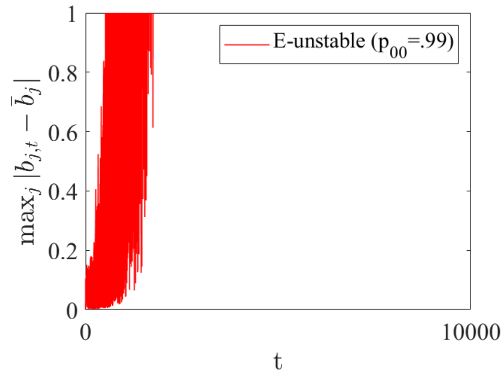
[31] The qualitative results reported in this section hold if agents do not contemporaneously observe x_t.

[32] The discrete-valued shocks are calibrated so that $E(i_t^*|s_t = 0) < -\bar{i}$ and $E(i_t^*|s_t = 1) > -\bar{i}$ where i^* denotes the shadow rate in the REE. To keep the speed of learning high in the simulations, we impose $\psi(k)_t = \max\{1/t_k^{2/3}, .04\}$, though convergence would occur more gradually under the alternative assumption of decreasing gain ($\psi(k)_t = 1/t_k^{\alpha}$). Finally, we impose $\sigma_d = \sigma_s = 0.0001$ to ensure that $R(k)_t$ is nonsingular in simulations.

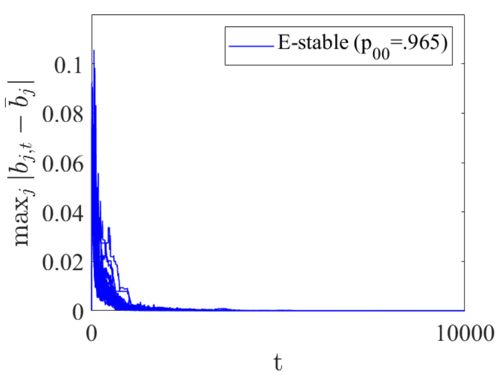
Figure 6: Learning and Convergence to the REE



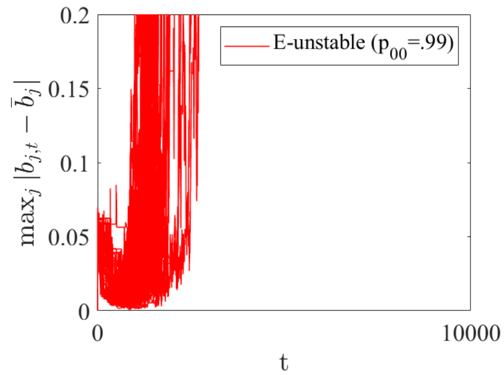
(a) Inflation Targeting ($p_{00} = 0.965$)



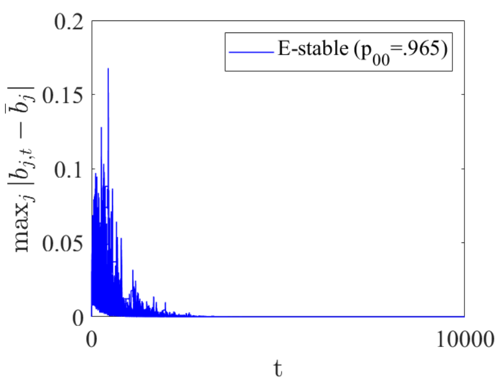
(b) Inflation Targeting ($p_{00} = 0.99$)



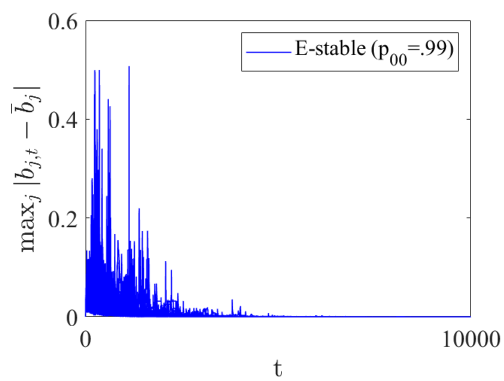
(c) Avg. Inflation Targeting ($p_{00} = 0.965$)



(d) Avg. Inflation Targeting ($p_{00} = 0.99$)



(e) Price Level Targeting ($p_{00} = 0.965$)



(f) Price Level Targeting ($p_{00} = 0.99$)

Note: The convergence results for various targeting rules are plotted for simulations. Convergence occurs if $\max_j |b_{j,t} - \bar{b}_j| \to 0$ as t increases where $b_{j,t}$ in (29) is the agents' current estimate of the coefficient $\bar{b}_j$ in the REE. The transition probability of p_{00} indicates the probability that the economy remains at the ZLB in the next period. Average inflation targeting has the target window of $m = 9$ for all the simulations. Some plotted simulations feature flat line segments that are a consequence of off-equilibrium beliefs being fixed.

converge to their REE values. This shows that price level targeting can promote real-time learning of REE more effectively than inflation targeting. In addition, the simulation for average inflation illustrates the importance of beliefs about transition probabilities; under $p_{00} = 0.965$ and average inflation targeting, beliefs converge to the REE values, but they diverge away from the REE if $p_{00} = 0.99$. The optimistic expectations of a shorter ZLB duration help promote E-stability. Following the same intuition, we find that the optimistic expectations (i.e., a shorter expected duration of the ZLB event such as $p_{00} = 0.965$) are associated with a quicker convergence for price level targeting, and a slower divergence for inflation targeting compared to the cases involving a longer expected ZLB duration such as $p_{00} = 0.99$. These simulation results confirm our findings in Section 5.1.

5.2.2 Expectations-Driven Liquidity Traps

If initial beliefs are sufficiently "local" to the rational beliefs, then E-stability predicts convergence to REE. However, if agents' initial inflation expectations are very low ("pessimistic") relative to the rational expectations then the ZLB may endogenously bind under learning in a state of the world where the REE interest rate would be positive. In such cases, the E-stability conditions may still prove useful for predicting the dynamic stability of inflation at the ZLB.

To illustrate this point, we set up a simple experiment in the model (9)-(10) paired with one of the policy rules, in which agents' initial pessimistic beliefs (i.e., *not* the fundamental shock to demand) causes the ZLB to bind at time $t = 0$. In this experiment, we shut down all fundamental shocks (i.e., $v_t = u_t = 0$) to isolate the role that regime-switching expectations play in preventing a deflationary spiral under learning. Thus, we restrict our attention to an "expectations-driven liquidity trap" which occurs because expectations are initially unanchored from rational expectations and not because of some fundamental shock to the economy.[33] Under these assumptions, the model can be cast in the form:

$$x_t = A(s_t)\hat{E}_t x_{t+1} + B(s_t)x_{t-1} + C(s_t) \tag{31}$$

[33] Our notion of expectations-driven liquidity trap should not be confused with the expectations-driven liquidity traps studied by Mertens and Ravn (2014) or Nakata and Schmidt (2020) which occur as a consequence of sunspots in a REE.

where $s_t = 0$ if the ZLB binds at time-t and $s_t = 1$ otherwise. To pin down s_t without fundamental shocks we must clarify the timing of temporary equilibrium: in the beginning of t agents assume $s_t = 0$ ($s_t = 1$) when forming time-t expectations, unless they observed positive (zero) interest rates at the end of $t-1$. We use $\tilde{s}_t = s_{t-1}$ to denote the agents' subjective beliefs about s_t. The agents believe the ZLB regime has persistence $p_{00} < 1$ and the unconstrained regime has persistence $p_{11} = 1$. We assume the PLM:

$$x_t = a(\tilde{s}_t)_{t-1} + \Omega(\tilde{s}_t)x_{t-1} + \epsilon_t \tag{32}$$

where $\Omega(s_t)$ is from the REE law of motion and $a(s_t)_t$ evolves according to a version of (29)-(30):[34]

$$a(k)_t = a(k)_{t-1} + \psi(k)R(k)_t^{-1}(x_t - \Omega(k)x_{t-1} - a(k)_{t-1}) \tag{33}$$

$$R(k)_t = R(k)_{t-1} + \psi(k)(1 - R(k)_{t-1}) \tag{34}$$

where $\psi(k) = \psi \in (0, 1)$ if $s_t = k$ and 0 otherwise, and $k = 0, 1$. We assume that ψ is constant for exposition's sake. We set $a(0)_{-1}$ to a non-zero (pessimistic) value which causes the ZLB to initially bind at $t = 0$, and we assume $a(1)_{-1}$ is the zero vector which implies that agents believe the economy will eventually return to the intended steady state with positive interest rates if the (perceived) transient ZLB event ends. Note that $a(0)_{-1} \neq 0$ is not rational since the absence of fundamental shocks implies that the economy is always in the intended steady state in the corresponding REE.[35]

After agents form time-t expectations using beliefs $a(k)_{t-1}$, equilibrium interest rates are set according to $\max\{i_t^*, -\bar{i}\}$, and this determines s_t. The temporary equilibrium is therefore:

$$x_t = \left(I - A(s_t)\hat{E}_t\Omega(\tilde{s}_{t+1})\right)^{-1}\left(A(s_t)\hat{E}_t a(\tilde{s}_{t+1})_{t-1} + B(s_t)x_{t-1} + C(s_t)\right).$$

[34] Conventional adaptive learning models assume that agents learn $\Omega(s_t)$. However, in the models we now consider, learning the intercept term is a far more demanding task for learning agents, and so we simplify the analysis and abstract from the role of initial beliefs about $\Omega(s_t)$ by assuming it is known by the agents. Without much loss of generality, one may assume the REE is the unique, mean-square stable MSV solution.

[35] In this section and throughout the paper, we abstract from the possibility that rational agents coordinate on the second steady state equilibrium with permanently binding ZLB.

Note that in our setup, $s_t = \tilde{s}_t$ when the economy is at the ZLB and in all periods after lift-off from the ZLB. Only in the period of lift-off will $\tilde{s}_t = 0$ and $s_t = 1$. Under these assumptions we ask: will a deflationary spiral take place in real time?

Through substitution we can reduce the law of motion for $a(0)_t$ during the ZLB regime to a simple VAR(1) process:

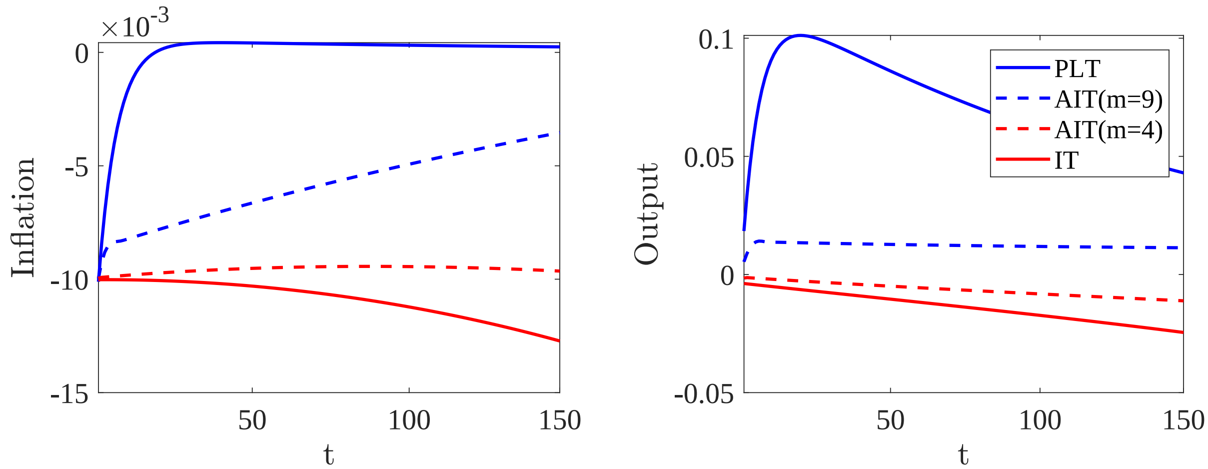
$$
\begin{aligned}
a(0)_t &= \left(I + \psi R(0)_t^{-1} \left(p_{00} F(0) - I\right)\right) a(0)_{t-1} + \\
&\quad \psi R(0)_t^{-1} \left((1 - p_{00}) F(0) a(1)_{t-1} + \left(I - A(0) \sum_{j=0}^{1} p_{0j} \Omega(j) \right)^{-1} C(0) \right) \\
F(0) &= \left(I - A(0)(p_{00}\Omega(0) + (1 - p_{00})\Omega(1))\right)^{-1} A(0).
\end{aligned}
$$

Since $R(0)_t \to 1$ almost surely and $a(1)_t = a(1)_{-1} = 0$ is fixed while agents are updating beliefs at the ZLB, $a(0)_t$ converges if the eigenvalues of $p_{00}F(0)$ are inside the unit circle. By inspecting (31)-(33) jointly, we can deduce two implications of convergence of $a(0)_t$. First, since $a(1)_t$ is fixed at the ZLB, x_t will converge if $a(0)_t$ converges. Second, x_t can converge towards levels of inflation and output for which the ZLB no longer binds if agents have sufficiently high expectations for long-run levels of inflation and output in the unconstrained monetary regime (e.g. if $a(1)_{-1}$ is the zero vector). Thus, the economy avoids a deflationary spiral and can escape the expectations-driven liquidity trap for *any* $a(0)_{-1}$ if the eigenvalues of $p_{00}F(0)$ are strictly inside the unit circle.[36] As it turns out, this condition on $p_{00}F(0)$ is a special case of the E-stability conditions applied to a model with ZLB regime persistence equal to p_{00} and an absorbing positive interest rate regime, which one can verify from the general E-stability conditions stated in Appendix A.4.[37] Therefore, E-stability is useful for predicting the possibility of diverging inflation dynamics under learning at the ZLB–even if agents' initial pessimism causes the ZLB to bind for an indefinite period of time. Figure 7 illustrates expectations-driven liquidity traps assuming the benchmark calibration and

[36] Note that if $p_{00}F(0)$ has eigenvalues outside the unit circle then the economy can still escape the endogenous liquidity trap, but only for *some* initial conditions on $a(0)_{-1}$ (e.g. see Mertens and Ravn (2014)). Also note that agents' initial beliefs about $\Omega(s_t)$ introduce additional initial conditions that affect convergence (see section 5.2.1).

[37] From Appendix A.4., E-stability obtains if the real parts of the eigenvalues of $p_{00}F(0)$ and $F(1) = \left(I - A(1)\Omega(1)\right)^{-1} A(1)$ are less than one. For our calibration, the eigenvalues of $F(1)$ are inside the unit circle.

Figure 7: Learning in an Expectations-Driven Liquidity Trap

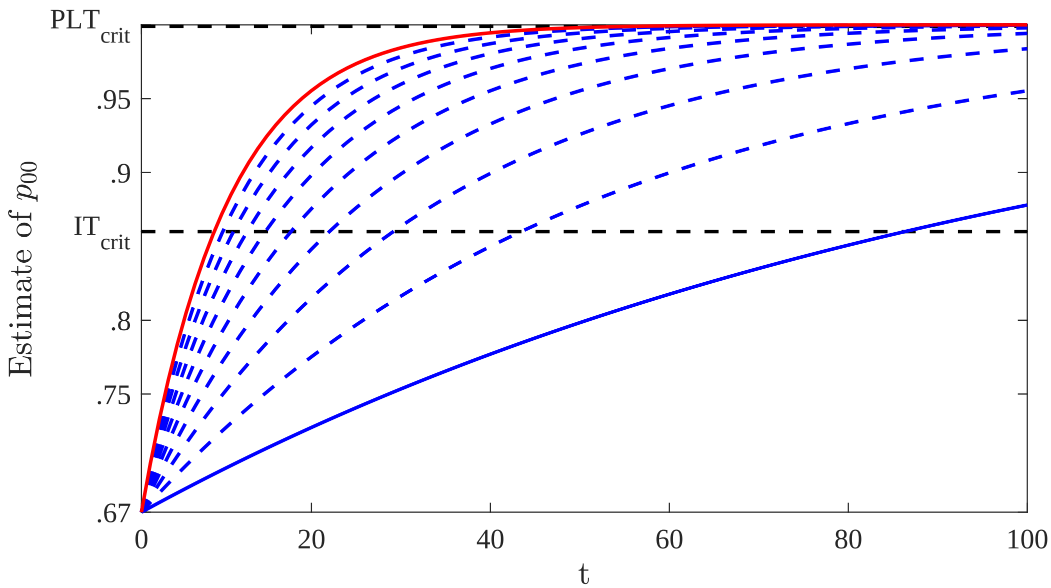


Note: The vertical axis represents the percentage deviation (e.g., 0.01 equals 1% quarterly inflation rate). Blue (red) lines depict E-stable (E-unstable) calibrations. In the E-stable simulations, the economy escapes the expectations-driven liquidity trap, whereas deflationary spirals occur in the E-unstable simulations. Note that PLT, AIT, IT denote price level targeting, average inflation targeting and inflation targeting, respectively.

$p_{00} = 0.965$. As expected, price level targeting rules and average inflation targeting rules with long averaging window outperform the simple inflation targeting rule. Deflationary spirals occur under inflation targeting and average inflation targeting with short averaging window.

The condition on $p_{00}F(0)$ reveals that agents' *beliefs* about the persistence of the ZLB, i.e., p_{00}, matter for stability under learning at the ZLB, and not the actual duration of the ZLB events per se. This result is a consequence of a basic self-referential feature of the model: today's beliefs influence temporary equilibrium, which in turn, influences future beliefs. For instance, low inflation expectations are confirmed by low equilibrium inflation in the absence of active monetary policy, which can result in even lower future inflation expectations, and eventually, a deflationary spiral. However, this self-referential feature of the model is diminished by the expectation that the economy may escape the ZLB (i.e., $p_{00} < 1$). To see this, consider (9)-(10) and (13), and note that $\hat{E}_t(x_{t+1}|s_t = 0) = p_{00}a(0)_{t-1} + (1-p_{00})a(1)_{t-1}$ under the correctly-specified PLM (21), where $a(1)_{t-1}$ is fixed while $s_t = 0$. Only $a(0)_{t-1}$ evolves at the ZLB, and $\partial x_t / \partial a(0)_{t-1} = A(0)p_{00}$, so that the actual economy, x_t, is more respon-

Figure 8: Learning the Transition Probability



Note: IT$_{crit}$ and PLT$_{crit}$ denote the maximum value of p_{00} that delivers an E-stable REE under inflation targeting and price level targeting, respectively. The solid red (blue) line shows learning dynamics when $\psi = 0.1$ ($\psi = 0.01$); dashed blue show dynamics for values of $\psi \in [0.01, 0.1]$.

sive to changes in beliefs, $a(0)_{t-1}$, for higher values of p_{00}. Thus, we mitigate destabilizing feedback from expectations to reality and therefore to future expectations by decreasing the expected ZLB duration, p_{00}. This intuition also rationalizes why persistent (a high value of p_{11}) and frequent (a low value of p_{00}) active monetary regimes are key to the existence of E-stable REE, and suggests that a good monetary policy should promote determinacy and E-stability over the largest possible set of transition probabilities, (p_{00}, p_{11}). Therefore we can conclude that the price level targeting rule (14) dominates inflation targeting (13) and average inflation targeting (15) in terms of stabilizing the economy.

5.3 Learning the Persistence of the ZLB

Our analysis indicates that agents' beliefs about the persistence of the ZLB are an important determinant of E-stability, and as a starting point, we assumed fixed beliefs about p_{00} and p_{11}. However, agents could be expected to revise their transition probability beliefs, for example, if they experience a particularly deep and persistent liquidity trap. Therefore, we relax the assumption of the fixed transition probability belief. In this thought experiment, agents are allowed to revise their beliefs about the transition probability using the following recursive estimator:

$$\hat{p}_{kk,t} = \hat{p}_{kk,t-1} + \psi(k)_{t-1}(\xi(k)_t - \hat{p}_{kk,t-1}) \tag{35}$$

where $\psi(k)_{t-1} > 0$ if $s_{t-1} = k$ and 0 otherwise, and $\xi(k)_t = 1$ if $s_t = k$ and 0 otherwise. In each period, agents also update their beliefs about the coefficients in their PLM (21) using (29)-(30), and time-t expectations are formed using recent estimates of the PLM coefficients and also $\hat{p}_{kk,t-1}$.[38] During a very prolonged liquidity trap, such as the episodes featured in Figure 7, agents' estimates of p_{00} could increase to levels where the E-stability conditions fail and inflation becomes dynamically unstable in real time. Figure 8 illustrates the evolution of beliefs about p_{00} according to (35) for different constant-gain parameters, $\psi(k_t) = \psi \in [0.01, 0.1]$ and given an initial belief equal to 0.67 in a ZLB event that lasts more than 100 consecutive quarters. The figure displays the maximum value of p_{00} that gives an E-stable REE under the inflation targeting and price level targeting rules, respectively, assuming the benchmark calibration and $p_{11} = 1$. Once the agents' estimate of p_{00} exceeds these maximum values, dynamically unstable dynamics may arise at the lower bound. It is evident from the figure that an inflation targeting rule poses risks to instability under learning about the transition probability. If agents have a high gain parameter (e.g., $\psi = 0.1$) then unstable inflation dynamics may arise at the ZLB in fewer than 10 quarters. On the other hand, the price level targeting rule ensures stable inflation for arbitrarily high values of $p_{00} < 1$, and hence evolving beliefs about the persistence of the ZLB will never pose a threat to stability under price level targeting. Again, our results are in favor of the price level targeting rule over its alternatives.

[38]From the perceived transition probability law of motion, (35), $p_{kk,t}$ is entirely determined by s_t, and therefore we can study the evolution of perceived transition probabilities in isolation from the remaining equations in the model.

6 Conclusion

This study evaluates policy rules that respond to average inflation and price level as well as output instead of the period inflation rate (a standard Taylor rule) using the criteria of determinacy of an REE and the learnability of the equilibrium in a standard New Keynesian model subject to persistent, recurring ZLB episodes. Our results are strongly in favor of the price level targeting framework as effective stabilization policy, which gives a unique, learnable equilibrium in models even with extremely persistent ZLB events. Thus, under price level targeting, policymakers should be less worried about sunspots, and deflationary spirals under learning. Nominal income targeting can be understood as a special case of the price level targeting rules when the reaction coefficients on price level and output are the same. We also find that average inflation targeting rules can promote determinacy and E-stability very effectively, provided that the measure of average inflation is sufficiently backward looking. However, standard Taylor rules that implement a simple inflation targeting policy are prone to indeterminacy and possibly E-instability. These findings have important implications for stabilization policy in the current low interest rate environment.

Several avenues for future work are available. For instance, we take the frequency of ZLB events as given in order to show that the expected duration and frequency of these events are key determinants of determinacy and E-stability under various policy rules. Future work might instead investigate whether some policy rules help to avoid liquidity traps altogether, or minimize their welfare costs when they do occur under learning. Future work should also address E-stability under the infinite-horizon learning approach of Preston (2005).

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Appendices

A.1. Proof of Proposition 1

Consider a slightly modified version of the Taylor rule (13):

$$i_t = \tilde{s}_t(\phi_\pi \pi_t + \phi_y y_t) - (1 - \tilde{s}_t)\bar{i}$$

where $\tilde{s}_t = \epsilon$ if $s_t = 0$ and ϵ is some arbitrarily small positive constant; and $\tilde{s}_t = 1$ otherwise. We introduce $\tilde{s}_t$ to ensure that the inflation process is well defined when we combine the modified Taylor rule with (26), yielding the following Markov-switching expectational difference equation for inflation:

$$\pi_t = (\phi_\pi \tilde{s}_t)^{-1} E_t \pi_{t+1} + (\phi_\pi \tilde{s}_t)^{-1} \left((1 - \tilde{s}_t)\, \bar{i} + \sigma u_t \right) \tag{36}$$

where $\phi_\pi > 0$ and $y_t = 0$ for all t is imposed in (13) with flexible prices.

Assume $\phi_\pi \geq 0$ and $p_{00} + p_{11} > 1$. From Cho (2021),[39] (36) is determinate if and only if

$$r(F) = r \begin{pmatrix} p_{11}(\phi_\pi)^{-2} & p_{10}(\phi_\pi)^{-2} \\ p_{01}(\epsilon)^{-2} & p_{00}(\epsilon)^{-2} \end{pmatrix} < 1$$

where $p_{10} = 1 - p_{11}$, $p_{01} = 1 - p_{00}$, and $r(F)$ denotes the spectral radius of the matrix F. The eigenvalues of F, λ_1 and λ_2, are the roots of the following quadratic equation:

$$f(\lambda) = \lambda^2 - (p_{00}(\epsilon)^{-2} + p_{11}(\phi_\pi)^{-2})\lambda + (p_{11} + p_{00} - 1)\phi_\pi^{-2}\epsilon^{-2} = 0$$

[39] See Appendix A.3 for further details.

As demonstrated on p. 28 of LaSalle (1986), both eigenvalues, λ_1 and λ_2, are inside the unit circle if and only if

$$\begin{aligned} |(p_{11}+p_{00}-1)\phi_\pi^{-2}\epsilon^{-2}| &< 1 \\ |p_{00}\epsilon^{-2}+p_{11}\phi_\pi^{-2}| &< 1+(p_{11}+p_{00}-1)\phi_\pi^{-2}\epsilon^{-2}. \end{aligned}$$

The first condition for determinacy, $|(p_{11}+p_{00}-1)\phi_\pi^{-2}\epsilon^{-2}| < 1$, is surely violated for $\phi_\pi < \infty$ as $\epsilon \to 0$. Hence, the model (36) is indeterminate when $\epsilon \approx 0$.

From McClung (2020), we obtain E-stability of the MSV solution to (36) if

$$r^e(A) = r^e \begin{pmatrix} p_{11}(\phi_\pi)^{-1}-1 & p_{10}(\phi_\pi)^{-1} \\ p_{01}(\epsilon)^{-1} & p_{00}(\epsilon)^{-1}-1 \end{pmatrix} < 0,$$

where $r^e(A)$ denotes the maximum of the real parts of the eigenvalues of A. Because the trace of A, $tr(A) = p_{11}(\epsilon)^{-1} + p_{00}(\phi_\pi)^{-1} - 2 > 0$ for small ϵ, at least one eigenvalue of A is positive as ϵ approaches zero. Hence, the MSV solution is E-unstable.

A.2. Proof of Proposition 2

Consider (28) and assume $\phi_p \geq 0$ and $p_{00}+p_{11} > 1$. Then, (28) is determinate and the unique REE is E-stable if and only if $\phi_p > 0$ and $p_{00} < 1$. From Cho (2021),[40] (28) is determinate if and only if

$$r(F) = r \begin{pmatrix} p_{11}(1+\phi_p)^{-2} & p_{10}(1+\phi_p)^{-2} \\ p_{01} & p_{00} \end{pmatrix} < 1$$

[40] See A.3 for further details.

where $p_{10} = 1 - p_{11}$, $p_{01} = 1 - p_{00}$, and $r(F)$ denotes the spectral radius of the matrix F. The eigenvalues of F, λ_1 and λ_2, are the roots of the following quadratic equation:

$$f(\lambda) = \lambda^2 - (p_{00} + p_{11}(1+\phi_p)^{-2})\lambda + (p_{11} + p_{00} - 1)(1+\phi_p)^{-2} = 0$$

As demonstrated on p. 28 of LaSalle (1986), both eigenvalues, λ_1 and λ_2, are inside the unit circle if and only if

$$\begin{aligned} |(p_{11} + p_{00} - 1)(1+\phi_p)^{-2}| &< 1 \\ |p_{00} + p_{11}(1+\phi_p)^{-2}| &< 1 + (p_{11} + p_{00} - 1)(1+\phi_p)^{-2}, \end{aligned}$$

which holds provided that $p_{00} + p_{11} - 1 > 0$, $\phi_p > 0$, and $p_{00} < 1$. From McClung (2020), E-stability of the MSV solution to (16) is obtained if

$$r^e(A) = r^e \begin{pmatrix} p_{11}(1+\phi_p)^{-1} - 1 & p_{10}(1+\phi_p)^{-1} \\ p_{01} & p_{00} - 1 \end{pmatrix} < 0$$

where $r^e(A)$ denotes the maximum of the real parts of the eigenvalues of A. Because the trace of A is negative (i.e., $tr(A) = p_{11}(1+\phi_p)^{-1} + p_{00} - 2 < 0$), and the determinant of A is positive (i.e., $det(A) = (1-p_{00})(1 - 1/(1+\phi_p)) > 0$) under the assumptions in Proposition 2, both eigenvalues of A have negative real parts. Thus, we have E-stability of the MSV solution to (28).

A.3. Regime-Switching Model with Intercept

The model (16) contains a regime-switching intercept term, $C(s_t)$. Although recent works (e.g., Bianchi and Melosi (2017)) have solved Markov-switching DSGE models with regime-switching intercept terms, those that discuss the determinacy properties of Markov-switching models typically assume $C(s_t) = 0$, for all s_t. Here, we show one way to handle the intercept term when solving the model using the forward method of Cho (2016), and we argue that the determinacy conditions of Cho (2016) and Cho (2021) can be applied to a model with $C(s_t) \neq 0$. Throughout the appendix we assume $v_t = 0$ but the same result obtains if $v_t \neq 0$.

A.3.1. Solution Method

The model (16) assumes the form:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t).$$

If a unique REE exists, it assumes the MSV form:

$$x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t).$$

The forward method of Cho (2016) obtains a solution for $\Omega(s_t)$. Refer to the original work for details. The intercept term, $\Gamma(s_t)$, must satisfy:

$$\Gamma(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1}\left(C(s_t) + A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Gamma(s_{t+1})\right) \quad (37)$$

Define $F(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} A(s_t)$,
$G(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} C(s_t)$, $G = (G(0)'\, G(1)')'$, $\Gamma = (\Gamma(0)'\, \Gamma(1)')'$, and

$$\Psi_F = \begin{pmatrix} p_{00}F(0) & p_{01}F(0) \\ p_{10}F(1) & p_{11}F(1) \end{pmatrix}.$$

Then, given $\Omega(s_t)$, the solution for Γ is unique and given by

$$\Gamma = (I - \Psi_F)^{-1} G$$

assuming $(I - \Psi_F)$ is non-singular.[41]

A.3.2. Determinacy

Consider the class of Markov-switching DSGE models that assumes the form:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} \tag{38}$$

where x_t is an $n \times 1$ vector of endogenous variables, s_t is an $(S+1)$-state Markov Chain, and $p_{ij} = Pr(s_{t+1} = j | s_t = i)$ is the (i,j)-th element of the transition probability matrix, P. From Cho (2016) or Farmer et al. (2011), we can express any REE of (38) as

$$x_t = \Omega(s_t)x_{t-1} + w_t \tag{39}$$

$$w_t = F(s_t)E_t w_{t+1}, \tag{40}$$

where w_t is any stochastic process satisfying (40) and

$$\Omega(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} B(s_t) \tag{41}$$

$$F(s_t) = \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} A(s_t). \tag{42}$$

Thus, any REE of (38) can be represented as the sum of an MSV component, $x_t = \Omega(s_t)x_{t-1}$, and a non-fundamental process, w_t. To assess whether (38) admits a unique mean-square stable REE (i.e., whether (38) is "determinate"), Cho (2016) and Cho (2021) propose the

[41] If $(I - \Psi_F)$ is singular, then Ψ_F has a unit eigenvalue. McClung (2020) shows that the real parts of Ψ_F must be less than one for the underlying equilibrium to be E-stable (in our numerical analysis, the eigenvalues of Ψ_F are always inside the unit circle when E-stability is satisfied). Furthermore, we can show that the underlying model is indeterminate if $r(\Psi_F) \geq 1$. Thus, we do not encounter singular $(I - \Psi_F)$ in cases where the REE is E-stable or the model is determinate.

following useful matrices:

$$\bar{\Psi}_{\Omega\otimes\Omega} = \begin{pmatrix} p_{00}\Omega(0)\otimes\Omega(0) & \dots & p_{S0}\Omega(0)\otimes\Omega(0) \\ \vdots & \ddots & \vdots \\ p_{0S}\Omega(S)\otimes\Omega(S) & \dots & p_{SS}\Omega(S)\otimes\Omega(S) \end{pmatrix}$$

$$\Psi_{F\otimes F} = \begin{pmatrix} p_{00}F(0)\otimes F(0) & \dots & p_{0S}F(0)\otimes F(0) \\ \vdots & \ddots & \vdots \\ p_{S0}F(S)\otimes F(S) & \dots & p_{SS}F(S)\otimes F(S) \end{pmatrix}.$$

Theorem 1 gives the determinacy criterion for (38).

Theorem 1 *Consider the model (38) and suppose $\Omega(s_t)$ exists and is real-valued. Then, (38) is a determinate model if and only if:*

1. $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$
2. $r(\Psi_{F\otimes F}) \leq 1$.

Proof: see Propositions 1 and 2 in Cho (2021). ■.

Intuitively, $r(\Psi_{F\otimes F}) < 1$ ensures that $w_t = 0$ and that $\Omega(s_t)$ is the only fixed point of (41) that gives a mean-square stable solution of (38); and $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ ensures that the MSV solution, $x_t = \Omega(s_t)x_{t-1}$, is mean-square stable. Hence, if $r(\Psi_{F\otimes F}) < 1$ and $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ then the MSV solution, $x_t = \Omega(s_t)x_{t-1}$, is the unique mean-square stable solution of (38).[42]

We examine a closely-related model, given by (16), which is reproduced here:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t).$$

[42]Note that Theorem 2 technically applies to a particular MSV solution ("minimum-of-modulus" (MOD) solution). However, if $r(\Psi_{F\otimes F}) < 1$ and $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ then the MSV solution, $x_t = \Omega(s_t)x_{t-1}$, is the MOD solution. Moreover, if $B(s_t) = 0$, then the MSV is unique (and therefore the MOD solution by default). See Cho (2021) for more information.

Following Farmer et al. (2011), we can recast the model in the form

$$\tilde{x}_t = \tilde{A}(s_t)E_t\tilde{x}_{t+1} + \tilde{B}(s_t)\tilde{x}_{t-1}, \tag{43}$$

where $\tilde{x}_t = (x_t', z_t)'$, z_t is a dummy variable satisfying $z_0 = 1$ and $z_t = z_{t-1}$, and

$$\tilde{A}(s_t) = \begin{pmatrix} A(s_t) & 0_n \\ 0_n & 0_n \end{pmatrix}$$
$$\tilde{B}(s_t) = \begin{pmatrix} B(s_t) & C(s_t) \\ 0_{n\times 1} & 1 \end{pmatrix}.$$

Given the form of (43) and following Farmer et al. (2011) or Cho (2016), any REE of (43) (and therefore any REE of (16)) can be expressed as

$$\tilde{x}_t = \tilde{\Omega}(s_t)\tilde{x}_{t-1} + \tilde{w}_t \tag{44}$$
$$\tilde{w}_t = \tilde{F}(s_t)E_t\tilde{w}_{t+1} \tag{45}$$

where $\tilde{w}_t$ is any stochastic process satisfying (40) and

$$\tilde{\Omega}(s_t) = \left(I - \tilde{A}(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\tilde{\Omega}(s_{t+1})\right)^{-1}\tilde{B}(s_t) \tag{46}$$
$$\tilde{F}(s_t) = \left(I - \tilde{A}(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\tilde{\Omega}(s_{t+1})\right)^{-1}\tilde{A}(s_t).$$

Given the restrictions on $\tilde{B}(s_t)$ and $\tilde{A}(s_t)$, one can now recast the solution (44) and (45), and therefore any REE of (16), as:

$$x_t = \Gamma(s_t) + \Omega(s_t)x_{t-1} + w_t \tag{47}$$
$$w_t = F(s_t)E_tw_{t+1} \tag{48}$$

where w_t is any stochastic process satisfying (48) and $\Omega(s_t)$ and $\Gamma(s_t)$ satisfy (41) and (42), respectively. Then $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)$ is a mean-square stable solution of (16). Define $r(\Psi_{F\otimes F})$ and $r(\bar{\Psi}_{\Omega\otimes\Omega})$ in terms of $\Omega(s_t)$ and $\Gamma(s_t)$ as above, and $r(\Psi_{\tilde{F}\otimes\tilde{F}})$ and $r(\bar{\Psi}_{\tilde{\Omega}\otimes\tilde{\Omega}})$ in terms of $\tilde{\Omega}(s_t)$ and $\tilde{\Gamma}(s_t)$. After disregarding the unit root associated with the definitional equation for z_t in $r(\bar{\Psi}_{\tilde{\Omega}\otimes\tilde{\Omega}})$, we can show $r(\Psi_{F\otimes F}) = r(\Psi_{\tilde{F}\otimes\tilde{F}})$ and $r(\Psi_{\Omega\otimes\Omega}) = r(\Psi_{\tilde{\Omega}\otimes\tilde{\Omega}})$. Thus, we can apply the methods of Cho (2016) and Cho (2021) to the transformed model (43) to determine whether (17) is the unique REE of (16).

We have now demonstrated one approach to applying Cho's conditions to assess the determinacy of (16); below, we demonstrate an alternative approach. Suppose $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)$ is not the unique mean-square stable solution of (16), but that $r(\Psi_{F\otimes F}) < 1$ and $r(\Psi_{\tilde{\Omega}\otimes\tilde{\Omega}}) < 1$. Then there must exist another mean-square stable solution:

$$x_t = \Gamma_t^* + \Omega(s_t)x_{t-1} \tag{49}$$

where Γ_t^* is some time-varying intercept term. We place no restriction on Γ_t^* other than it is $n \times 1$ and that it depends on information available to the agents at time t.[43] Importantly, $r(\Psi_{F\otimes F}) < 1$ implies that $\Omega(s_t)$ is the unique lagged REE coefficient matrix in any REE of (16). Now, define $\hat{\Gamma}_t = \Gamma_t^* - \Gamma(s_t)$. Then, by substituting (49) into (16) and rearranging, we have

$$\begin{aligned}
\Gamma_t^* &= F(s_t)E_t\Gamma_{t+1}^* + \left(I - A(s_t)\sum_{s_{t+1}} p_{s_ts_{t+1}}\Omega(s_{t+1})\right)^{-1} C(s_t) \\
\hat{\Gamma}_t + \Gamma(s_t) &= F(s_t)E_t\left(\hat{\Gamma}_{t+1} + \Gamma(s_{t+1})\right) + \left(I - A(s_t)\sum_{s_{t+1}} p_{s_ts_{t+1}}\Omega(s_{t+1})\right)^{-1} C(s_t),
\end{aligned}$$

where $F(s_t)$ is defined in (42). Substituting (37) into the last equation yields

$$\hat{\Gamma}_t = F(s_t)E_t\hat{\Gamma}_{t+1}.$$

[43]Note that Γ_t^* could depend on an arbitrary number of lags of the Markov state, s_t.

It follows from $r(\Psi_{F\otimes F})<1$ that $\hat{\Gamma}_t=0$. Hence, $\Gamma^*_t=\Gamma(s_t)$, and there is a contradiction.[44] We conclude that Theorem 2 also ensures determinacy of (16).

A.4. E-stability Conditions: Contemporaneous Information

Consider (17) and define

$$F(s_t) \quad = \quad \left(I - A(s_t)\sum_{s_{t+1}} p_{s_t s_{t+1}}\Omega(s_{t+1})\right)^{-1} A(s_t)$$

where all matrices are from the model (16) and the model equilibrium (17) under study. Under the contemporaneous information assumptions discussed in section 2, (17) is E-stable if the real parts of the eigenvalues of

$$\Psi_{\Omega'\otimes F} \quad = \quad \begin{pmatrix} p_{00}\Omega(0)'\otimes F(0) & p_{01}\Omega(0)'\otimes F(0) \\ p_{10}\Omega(1)'\otimes F(1) & p_{11}\Omega(1)'\otimes F(1) \end{pmatrix}$$

$$\Psi_F \quad = \quad \begin{pmatrix} p_{00}F(0) & p_{01}F(0) \\ p_{10}F(1) & p_{11}F(1) \end{pmatrix}$$

are less than one. See Proposition 1 of McClung (2020) for a formal proof.[45]

[44] Note that $r(\Psi_{F\otimes F})<1$ implies $r(\Psi_F)<1$ and this specifically precludes the existence of non-fundamental regime-switching intercept terms. See McClung (2020) for more details.

[45] Note that the proof of Proposition 1 of McClung (2020) assumes $C(s_t)=0$. However, it is straightforward to show that $C(s_t)\neq 0$ does not affect any E-stability computations in the proof. Therefore, the value of the regime-switching intercept is irrelevant to E-stability.

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