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Market Operations Department
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The Term Structure of Interest Rates: Estimation and Interpretation

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Abstract

This document reports the currently used term structure estimation method at the Bank of Finland ans discusses interpretation of the results it generates. We start by introducing two widely used term structure estimation methods: the Cubic Spline Function method and the Nelson-Siegel approach. We compare their results, paying special attention to the smoothness of forward interest rates and distribution of pricing errors. Next, we introduce the Bank of Finland's method, commenting on its strenghts and weaknesses. Finally, we discuss interpretation of the term structure of interest rates with emphasis on the inflation expectations and the role of the time-varying risk premia.

Key words: term structure of interest rates, cubic splines, Nelson-Siegel, forward interest rates, relative value, inflation expectations, time-varying risk premia.

Tiivistelmä

Tämän keskustelualoitteen tarkoituksena on esitellä Suomen Pankissa käytössä oleva korkorakenteen estimointimalli sekä arvioida sen tuottamia tuloksia. Tutkimuksen ensimmäisessä osassa tarkastelemme kahta yleisesti käytettyä korkorakenteen estimointitapaa: ns. Cubic-Spline -mallia ja Nelson-Siegel -lähestymistapaa. Vertailemme mallien tuloksia kiinnittäen erityistä huomiota termiinikorkojen siistiin käyttäytymiseen sekä hintavirheiden jakaumaan. Tutkimuksen toisessa osassa esittelemme Suomen Pankissa käytetyn mallin, vertaillen mallin vahvuuksia ja heikkouksia. Lopuksi käsittelemme korkorakenteen tulkintaa inflaatio-odotusten ja ajassa muuttuvien riskipreemioiden kannalta.

Asiasanat: korkorakenteen estimointi, kuutiosplinit, Nelson-Siegel, termiinikorot, suhteellinen edullisuus, inflaatio-odotukset, ajassa muuttuva riskipreemio.

Contents

Chapter 1

Introduction

One of the oldest problems in economic theory is the interpretation of the term structure of interest rates. It has been long recognized that the term structure of interest rates conveys information about economic agents' expectations about future interest rates, inflation rates and exchange rates. Indeed, it is widely seen that the term structure is *the best* source of information about the economic agents' inflation expectations for one to four years ahead.[1] Since it is usually recognized that the monetary policy can only have effect with "long and variable lags" as Friedman (1968) put it, the term structure provides an invaluable source of information for the monetary authorities.[2] Moreover, recent empirical studies[3] indicate that the term structure predicts consumption growth better than vector autoregressions or leading commercial econometric models.

The empirical success above is, unfortunately, diminished by the fact that currently there does not exist a theoretical model which could explain all the implications of the term structure. This means that currently it is not possible to obtain *exactly* all information that is hidden in the term structure. This report tries to give the reader a perspective about what is known about the term structure, and how we can use the information it contains.

Before we can turn to the question of the interpretation of the term structure of interest rates we have to specify what we mean by the term structure. The term structure is something we cannot observe; we observe only some of its implications such as yields to maturity on coupon-bearing bonds. Thus,the question of how to pull back the underlying term structure from the market data is a non-trivial one. Different estimation methods may give very different term structures, forward interest rates, and, ultimately, indicate contradictory

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[1]See, eg, Fama (1975, 1990), Mishkin (1981, 1990a, 1992) for studies about the inflation expectations and the term structure of interest rates using U.S. data. Mishkin (1991) and Jorion and Mishkin (1991) use international data. Abken (1993) and Blough (1994) provide nice surveys of the literature.

[2]Svensson (1994ab) offers excellent discussions about monetary policy and the role of the term structure of interest rates as a source of information.

[3]See, e.g., Harvey (1988), Chen (1991), and Estrella and Hardouvelis (1991).

or incongruent implications.

This document reports the currently used method for estimating the term structure of interest rates at the Bank of Finland, and discusses how we interpret the results. We start in Chapter 2 by introducing the main concepts in section 2.1 and two widely used term structure estimation methods: the Cubic Spline Function method in section 2.3 and the Nelson-Siegel approach in section 2.4. We compare the results they produce in section 2.5, paying special attention to the smoothness of forward interest rates and distribution of pricing errors. Next, we introduce the Bank of Finland's method in section 2.4.2 and discuss its strengths and weaknesses.

Finally, we discuss about the interpretation of the term structure of interest rates in Chapter 3. In section 3.1 we explain what should be found from the forward interest rates: expectations and different term premia. When there are no premia, obtaining interest, inflation, and exchange rate expectations is a trivial task as shown in section 3.2. With non-zero term premia the task is non-trivial and currently unsettled issue in the literature as is emphasized in section 3.3. That section is a bit more technical than other sections of the paper, and may be skipped by the reader not interested in empirical research. The main findings are restated in Chapter 4.

Chapter 2

Estimation of the Term Structure

2.1 Main concepts

In the introduction we stated that the term structure is something we cannot observe. We only observe its implications: yields to maturity on coupon-bearing bonds. Before we continue this discussion, defining some key terms may be in order. *Yield to maturity* on a coupon-bearing bond is its *internal rate of return* that will set the present value of the bond equal to its price

$$P_j = \sum_{i=1}^{N} \frac{CF_i}{(1+y_j)^{t_i}}, \tag{2.1}$$

where P_j is the price of bond j with N cash flows, CF, so that cash flows i takes place after t_i periods. Such y_j that solves the above equation given above parameters is called bond j's yield to maturity. It should be noted that Eq.2.1 is an implicit function in y_j, and numerical methods (e.g., Newton-Raphson procedure) are needed to solve for y_j.

Cash flows usually consist of coupon payments, C_i, $i = 1, \ldots, N-1$, and the final repayment, $C_N + F$, where F is the nominal value of the bond. To normalize different payment structures so that they can be compared, it is common to speak about *spot rates*, which corresponds to yields to maturity on zero-coupon bonds

$$P_j = \frac{F_j}{(1+s_j)^{t_N}}, \tag{2.2}$$

where s_j is the spot rate for the zero-coupon bond j using annual compounding. The discount function and spot rates are now related in the following way:

$$d(t) = \frac{1}{(1+s(t))^t}, \tag{2.3}$$

or with continuous compounding

$$d(t) = e^{-t*s(t)}, \tag{2.4}$$

Assume you have a zero-coupon bond maturing at time $t-1$ in the future, and that you want to increase your investment horizon from $t-1$ to t. There are two ways to accomplish this. You can either sell your current bond and buy yourself a new bond maturing at time t, or you can keep your current bond and buy yourself a contract for a period between $t-1$ and t. Since both investments are determined now and the interest rate risk from both investments is the same, the no-arbitrage principle dictates that the rate of return from both investments must be the same

$$\begin{aligned}
\frac{(1+s_t)^t-(1+s_{t-1})^{t-1}}{(1+s_{t-1})^{t-1}} &= \frac{(1+s_{t-1})^{t-1}(1+f_{t-1,t})-(1+s_{t-1})^{t-1}}{(1+s_{t-1})^{t-1}} && \Rightarrow \\
(1+s_t)^t &= (1+s_{t-1})^{t-1}(1+f_{t-1,t}) && \Rightarrow \\
f_{t-1,t} &= \frac{(1+s_t)^t-(1+s_{t-1})^{t-1}}{(1+s_{t-1})^{t-1}}, && (2.5)
\end{aligned}$$

where $f_{t-1,t}$ is the internal rate of return for a contract between periods $t-1$ and t. Eq.2.5 gives an expression for a *forward rate* between periods $t-1$ and t given the spot rates for period $t-1$ and for period t. Generalizations to different horizons are obvious. Forward rate is an interest rate determined now (trade date) for an investment beginning in the future (settlement date) and ending further in the future (maturity date).

If we let the difference between the maturity time and the settlement time approach zero, we obtain the instantaneous forward rate. The instantaneous forward rate may be seen as the marginal increase in the rate of return from a marginal increase in the investment horizon. Hence, the spot rate and the (instantaneous) forward rate are related in the same manner as marginal and average cost of production are related such that quantity producedcorresponds to time to maturity. The spot rate at maturity m is the average of the forward rates

$$s(m) = \frac{1}{m}\int_0^m f(t)\,dt, \tag{2.6}$$

where $f(t)$ denotes the instantaneous forward rate at maturity t.

The meaning of *yield curve* depends on the author. Here yield curve means a collection of bond-specific yields versus bond-specific maturities. This is something that can be unambiguously calculated from the market prices. By the term structure of interest rates we mean *spot curve*, or the internal rate of return for any zero-coupon bond maturing at any time in the future.

In practice, we do not have bonds in general or zero-coupon bonds especially for each date in the future. Hence, we need first to estimate the zero-coupon bonds from bonds on the market, then the spot rates from these "synthetic" zero-coupon bonds for each date in the future, and finally forward rates from spot rates. The first and the last step are unambiguous. The problem is the middle step: How do we fit a finite number of zero-coupon bonds into a continuous and *well-behaving* spot curve? It is very easy to obtain perfect fit, but that will create widely fluctuating forward curve with possible negative or infinite forward rates.

Finally, it should be mentioned that when we discuss the term structure we mean spot curve estimated from liquid government nominal bonds. Illiquid bonds or bonds issued by private banks have risk premia of their own, which would complicate the discussion in section 3.

The basic formula that is used to estimate the discount function or the spot rates from bond prices is as follows. The (weighted) sum of bond pricing errors ϵ_j is minimized

$$E = \min \sum_{j=1}^{N} \omega_j * {\epsilon_j}^2 =$$

$$\min \sum_{j=1}^{N} \omega_j * \left[\hat{P}_j - P_j\right]^2 =$$

$$\min \sum_{j=1}^{N} \omega_j * \left[\sum_{i=1}^{M} \left(\hat{d}(t_i) * Cf_{ij}\right) - P_j\right]^2 = \tag{2.7}$$

$$\min \sum_{j=1}^{N} \omega_j * \left[\sum_{i=1}^{M} (\exp(-t_i * \hat{s}(t_i)) * Cf_{ij}) - P_j\right]^2, \tag{2.8}$$

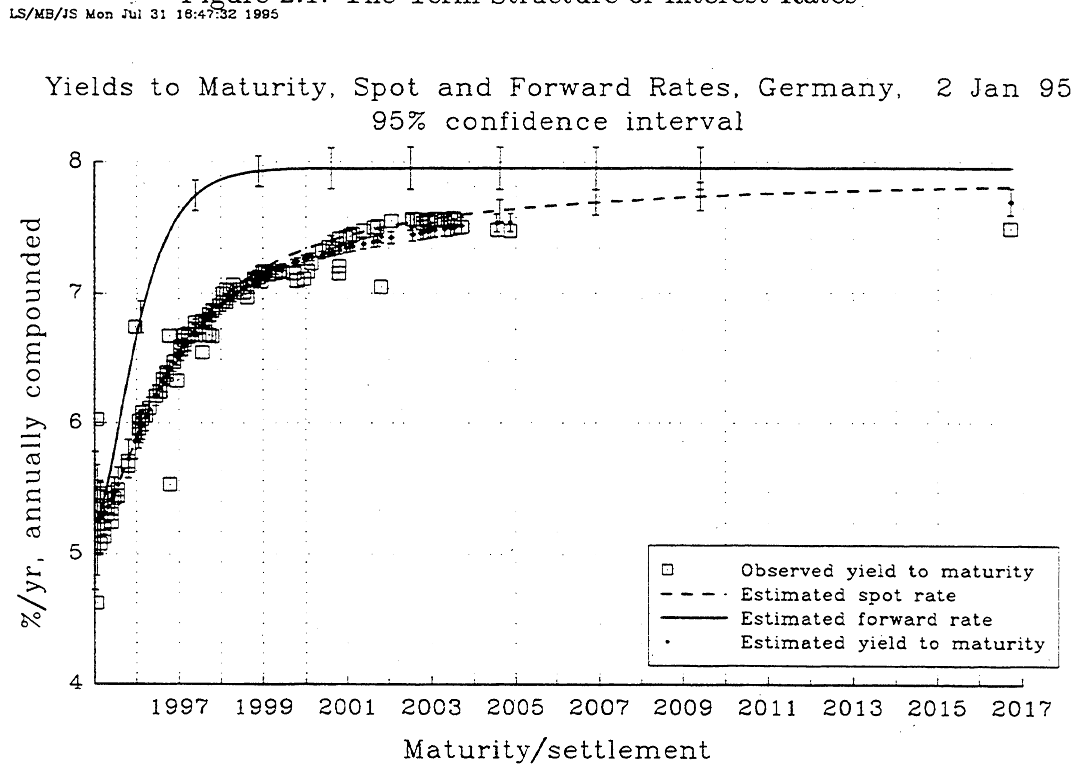
where N is the number of bonds, $\hat{P}_j$ is the estimated price and Cf_{ij} is the i:th cash flow of the bond j.

Figure 2.1 presents an example of the term structure. It shows the situation for Germany on 2 January 1995. The boxes represent observed data from the market, the straight line represents the estimated term structure, the dotted line the (instantaneous) forward rate implied by the spot curve, and finally dots show "theoretical" yields to maturity implied by the spot curve. That is, once we know the theoretical spot curve and the cash flow structure of the bonds at market, we can pull back the theoretical price and yield to maturity for each bond using Eq. 2.1. Figure presents also 95% confidence interval for the estimates.

2.2 Other works

The most straightforward method to find the term structure is to use a simple bootstrapping or recursive method. The rates are defined recursively from the shortest instrument onwards. Early attempts to fit the term structure were made by Carleton and Cooper (1976), and Cohen, Kramer and Waugh (1966) using regression techniques. These discrete point estimation methods resulted

Figure 2.1: The Term Structure of Interest Rates



in a good fit, although the rates were not smooth and did not satisfy no-arbitrage arguments. Further, these methods are only applicable with carefully selected situations (eg selected maturity dates).

McCulloch was the first to use spline techniques to smooth the rates between coupon dates. Initially he used a quadratic spline technique (1971) and later he introduced the cubic spline method (1975) (see following section). Vasicek and Fong (1982) used exponential splines in their work, and argued that because the curvature of the discount function is of exponential nature, it cannot be approximated properly with polynomial functions.

Attempts to find methods for estimating the term structure directly, as opposite to discount function methods, were made by Fama and Bliss (1987) and by Nelson and Siegel (1987).

A quite different approach has been proposed by Delbaen and Lorimier (1992). First they estimate the initial term structure of forward rates (day 1), then they try to minimize the overnight fluctuation of the forward rates subject on the condition that the pricing errors remain reasonable.

All previous models lack a particular theoretical background beyond the basic bond mathematics and some empirical findings based on the behaviour of the rates. The theorethical models that are applicable to the estimation of the term structure form a different category. Among the most important models are the one proposed by Cox, Ingersoll and Ross (1985) and the Longstaff-Schwartz model (1992). The model proposed by Dillen (1994) has also gained attention recently.

In the real world many models are based on a variety of risk factors such as duration and convexity. Additionally straightforward polynomial approximation methods are also quite popular.

2.3 The Cubic Spline Function Method

2.3.1 B splines

The various spline methods utilize splines as the building blocks of the discount function. We present here a piecewise polynomial approximation model based on B splines. Splines provide an extremely high degree of flexibility in terms of the shapes of the curves. B splines have reasonably good convergence properties and provide a high degree of derivative continuity. A more detailed description of the B spline method can be found in Steeley (1991) and Shea (1985). Spline methods are probably the most widely used methods both in practice and in theoretical academic studies.

A cubic spline is a piecewise cubic polynomial joined at so-called "knot points. At each point the polynomials that meet are restricted so that the level and first two derivatives meet. Because of derivative continuity each additional knot point in the spline adds only one independent parameter. By increasing the number of knots, splines provide an increasingly flexible functional form (Fisher, Nychka, Zervos, 1995).

The functional form of a k-order (for cubic spline $k = 3$) B spline is as follows:

$$B_p^k(t) = \sum_{l=p}^{p+k+1} \left[\prod_{h=p,h\neq l}^{p+k+1} \frac{1}{(t_h - t_l)} \right] * (t - t_l)_+^k \quad -\infty < t < \infty \tag{2.9}$$

The subscript p is the spline index number and it denotes that the spline is non-zero only between the interval $[t_p, t_{p+k+1}]$. The vector $[t_1, t_2, t_3 ... t_{P+k+1}]$ is called the knot placement and defines the piecewise use of splines in the maturity axis. Function $(t - t_l)_+^k$ is a truncated power function that has the property

$$(t - t_l)_+^k = \max((t - t_l)^k, 0). \tag{2.10}$$

With a sufficient number of piecewise splines any continuously differentiable function can be approximated in an interval within an arbitrary error. So we can write the discount function

$$d(t) = \sum_{p=1}^{P} w_p * B_p(t) \tag{2.11}$$

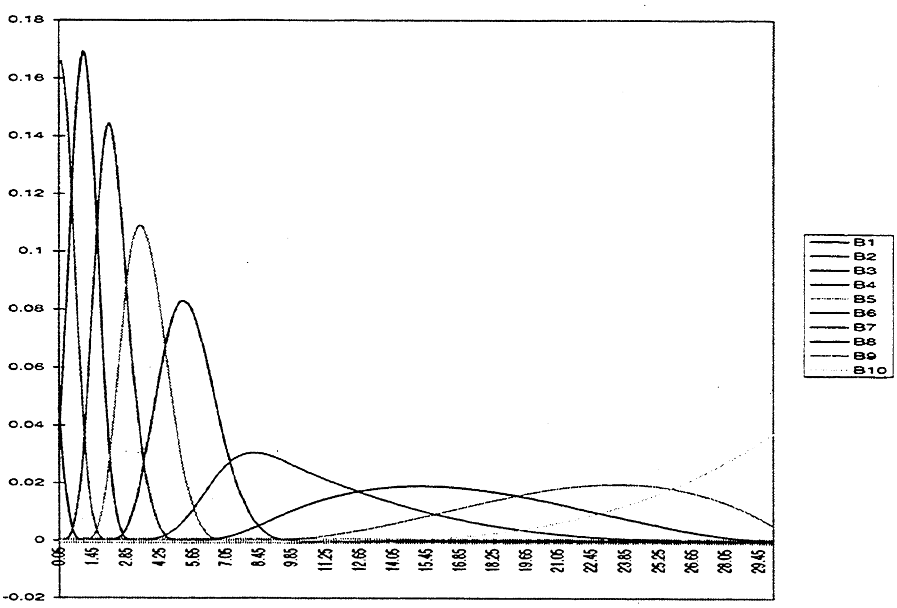
where P is the number of the B splines and w_p are the coefficients (weights) of the splines. If we insert Eq.2.11 into Eq.2.7, we get

$$P_j = \sum_{i=1}^{N} \sum_{p=1}^{p=P} CF_i * w_p * B_p(t) + \epsilon_j, \tag{2.12}$$

where the unknowns are the w_p and the pricing errors ϵ_j of the bonds.

Figure 2.2 presents an example of B-splines as approximating functions of the discount function.

Figure 2.2: B-splines with knot placement vector (1,2,3,5,8,15,23) (years)



The B spline base can, in fact, be a too flexible specification for the estimation of the discount function. The usual problem with spline approximations is that the derived forward rates tend to be unstable and widely fluctuating. As a result advanced market participants have turned their attention to other functional specifications. However it is possible to avoid the "overfitting problem hrough the use of "smoothing splines. (Fisher, Nychka, Zervos, 1995)

2.3.2 Knot points

Although the spline methods are quite straightforward to apply, one has to make some critical decisions on the quantity and location of the knot points. The placement of the knot points defines the explanatory power of the spline approximation. With respect to knot selection in Poirier (1976, s.151) it is said that

> "... the choice of knot positions corresponds very closely to the selection of functional type in an ordinary curve fitting problem. ... the knots in spline should not be seen as ordinary free parameters, but their specification should rather be seen as analogous to

the choice of functional type. Hence, the knots should be chosen as to correspond to the overall behaviour of the data (number of observations, positions of maxima and minima, etc.)...

Based on this, the following rules-of-thumb can be given when using cubic splines (Poirier, 1976):

1. Use as few knots as possible to avoid overfitting.
2. Use only one extremum point and one inflexion point per interval.
3. Center extremum points in the intervals.
4. Place inflexion points close to the knots.

Quantity of knots Technically as the quantity of knot points increases the explanatory power of the discount function also increases. As a result, the quality of fit increases. Simultaneously, however, the stability of the curves decreases. If the number of the knots is too small the pricing errors tend to be highly autocorrelated.

One way to determine an appropriate number of knot points is to use the well-known rule-of-thumb that the degrees of freedom of the approximation should equal the square root of the number of observations.

Another method to determine the number of knot points would be to use economic argumentation, stating that the term structure consists of distinct "quasi-independent sectors that follow separate patterns of behaviour. Such natural sectors for example would be the short end, intermediate and long end of the curve. (Litzenberger and Rolfo, 1984). Based on extensive testing of US Treasury data, Langetieg and Smoot (1988) recommended the use of 3 or 4 carefully selected knots. (See section 2.5.5).

The most structured way would perhaps be to use an information criteria similar than is described in section 2.5.3 and then maximize the adjusted coefficients of determination subject to the different models.

Location of knots The choice of knot locations has to be done in conjuction with choosing the number of knots. McCulloch (1971) proposed a location scheme whereby there are an equal number of observations (bonds) in each sector. The resulting curves are then homogeneous in degrees of freedom.

If the number of knots is selected based on economic arguments, then the natural next question is how the boundaries of the sectors are defined. Litzenberger and Rolfo (1984) used maturities of 1, 5 and 10 years as a set of natural knot locations. Langetieg and Smooth (1988) fine-tuned the short end of the curve by adding a knot at 0.5 years. To determine the optimal static location scheme properly, one would need to study the historical shapes of the term structure of interest rates. Unfortunately such studies should then be reviewed from time to time as the behaviour of the market participants changes and markets develope structurally and technically.

Tests made by Langetieg and Smooth (1988) indicate that the use of a static knot point location scheme resulted better estimates than the McCulloch variable location scheme.

The third possibility would be a variant of the McCulloch method, using optimization to determine the locations. An applicable algorithm can be found in deBoor (1978 pp. 218 - 222) and Powell (pp. 298 - 311). However if the location is determined by an optimization algorithm, the possibility to "intuitively interpret the parameters disappears. Also, the set of parameters increases and they are far less stable.

Model deficiencies Most criticism against the spline models has focused either on the fluctuation of estimated forward rates, the asymptotic behaviour of the rates or the lack of theory giving it a reputation as a "practitioners approach. In response we can state that the fluctuation of the rates can be controlled via good choice of knots and locations. However, the asymptotic behaviour is inevitable: the discount function is a sum of polynomials, so it is not finite when the term to maturity increases.

2.4 Nelson-Siegel model

The idea of cubic splines was to obtain as good fit as possible to the market data. It is as atheoretical as possible: the prices reflect all information that is relevant to the term structure. On the other hand, Nelson and Siegel (1987) attemted to minimize the number of parameters to be estimated by assuming that the instantaneous forward rate follows a simple, yet flexible, deterministic process: second-order linear differential equation with real and equal roots.[1] Hence, the instantaneous forward rate at maturity m is expressed as

$$f(m) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau}\right) + \beta_2 \frac{m}{\tau} \exp\left(-\frac{m}{\tau}\right), \tag{2.13}$$

where τ is time constant associated with the equation, and β_0, β_1, and β_2 are determined by initial conditions. In practice, these four parameters are estimated daily. Calling equation 2.6 that the spot rate is the average of the forward rates

$$s(m) = \frac{1}{m} \int_0^m f(t)\, dt.$$

By integrating, we obtain

$$s(m) = \beta_0 + (\beta_1 + \beta_2) \frac{1 - \exp(-\frac{m}{\tau})}{\frac{m}{\tau}} - \beta_2 \exp\left(-\frac{m}{\tau}\right). \tag{2.14}$$

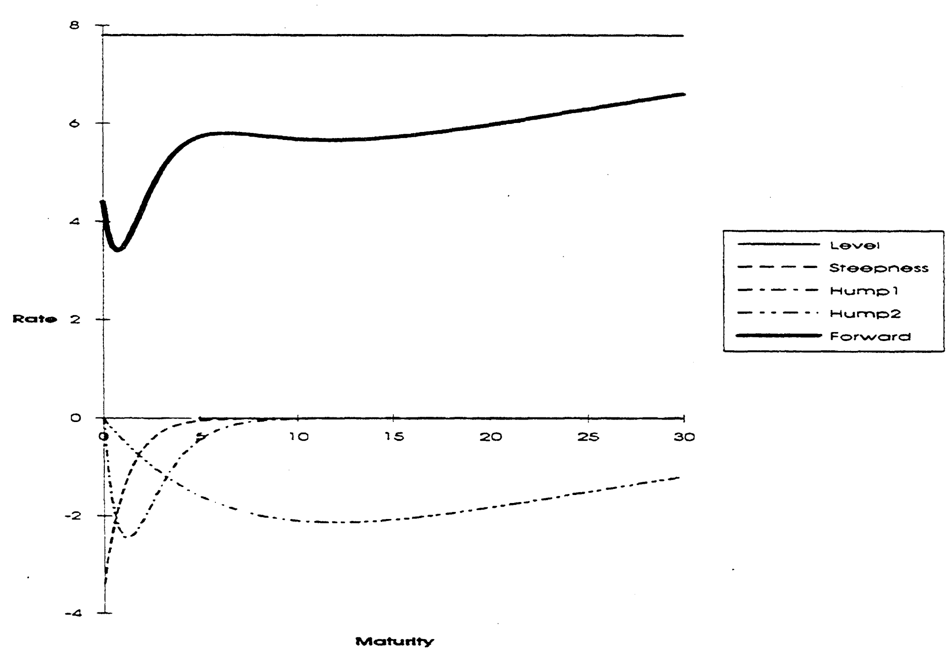
[1] Originally Nelson and Siegel assumed that the roots are unequal, but they noticed quickly that fitting an unequal roots model to the data lead to an overparameterized model.

The main advantage of the Nelson-Siegel method are its "intuitive" asymptotic properties

$$\lim_{m\to\infty} f(m) = \lim_{m\to\infty} s(m) = \beta_0$$
$$\lim_{m\to 0} f(m) = \lim_{m\to 0} s(m) = \beta_0 + \beta_1.$$

That is, both spot and forward rate approach a constant for both long and short maturities. Figure 2.3 presents the component functions of the Nelson - Siegel specification.

Figure 2.3: Components of 6 parameter Nelson-Siegel model



Another advantage of their approach is that there seems to be a close correspondence between the components of the Nelson-Siegel model and the findings of Litterman and Scheinkman (1991), whofound using factor analytic approach that three factors explain most of the observed variation in bond returns. These factors were *"level, "steepness and "curvature*[2].

Nelson and Siegel estimated their model by fixing τ, after which ordinary least square estimation can be used to get the βs. The procedure was repeated for a wide number of τs, and the one with best fit was chosen. Another

[2]The "level factor explains 89.5% of the total variation in US Government Bond sector returns, "steepness explains 8.5% and "curvature explains 2.0%. For other international government bond markets the figures are quite similar.

approach is to estimate all parameters simultaneously using a nonlinear estimation method (maximum likelihood, nonlinear least squares, generalized method of moments). Section 3 explains in detail how these methods are related.

2.4.1 Extended models

The Nelson-Siegel specification could be criticized that it is not flexible enough to describe all the detailed shapes of the curves. Thus a number of extended specifications has been proposed. Svensson (1994 a) increased flexibility by introducing an additional hump term to the original specification. He used the following extension:

$$f(m) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right), \tag{2.15}$$

Bliss (1991) used the following five parameter version

$$f(m) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right). \tag{2.16}$$

In the Bliss extension "steepness and "hump terms are completly independent compared to the original specification. It is a simplyfied version of the six-parameter form where the third component is dropped off.

2.4.2 The Bank of Finland Method

The old Bank of Finland model used a B-spline approach to construct the discount function. Due to poor asymptotic behaviour and difficulties in interpretation of model parameters which will be discussed in the following section 2.5.5 we decided to switch to a modified Nelson - Siegel model. As the functional form we selected the extended form proposed by Svensson (1994), where the spot rates are

$$\begin{aligned} s(m) &= \beta_0 + \beta_1 * \frac{1-\exp(-\frac{m}{\tau_1})}{\frac{m}{\tau_1}} + \beta_2 * \left[\frac{1-\exp(-\frac{m}{\tau_1})}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right)\right] \\ &+ \beta_3 * \left[\frac{1-\exp(-\frac{m}{\tau_2})}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right)\right] \end{aligned} \tag{2.17}$$

The optimization is done using a variable metric method, namely an algorithm called *Broyden – Fletcher – Goldfarb – Shanno(BFGS)*. The model is written in standard C language for a SUN UNIX environment.

We also tested the implementation made by Svensson(1993), where the Nelson-Siegel method was implemented in its basic four-parameter configuration as well as in "enhanced six-parameter configuration. The implementation was done on GAUSS using maximum likelihood methodology. [3]

[3] We would like to thank L.E.O Svensson for providing his GAUSS code for us.

The original GAUSS code was not able to handle zero-coupon bonds outside the money markets (STRIPS) with maturing over one year. We fixed this shortcoming. Also the computer code was not able to handle instruments that are in their ex-dividend period. This feature made it impossible for us to test the model with UK data properly.

We found the implementation quite slow for our purposes: computing time increased exponentially with the number of instruments. Further the enhanced six-parameter form often failed to converge. Finally, we found that the estimation methodology (and probably convergence) was highly sensitive to pricing errors in the input data.

2.5 Comparison of methods

As stated at the start, there is no absolutely correct way to estimate the term structure. The purpose of the estimation dominates the selection of estimation methodology. One can only say that there may be correctly estimated term structures for specific purposes.

2.5.1 General

The properties that are required from a good estimation methodology are

1. Good fit; ie accuracy and precision
2. Parsimony, intuitivity, logicality, opaqueness
3. Robustness and stability.

Bliss(1991) presents an excellent study on testing different estimation methodologies from the bond pricing point of view. He first analyzed the methods based on the coefficients of determination (R^2) and *hitratios*. Then he split his datasets into two blocks and performed extensive out-of-the-sample tests to analyze the sensitivity of the estimates. In his tests, the *Fama - Bliss adjusted bootstrapping method* and *extended Nelson - Siegel-method* ranked highest. Spline methods tended to overfit bond prices and their rates were unstable in out-of-the-sample tests.

Buono, Gregory-Allen and Yaari (1992) generated four "typical shapes of the term structure of forward rates: flat, increasing, inverted and humped (with some realistic restrictions). Then a set of bonds was priced based on these term structures with a small stochastic bond specific error, allowing them to use Monte-Carlo simulation to test which of the estimation methodologies produced the closest forward rate estimates. They found that discrete point methods produced more exact estimates than the exponential polynomial method. The methods presented in our paper were not included in their work.

2.5.2 Modeling considerations

Basically there are three major "structural decisions to be made before the estimation can be done.

First, one must have some underlying assumptions about the functional form of the discount function or the spot rates. These define the smoothing principle between rates, and may imply some fundamental mathematical restrictions on asymptotic behaviour of the rates. There are theoretical reasons [4] to select such a functional form that produces smooth second derivatives and finite and even bounded asymptotes. Beyond these, selection is fairly arbitrary and mainly done *ad hoc*.

Second, the weighting of the observations must be decided. This issue is fundamental to the homogeneity (or quality) of the curve. Because of the nonlinear relationship between (observed) prices and (calculated) curves, the minimization of pricing errors leads to heteroschedastic yield errors, and hence, unreliable yield estimates in the short end of the curves. More homogeneous curves in the yield terms could naturally be reached by minimizing yield errors, but then the price errors are heteroscedastic. The third possibility is to weight the bonds by reciprocal of the bid-ask spread, which gives us the "benchmark or "on the run (or most liquid) curves.

Finally, there is the question of the robustness of the estimation. Especially when estimated curves are used to provide information for pricing tools in a realtime environment, they should be robust in the sense that individual errors in the bond input data do not cause radical change in the results. The obvious choices are to use standard least squares estimation or more robust estimators based on median estimation error.

2.5.3 Methodology of comparison

Although the selection of an estimation method is usually made on the basis of good fit, there are several other items in the background that are strongly present in the decision situation. When selecting the methodology to estimate the term structure of interest rates the alternatives should be thought at least from the following points of view:

1. Goodness of fit
2. Quality of curves
3. Asymptotic behaviour of rates
4. Stability of parameters
5. Compactness of the model

[4]Smoothness, continuity and non-negativity of the forward rates are the typical economic characterizations of a "sensible term structure.

Goodness of fit The ability to explain bond prices properly is the most important feature of the estimation method. It is also the easiest to quantify. One can use standard econometric measures such as the coefficient of determination (R^2), rooted mean square error ($RMSE$) or mean absolute deviation (MAD). $RMSE$ and MAD should be adjusted. Because the bid-ask spread produces a range of feasible "zero-error prices, the possibility to fit prices properly increases and an additional measure of goodness-of-fit can be used. $Hitrate$ is defined as the fraction of bonds with zero error compared to the total amount of bonds in the sample.

Quality of curves This could be characterized by the amount of fluctuation or smoothness of the (forward) curves. According to Adams and van Deventer (1994), the smoothness of the forward curve can be measured by a functional

$$Z = \int_0^T [f''(s)]^2 \, ds \tag{2.18}$$

One should note that this measure penalizes the changes of the slope of the forward rate in an equal manner in every part of the forward rate. This might be a little unacceptable based on the assumption that we have more information or stronger expectations on the behaviour of the curves in a short time horizon (eg 1-24 months) than longer horizon (eg 15-30 years). Instead the non-fluctuating criterion should be modified in a way that it is less affected by strong movements at the short end but is very strict at the long end of the curve. If we slightly modify equation 2.18 we get a functional

$$Z = \int_0^T [s * f''(s)]^2 \, ds \tag{2.19}$$

The result is that the amount of fluctuation is weighted by maturity and we achieve the desired flexibility at the short end.

Asymptotic behaviour of rates The mathematical characteristics of the functional form tend to have a strong impact on estimated rates, especially in situations where the data sample is limited or heavily concentrated in certain parts of the maturity axis. This is almost always the case when the rates beyond 10 years are estimated for markets other than the US.

Stability of parameters The model parameters should be "stable and somehow predictable in their behaviour. This is normally the case when the model is properly specified. In the case of an overparametrized model the fluctuations of the parameters can be large.

Compactness of the model The compactness of the model can be measured by information criteria. Consider equations (2.1) and (2.2). One can easily see that if the spot rates $s(t)$ in equation (2.2) and yields y_j are exact transformations of each others the error term ϵ_j vanishes. This situation can be achieved when either the markets are perfectly efficient or when the estimation model is too flexible and thus overspecified. To avoid such overfitting, the quality of fit eg the accuracy of the model is measured by adjusted coefficient of determination $\tilde{R}$. The first equation is used when standard mean squared error methodology is used, the second equation gives the adjustment when a more robust mean absolute price error methodology is used.

$$\tilde{R}_j^2 = 1 - \frac{n + K_j}{n - K_j} * (1 - R_j^2) \tag{2.20}$$

$$\tilde{R}_j = 1 - \frac{n + K_j}{n - K_j} * (1 - R_j) \tag{2.21}$$

where n is the number of observations (bonds), K_j is the number of parameters in model j and R_j is the coefficient of determination of model j.

2.5.4 Data description

Our test data consists of the prices of USD, DEM, FRF, GBP and JPY government securities. From each of these markets we have included only plain vanilla domestic issues, ie those with no special characteristics such as embedded options, exchangeable issues or bonds with abnormal coupon periods. We have separated the strips data (STR) from the US bond data (USD) as an individual data set, which means that we have actually fitted two different curves to our US data: US strips and US coupons and bills. The French strips are not separated from the coupons.

Our bimonthly data covers the period from 25 November 1993 to 17 January 1995 with some dates missing during summer 1994 due to technical problems. The total amount of sample dates is 23. We have collected the data from Bloomberg information system using "the most reliable price contributor. We have not applied any price filtering procedure to the data, so the data may contain pricing errors and could be characterized as "dirty data. Table 2.1 describes the key characteristics of our data sets[5].

The DEM data sets are heavily concentrated below the 10-year sector of the curve. This is due to the issuing policy of Bunds and Bobls (German government securities). 10-year Bunds and 5-year Bobls are issued irregularly, but typically from three to five new issues are introduced during a calendar year. We included the instruments issued by Treuhandanstalt into our data sets.

[5]The Bonds, Maturity, Coupon and Spread columns give the average figures over our sample period. MaxMat gives the maximum maturity for the whole sample.

Currency	Bonds	Maturity	MaxMat	Coupon	Spread
DEM	114.4	3.9	22.8	7.0	0.083
FRF	98.6	10.1	29.4	3.5	0.161
GBP	44.2	6.5	23.8	10.3	0.087
JPY	107.4	6.8	20.3	5.2	0.001
STR	116.1	14.5	29.7	0	0.268
USD	208.2	5.5	29.7	6.7	0.059

Table 2.1: Characteristics of the data sets

The FRF data consists of four "subsectors: BTFs (treasury bills 1-4 instruments per month), BTANs (2-5 year coupons; 2-4 instruments per year) and OATs (old 10-30 year coupons, 1-2 per year below the 10-year sector and 1 issue every 4th year beyond the 10-year) and OATstrips (two instruments per year up to 30 years). The bid-ask spreads of the OATstrips are typically wider than others.

The GBP data set is the smallest, in part because the market has quite many excluded issues with special characteristics. The distribution of the instruments is similar to DEM data.

The JPY data set is exceptional among our test sets because we were unable to get two-side prices in electronic format for most issues. This can be seen from the narrowness of the bid-ask spreads.

The STR data consists of evenly distributed quarterly instruments up to 30 years. The spreads are widest which is later reflected in the high hitratios.

The USD data set is the largest and the best in terms of quality.

Shapes of the term structures At the beginning of the sample period the DEM and FRF curves were humped, with USD, STR and GBP curves increasing in "normal way. During the period the steepness of the curves in European markets increased significantly, while the USD curves flattened. At the end of the period GBP, USD and STR rates were all quite flat.

The DEM and STR curves are usually humped. In the DEM curve it is due to the futures market. The bond issues that are deliverable as BOBL- and BUND-futures always tend to be expensive against the neighbouring issues in the cash market. In the STR curve the hump is due to the high convexity of the long STRIPs. High convexity means greater possibilities of additional return and that is penalized by lower yield.

Generally speaking, the shapes of the USD and STR curves are very similar. This is due because of the constant coupon stipping and reconstitution activity that prevents these two markets from deviating very far from each other.

The figures in Appendix A presents the development of rates during the sample period in different markets.

2.5.5 Results of comparison

The models that we compared in our tests were B-spline models with variety of different knot specifications and a Nelson-Siegel approach using the basic four parameter form.

Goodness of fit Goodness of fit is here measured by four measures: the mean absolute deviation (MAD), rooted mean square error and hitratio. The fourth measure is non-zero pricing error (NZRMSE), that is calculated as RMSE of non-zero errors. The underlying idea is that hitratio captures all perfect fits so NZRMSE should be more sensitive to remaining, true errors. (This is naturally affected by functional specification and degrees of freedom). For the spline approach, this means the number and the location of knot points, in the case of the Nelson-Siegel specification, whether the extended (five-six parameter) version or the basic model is used.

Table 2.2 presents the RMSE, NZRMSE, Hitratios and MAD for different methodologies sorted by currency. The general differences between the markets are quite easily seen. The USD and STR markets are the most efficient, while in the GBP and JPY markets the errors are high due to taxation and par effects. Typically, the B-spline method is superior here because of its higher flexibility. This can be clearly seen from the hitratios. One should, however, note the small differences, especially in the NZRMSEs reflect the good ability of the NS4 method to price all issues relatively well.

Quality of the curves Table 2.3 presents a summary of the smoothness statistics of the estimates. The NS4 methodology gives good results in all markets. B-spline results are reasonable as long as the model is properly parametrized.

Asymptotic behaviour of forward rates Table 2.4 summarizes the asymptotic behaviour of forward rates in different specifications. One can easily see that the spline method should never be used for extrapolation due to its poor asymptotic properties. Note that, although the rates in the Nelson-Siegel approach are always limited, there are no limitations to guarantee non-negative forward rates.

Stability of parameters Figures 2.4 and 2.5 present the time path of the model parameters. Interpretation of the parameters of the NS4 model is fairly straightforward, especially when the parameters are compared to the evolution of rates. Note the changes in the level of the yield curve, the increase in the steepness of the curve in early 1994 and the corresponding changes in parameters β_0 and β_1.

Currency	Label	Bsp4	NS4 SP
DEM	RMSE	4.4097	4.5632
	NZRMSE	5.8246	5.2864
	HITRATIO	0.2383	0.137
	MAD	17.6346	23.2465
FRF	RMSE	3.192	3.4197
	NZRMSE	5.0513	4.1233
	HITRATIO	0.3893	0.1952
	MAD	13.7407	20.19
GBP	RMSE	8.786	8.2285
	NZRMSE	10.281	9.5544
	HITRATIO	0.139	0.1324
	MAD	29.8072	32.48
JPY	RMSE	7.6059	7.4933
	NZRMSE	7.6133	7.4981
	HITRATIO	0.0012	0.0004
	MAD	46.9511	52.31
STR	RMSE	0.8751	2.8899
	NZRMSE	3.4449	4.0173
	HITRATIO	0.6864	0.2874
	MAD	2.1235	16.15
USD	RMSE	1.5798	2.2352
	NZRMSE	2.1604	2.4887
	HITRATIO	0.2621	0.096
	MAD	7.4257	16.24

Table 2.2: Goodness of Fit

Currency	Bsp4	NS4 SP
DEM	0.0485	0.0418
FRF	0.1006	0.0098
GBP	0.2258	0.0511
JPY	0.0422	0.0327
STR	0.1443	0.0085
USD	0.0957	0.0211

Table 2.3: Quality of the rates

Model	$fwd(\infty)$	determination
$B_N^3 spline$	$\pm\infty$	w_N
NS4	limited	β_0

Table 2.4: Asymptotic behaviour of the models

Figure 2.4: DEM Parameters using NS4 model

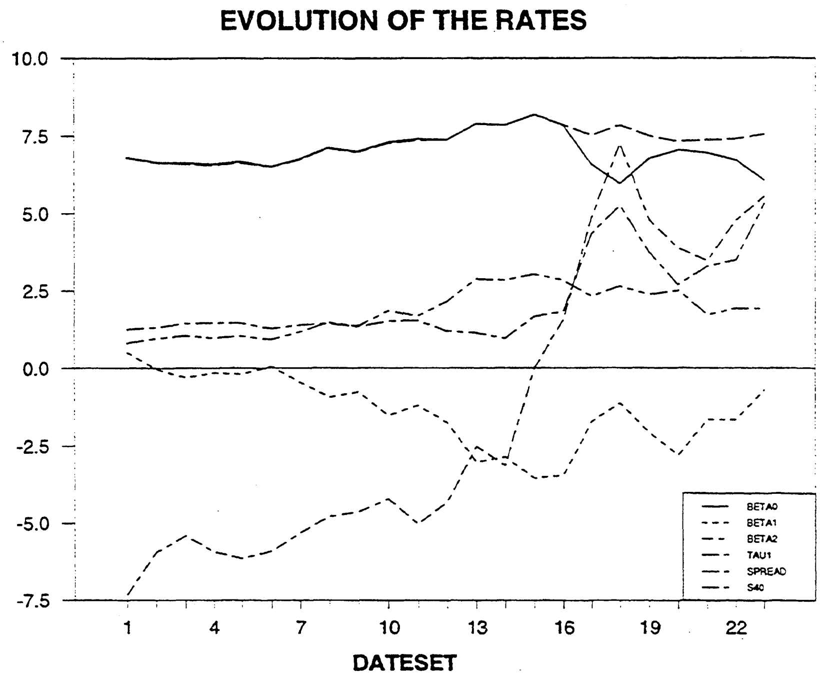
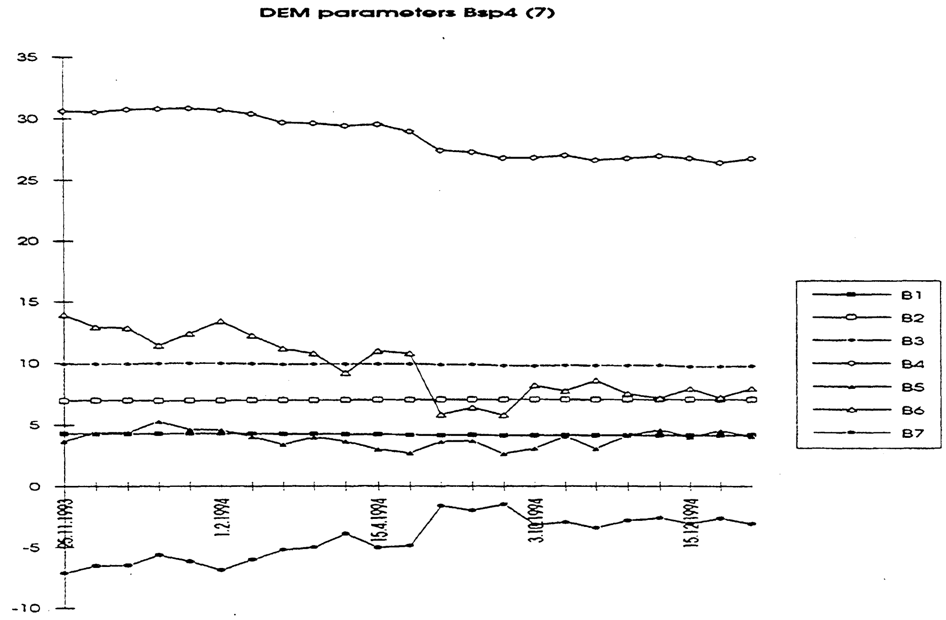


Figure 2.6 presents the results using Bsp approach when the number of knot points are increased. The location of the knot points followed McCulloch strategy. The hitratios, MAD and fluctuation are plotted against the left axis and R, AdjustedR and AmemiyaR are plotted against the adjusted righthand scale (R-99.0). The AdjustedR figures show where the point of the maximum parametrization has been reached. The overfitting is even more dramatically reflected in the quality of curves measures.

Table 2.5 presents the maximum parametrization of the spline models in our sample using different "decision rules (see section 2.3). It can be seen that the McCoulloch location method is not optimal as the adjusted coefficient figures always give smaller parametrization. One empirical observation: we found that the knots should be located farther out than McCulloch proposed.

Information criteria Table 2.6 summarizes our results as far as the information criteria is concerned. Both methods give extremely good results in absolute terms. In relative terms, the differences are surprisingly small.

Figure 2.5: DEM Parameters using the Bsp4 model



2.5.6 Discussion

Estimated rates are always a function of the input data, which means that the input data should be as accurate and as homogeneous as possible. Because the rates are a complex average of the bond data, interpretation of the rates is justified only if there are enough observations in that sector of the term structure.

Because both of our estimation methodologies lack theorethical justification, the most basic difference between them is probably that in the B-spline method the curves are constructed from vertical pieces (the maturity axis is splined !) whereas in the Nelson-Siegel method curves are constructed from

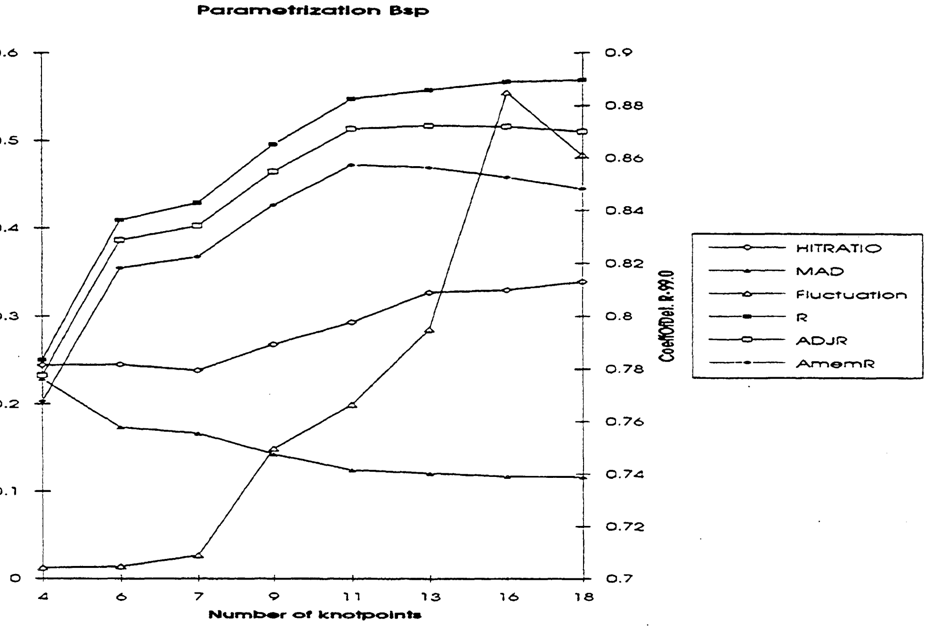
Currencycy	McCulloch	AdjRSQ
DEM	14	6-9
FRF	12	6-7
GBP	10	6
JPY	13	11
STR	14	11
USD	17	11

Table 2.5: Maximum number of component functions

Currency	Label	Bsp4	NS4 SP
DEM	RSQ	99.99707	99.9969
	ADJRSQ	99.99691	99.99681
	AmemR2	99.99669	99.99667
	R	99.82311	99.76584
	ADJR	99.81319	99.75945
	AmemR	99.79997	99.74881
FRF	RSQ	99.99846	99.9984
	ADJRSQ	99.99836	99.99835
	AmemR2	99.99822	99.99826
	R	99.8623	99.79843
	ADJR	99.85322	99.79199
	AmemR	99.84112	99.78127
GBP	RSQ	99.99605	99.99672
	ADJRSQ	99.99541	99.99647
	AmemR2	99.99455	99.99606
	R	99.70035	99.67389
	ADJR	99.65176	99.64944
	AmemR	99.58697	99.60867
JPY	RSQ	99.99244	99.99294
	ADJRSQ	99.99198	99.99274
	AmemR2	99.99138	99.9924
	R	99.52615	99.47452
	ADJR	99.49772	99.45921
	AmemR	99.45981	99.4337
STR	RSQ	99.99987	99.99893
	ADJRSQ	99.99987	99.9989
	AmemR2	99.99986	99.99886
	R	99.97873	99.83792
	ADJR	99.97756	99.83358
	AmemR	99.976	99.82635
USD	RSQ	99.99918	99.99874
	ADJRSQ	99.99915	99.99872
	AmemR2	99.99912	99.99869
	R	99.92558	99.83762
	ADJR	99.92336	99.83523
	AmemR	99.9204	99.83125

Table 2.6: Information criteria

Figure 2.6: Effects of parametrization changes in the B-spline model



horizontal components. This leads the emphasis in the spline methods to be on flexibility and accuracy, while in the Nelson-Siegel approach the asymptotic behaviour and the continuity (in its broad sense) of the rates is emphasied. This difference is also reflected in how the model parameters are interpreted.

One should bear in mind that the ultimate test of the estimation model is its feasibility as a trading tool. If the model reflects reality, then rates can be interpreted in a way that is presented in chapter 3.

2.5.7 Enhanced model specification

There are some small shortcomings to our current model. Most of these features relate to some market-specific structural feature, which in many cases due to legislation or taxation. These features could also be characterized as causes of heterogenities in the input data.

Coupon effect In GBP markets it is commonly known that high coupon instruments (at least used to) trade at significant discount compared to the low coupon instruments. The coupon effect explains about 10 % of the remaining estimation error. This effect has been very stable and is due to the different tax treatment of coupon income vs. capital gains for certain market participants.[6] The same feature can also be seen in DEM markets to a more limited extent.

Par effect Both the results of Kikugawa and Singleton (1994) and our findings indicate that in the JGB markets there are many market participants who prefer instruments that trade close to (or just below) par. This is due to the tax reasons. These *par effects* typically explain around 30-50 % of the remaining price errors otherwise unexplained by our estimation model. Unfortunately, the functional form of the par effect is most obviously non-linear and tends to be time variant as well.

Benchmark effect All markets feature certain *benchmark bonds*, which are highly liquid and typically, most recently issued. They tend to have lower transactions costs, so they are used for market trading purposes. Benchmark bonds are also typically "special in repo markets.

Repo market specials In all markets where well-established repo market exists (namely USD, STR, JPY, FRF and DEM) some papers (usually benchmarks or papers belonging to futures deliverable basket) might trade at a significant premium compared to their theoretical value. At the same time, however, they tend to trade "special[7] at the repo market, so the total holding period returns of these papers are similar or even higher than for issues in their neighborhood.

Convexity adjustment Highly convex instruments look often expensive against their theoretical value. High convexity means greater possibilities for additional return in exchange for lower initial yield. Without any convexity adjustment, the rates calculated by models tend to be strongly downward sloping at the long end of the markets. The value of convexity can be approximated by

$$V_{Cx} = 0.5 * Cnvx * E(\Delta y)^2, \tag{2.22}$$

where V_{Cx} is the price value of convexity, $Cnvx$ is the convexity of the bond and Δy is the annual yield volatility. [8]

[6] In 1996, the UK tax system was changed so that domestic institutions will also become neutral as to coupon income and capital gains. The only exception thereafter will be private individuals holding less than GBP 200 000 in British government bonds (gilts). This group only accounts 0.25% of gilt turnover.

[7] If an instrument trades "special ("special collateral) an investor holding the instrument can lend the instrument and borrow money against it at a very low "special rate and invest the borrowed money either to purchase a normal instrument ("generic collateral) or to deposit markets.

[8] The size of the convexity bias can be quite large. If the annual volatility is 100 bp and the convexity of 25 year Strip is 6.45, the yield is 8.1% and the rolling yield 6.95% (downward sloping yield curve), the value of convexity is 2.88% and the expected one-year return is 6.95% + 2.88% = 9.83% (Ilmanen, 1995)

Variable	Mean	Std Dev	Minimum	Maximum
Mean	0.5641	1.3143	-4.7168	2.5136
Std Dev	49.1261	26.6884	17.6891	107.5665
Median	0.2689	2.5403	-6.2	9.3
Minimum	-303.127	313.0189	-842.51	-34.63
Maximum	167.0104	83.4142	58.67	366.3
Skewness	-1.6128	4.2527	-9.28	5.7124
Kurtosis	27.3405	30.8846	0.8895	95.4287

Table 2.7: Characteristics of distribution of DEM pricing errors over observation days using the NS4 model

2.6 The distribution of pricing errors

2.6.1 Properties of the error

The distribution of the pricing errors is dependent on the quality of input data as well as the functional form of the approximation, the weighting of the observations and the norm used in the approximation.

In order to find the market spot rates that describe the bond data well, we must calculate the pricing error between the observed price and the model price. Observed price is not a single point, but rather a range of possible prices bounded by bid and ask prices for each instrument. The range between the bid and ask prices, or spread, gives traders a buffer against short-term market fluctuations or uncertainties in the market. Quite often the estimation of the term structure of interest rates is based on the "one price law. Thus the true market behaviour and the information presented in the spread is missed. We define the price error to be

$$\epsilon_j = \begin{cases} P_j^a - \tilde{P}_j, & \text{if } P_j^a > \tilde{P}_j \\ P_j^b - \tilde{P}_j, & \text{if } P_j^b < \tilde{P}_j \\ 0, & \text{if } P_j^b < \tilde{P}_j < P_j^a \end{cases} \tag{2.23}$$

Because of the error term definition, the "dirty data issues discussed in the section 2.5.4 and the shortcomings of the model discussed in the last section 2.5.7, the distribution of the pricing errors does not follow normal distribution. Table 2.6.1 shows the typical distribution of the pricing errors using Nelson Siegel four-parameter approach. The high kurtosis can clearly be seen, but the distribution is not significantly skewed. The distribution is heavily concentrated towards the center but the tails are fat.

Based on the definition of the error term, one could assume that the probability distribution is close to the double exponential distribution in general shape although the tails should be thinner.[9] Empirically, we found that the

[9]A double exponential distribution is basically a normal distribution that is cut, but because of the trading spread (that affects to our error definition) the middle part of the distribution is concentrated towards the centre with the tails moved inwards.

probability distribution function to be

$$p(x;\lambda) = \lambda * e^{-\lambda * \sqrt[3]{x}} \tag{2.24}$$

which hence indicates much thicker tails than one would expect.

In such of situations, a robust estimation methodology is needed. One can use the sample median as a description of the center of the distribution in favour of the mean. The same issue is reflected in confidence intervals. The width of the confidence interval range of the median are in our case about 1/4 of the confidence intervals of the mean.

We used the more robust approach in our spline model. When the L_1-norm is used in approximation the linear programming approach can be used. As Powell (1980, pp 183-186) has demonstrated the equality of these approaches, we utilized it with our spline model. Gonin and Money (1989) review different guidelines on how to choose the optimal exponent p in general L_p-norm estimation. Most of these guidelines are based on the kurtosis β_2 of the distribution. As an example, the rule proposed by Harter recommends the use of L_1 if kurtosis $\beta_2 > 3.8$, L_2 if $2.2 < \beta_2 < 3.8$ and L_∞ if $\beta_2 < 2.2$. Other rules yield fairly similar results.

Sometimes the estimation errors for the short-term instruments are significantly skewed. This is due to the fact that the cash flows of these instruments are discounted at the same rate as the coupon cash flows of the longer ones. Note that the individual cash flows of the long papers are not as liquid as the short instruments. Chambers, Carleton and Waldman (1984) have reported similar observations.

2.6.2 Relative value models

Relative value arbitrage models used to identify small temporary price discrepancies are mainly based on the assumption that the pricing errors of each individual bond are mean reverting. It is assumed that all instrument-specific features (as discussed in section 2.5.7) determine an individual "fair yield spread $\bar{\xi}_j(t)$ at time t. It is calculated as an average of the bond pricing errors $\epsilon_j(t) ... \epsilon_j(t-T)$ in *yield terms* from the last T observations. The relative value of an individual bond j at time $t+1$ is given by the Studentized deviation

$$z_j(t+1) = \frac{\xi_j(t+1) - \bar{\xi}_j(t)}{\sigma_{\bar{\xi}_j(t)}}. \tag{2.25}$$

where $\sigma_{\bar{\xi}_j(t)}$ is the standard deviation of the previous *yield errors*. Several trading rules can be formed based on the relative value measures. The performance of the rules is, of course, highly dependent on transactions costs, because the deviations are surprisingly small in the marketplace. One can, however, say that the ultimate test of a term structure model is its applicability as a trading model.

Chapter 3

Interpretation of the Term Structure

3.1 Expectations and premia

3.1.1 Expected consumption growth

In the introduction we mentioned that recent empirical studies indicate that the term structure predicts consumption growth better than vector autoregressions or leading commercial econometric models. This may sound surprising, but actually it is simple consequence of basic (neoclassical) economic models. Almost all dynamic macroeconomic models produce a Euler equation which links the current consumption with the future consumption according to a relation[1]

$$u'(c_t) = \beta R_t E_t u'(c_{t+1}), \tag{3.1}$$

where E_t denotes the expectation operator with respect to I_t, the common information set of the agents at period t, c_t is the consumption at period t, $\beta \in (0,1)$ is a constant discount factor, $u(\cdot)$ is the household's one-period utility function, and R_t is the I_t measurable (risk-free real) gross rate of return on bond holding.

The economic content of Eq. 3.1 is same as in all economic models with optimizing agents: the decisions are varied until the marginal losses (costs) are same as marginal gains (returns). That is, Eq. 3.1 states that the current consumption decision is optimal if the marginal loss in utility today (left-hand side of equation), ie when one unit of consumption is allocated from today to tomorrow, is same as the marginal gain in utility tomorrow (right-hand side of equation).

By rearranging Eq. 3.1, we obtain

$$1 = \beta R_t E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]. \tag{3.2}$$

[1]This approach to asset pricing was pioneered by Lucas (1978). For a very readable introduction, see Sargent (1987, ch. 3).

This is justified as $u'(c_t)$ is in the agents' information set at time t. Moreover, we assume that the agents have a constant relative risk-averse utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},$$

where σ is the agents' constant coefficient of relative risk-aversion. It is usually assumed[2] that σ is a constant between 1 and 10. Using this specification, Eq. 3.2 reduces to

$$1 = \beta R_t E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \right]. \tag{3.3}$$

Suppose now that the agents expect that c_{t+1} will go down relative to c_t or it is expected that $(c_t/c_{t+1})^{\sigma}$ will go up. Since β is constant, the only way that (3.3) can hold is if R_t goes down. Economic reasoning is simple: if interest rates are high when consumption is high relative to the future consumption, everyone wants to save. In aggregate they cannot, because the endowment of the economy is fixed. Therefore, they will bid the interest rates down until everyone is happy consuming c_t. Thus, we can make a simple rule-of-thumb: *upward-sloping term structure forecasts economic recoveries, downward-sloping term structure forecasts economic recessions.*

3.1.2 Expectations and risk premia

The oldest and simplest theory about the information content of the term structure is the (pure) expectations hypothesis. According to the pure expectations theory, forward rates are unbiased predictors of future spot rates. It is also common to modify the theory so that constant risk-premium is allowed—this is usually called the expectations hypothesis. Next we will investigate this assumption using modern macroeconomic theory.

Assume that the representative agent has access to both one-period and two-period bonds.[3] The Euler equations associated with them are

$$\begin{aligned} \beta R_{1t} E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right] &= 1 \\ \beta^2 R_{2t} E_t \left[\frac{u'(c_{t+2})}{u'(c_t)} \right] &= 1, \end{aligned}$$

[2] See, e.g., Mehra and Prescott (1985) for references.

[3] Some of the discussion below follows Sargent (1987, section 3.5) very closely. For a very readable continuous-time discussion, see Ingersoll (1987). The most important continuous-time models are by Cox, Ingersoll, and Ross (1985) and Longstaff and Schwartz (1992). Duffie (1992) presents a general framework for continuous-time models. Den Haan (1995) compares continuous-time and discrete-time models. For other important theoretical discussions of the term structure, see Cox, Ingersoll, and Ross (1981), Breeden (1986) and Campbell (1986).

where R_{1t} and R_{2t} are I_t measurable gross rate of return on one-period and two-period bonds, respectively. These imply

$$\begin{aligned} R_{1t}^{-1} &= \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] \\ R_{2t}^{-1} &= \beta^2 E_t\left[\frac{u'(c_{t+2})}{u'(c_t)}\right]. \end{aligned}$$

Recalling the definition of the spot rate Eq. 2.2

$$P_j = \frac{F_j}{(1+s_j)^{t_N}}.$$

By normalizing the nominal value of bond, F_j, for both bonds to one, and letting $R_{1t} \equiv (1+s_1)$ and $R_{2t} \equiv (1+s_2)^2$, we obtain the following pricing relations

$$P_{1t} = \beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] \tag{3.4}$$

$$P_{2t} = \beta^2 E_t\left[\frac{u'(c_{t+2})}{u'(c_t)}\right], \tag{3.5}$$

Next, using Eqs.3.4, 3.5, and the law of iterated expectations, $E_t[E_{t+1}[x_{t+2}]] = E_t[x_{t+2}]$, we obtain

$$\begin{aligned} P_{2t} &= \beta^2 E_t\left[\frac{u'(c_{t+2})}{u'(c_t)}\right] \\ &= \beta^2 E_t\left[\frac{u'(c_{t+1})}{u'(c_t)} \cdot \frac{u'(c_{t+2})}{u'(c_{t+1})}\right] \\ &= E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)} \cdot \beta\frac{u'(c_{t+2})}{u'(c_{t+1})}\right] \\ &= E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)} \cdot P_{1t+1}\right]. \end{aligned} \tag{3.6}$$

Eq.3.6 can be further decomposed using the definition of conditional covariance, $\text{cov}_t[x_{t+1}, y_{t+1}] = E_t[x_{t+1}y_{t+1}] - E_t[x_{t+1}]E_t[y_{t+1}]$, and eq. 3.4

$$\begin{aligned} P_{2t} &= E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\right] E_t[P_{1t+1}] + \text{cov}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}, P_{1t+1}\right] \\ &= P_{1t}E_t[P_{1t+1}] + \text{cov}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}, P_{1t+1}\right]. \end{aligned} \tag{3.7}$$

Eq. 3.7 is a generalized version of the expectations theory of the term structure. The first term is the expectations model. The second term is a *risk, liquidity* or *term premium.* If one-period bonds have a high price when $u'(c_{t+1})/u'(c_t)$ is low (note that $\text{cov}_t[\beta u'(c_{t+1})/u'(c_t), P_{1t+1}]$ is negative), and hence consumption growth is high, then holding a two-period bond is not a good strategy. If at $t+1$ you do happen to get a negative income shock, you would like to sell your bond and consume a little more. But with negative correlation the price of two-period bond will be especially low. It would have been better to buy two one-period bonds instead. Hence, when there is positive covariance between the consumption growth and the price of the one-period bond, the two-period bond price is driven down (it becomes worth less than earlier). Using eq. 2.2, this means that the two-period yield is driven up. Two-period bonds have to promise a higher yield to compensate for this risk. Summarizing: *when there is positive covariance between the consumption growth and the short-term bond prices, long-term bonds will carry a positive risk premium.* Moreover, as conditional covariance is taken with respect to all available information at time t and as this information changes over time, so will the conditional covariance. That is, *risk premium by its nature must vary over time.*

Eq.3.7 implies also that the expectations model holds only in special cases. One special case is when the utility function is linear in consumption. That is, people are risk neutral with respect to consumption. This means that $u'(c_{t+1})/u'(c_t) = 1 \; \forall t$ and $\text{cov}_t[\beta u'(c_{t+1})/u'(c_t), P_{1t+1}] = 0$. A second case is when there is no uncertainty: $\text{cov}_t[\beta u'(c_{t+1})/u'(c_t), P_{1t+1}] = 0$. Hence, *as long as people are risk averse, the world is uncertain and bond prices correlate with consumption growth, bond prices will carry a risk premium.*

3.1.3 Convexity bias

Suppose now, for the sake of an argument, that people are risk-neutral. Eq. 3.7 reduces to

$$P_{2t} = P_{1t} E_t[P_{1t+1}],$$

or

$$E_t[P_{1t+1}] = \frac{P_{2t}}{P_{1t}}. \tag{3.8}$$

Let $F_{1t} - 1$ denote the forward rate at period t of one-period bond from period $t+1$ to period $t+2$. Using eqs. 2.2 and 2.5, we obtain

$$E_t\left[\frac{1}{R_{1t+1}}\right] = \frac{R_{1t}}{R_{2t}} = \frac{1}{F_{1t}},$$

or

$$F_{1t} = \frac{1}{E_t\left[\frac{1}{R_{1t+1}}\right]}. \tag{3.9}$$

Note that due to Jensen's inequality ($E[x^{-1}] > (E[x])^{-1}$ for $x \in (0, \infty)$)

$$E_t\left[\frac{1}{R_{1t+1}}\right] > \frac{1}{E_t[R_{1t+1}]},$$

which implies

$$\frac{1}{E_t\left[\frac{1}{R_{1t+1}}\right]} < E_t[R_{1t+1}].$$

Hence, the implication of eq. 3.9 is that even when the agents are risk neutral,

$$F_{1t} < E_t[R_{1t+1}].$$

The result is called *convexity premium* or *bias*. Due to the convex relationship between the bond price and the bond yield, forward rates are not equal to the expected spot rates even when we assume risk-neutral investors.

3.1.4 Summary

Summarizing the results of this section, we note that the forward rates are equal to the sum of expected spot rates, risk premium, and the convexity bias. Risk premium will tend to make forward rates higher than expected spot rates, whereas the convexity bias will tend to make forward rates lower than expected spot rates.

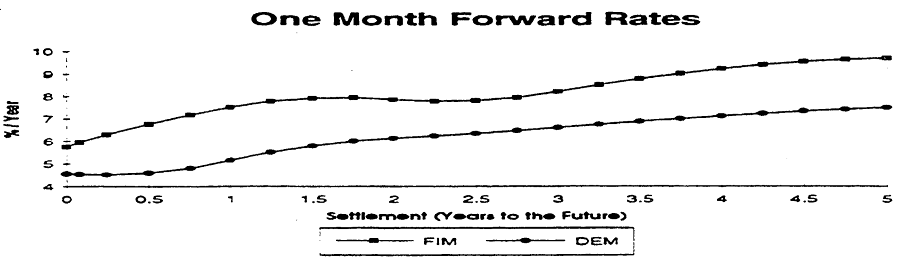
3.2 Forward interest rates as rough indicators

3.2.1 Interest rate expectations

In this section we will assume that neither risk premia nor convexity bias exist. Forward interest rates are unbiased predictors of future spot rates. We will show how to read interest, inflation, and exchange rate expectations in this unrealistic case. However, keeping in mind that the premia do exist, some useful intuition about the real expectations should be possible to obtain using (pure) expectations hypothesis as a guide.

Figure 3.1 shows one-month forward interest rates for Germany and Finland as of 10 August 1995. The x-axes show the settlement day as years ahead. That is, settlement day 1 corresponds to 10 August 1996. If we assume that the pure expectations hypothesis holds, on 10 August 1996 the one-month interest rate in Finland will (or, to be more precise, is *expected to be*) roughly 5% in Germany and roughly 7.5% in Finland. Moreover, the Figure shows the whole *time-path* of one-month interest rates: settlement 0 is the current one-month interest rate, 4.5% in Germany and 5.8% in Finland, and from then on they are expected to rise to 7% in Germany and to 9% in Finland in five years. Their difference is shown in Figure 3.2. Again, if we assume that the term premia are zero, we get the expected time-path of difference between Finnish and German one-month interest rates.

Figure 3.1: Forward rates as indicators



However, we probably can talk with more confidence about the expectations when we consider differences. Remember from the previous section that the forward rates are equal to the sum of expected spot rates, risk premium, and the convexity bias

$$F_{1t} = E_t[R_{1t+1}] + \varphi_{t,t+1},$$

where $\varphi_{t,t+1}$ denotes the *term premium*, the sum of risk premium and the convexity bias, for one-period bond from period t to period $t+1$. Now, let *'s denote foreign (German) variables. The difference between the Finnish forward rate and German forward rate will be

$$F_{1t} - F^*_{1t} = E_t[R_{1t+1}] + \varphi_{t,t+1} - \left[E^*_t[R^*_{1t+1}] + \varphi^*_{t,t+1}\right],$$

or

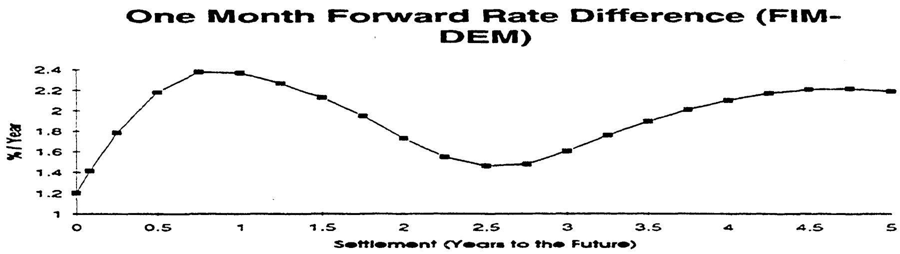
$$E_t[R_{1t+1}] - E^*_t[R^*_{1t+1}] = F_{1t} - F^*_{1t} - \left[\varphi_{t,t+1} - \varphi^*_{t,t+1}\right], \tag{3.10}$$

If we assume that $\varphi_{t,t+1} \approx \varphi^*_{t,t+1}$ we get

$$E_t[R_{1t+1}] - E^*_t[R^*_{1t+1}] \approx F_{1t} - F^*_{1t}.$$

Eq. 3.10 show that, assuming that the term premia are roughly the same in Finland and Germany, the expected time-path of difference between the Finnish and German one-month interest rates can be seen from Figure 3.2.

Figure 3.2: Forward rate differentials as indicators



3.2.2 Inflation rate expectations

Assume now that we have a market for real or index-linked bonds, ie bonds whose interest rate depends on the current inflation rate. Using the analysis of previous section, *'s denoting the index-linked variables, we observe that the difference between the forward rate of the nominal and the real bond is approximately the expected difference between their future spot rates. Since the nominal and index-linked bonds are issued by the same government, it may seem reasonable to assume that the only source of difference between their expected spot rates is due to the inflation expectations. In other words, the Fisher equation would hold and the expected future inflation rate would be the difference between the expected nominal interest rate and the real rate. However, as Svensson (1993) points out, just as forward rates have a risk premia over expected spot rates, nominal bonds have a *inflation risk premium* over real bonds.

Moreover, the analysis is complicated by the fact that currently only the Bank of England issues index-linked bonds. British real rates have usually fluctuated between 3 and 4 per cent. Hence, Svensson (1994b) assumes that an expected future Swedish short real rate is around 4%, and the inflation risk premium is zero. If we make the same assumptions, by subtracting 4% from the lines in Figure 3.1 we obtain the expected time-path of future inflation rates in Germany and Finland. In any case, as Barro (1995) states

> "The best and most objective sign that inflation is about to rise is a rise of yields on conventional gilts relative to those on indexed gilts. (...)
>
> Such information is available to inform policy because the Bank of England is the world leader at issuing, studying and perfecting index-linked securities. I wish the US Federal Reserve was as advanced."

3.2.3 Exchange rate expectations

Finally, assuming that in addition to zero term premia uncovered interest parity also holds—ie the risk premium from foreign exchange rate risk is zero—the differences between forward rates of two countries equals the expected future depreciation rate of the domestic country relative to the foreign country. Under above assumptions Figure 3.2 shows the time-path of how much Finnish markka is expected to depreciate relative to German mark.

3.3 Significance of premia

The empirical reserch on the term structure of interest rates has concentrated on the (pure) expectations hypothesis. That is, the question has been if forward rates are unbiased predictors of future spot rates. The most popular way to test the hypothesis has been running a linear regression (error term omitted)

$$s_{t+1} - s_t = a + b(f_{t+1,t} - s_t).$$

The pure expectations hypothesis implies that $a = 0$ and $b = 1$. Rejection of the first restriction, $a = 0$, gives the expectations hypothesis: term premium is nonzero but constant.

Yet, from the earlier discussion we have seen that even in principle this should not be the case. By and large the literature dismisses both restrictions.[4] Rejection of the second restriction, $b = 1$, requires, under the alternative, a risk premium[5] that varies through time and is correlated with the forward premium, $f_{t+1,t} - s_t$. Both implications are consistent with the theory presented above, and most studies–(eg Fama and Bliss (1987) and Fama and

[4]The literature is huge. Useful surveys are provided by Melino (1988), Shiller (1990), and Mishkin (1990b). The most important individual studies are probably Shiller (1979), Shiller, Campbell, and Schoenholtz (1983), Fama (1984, 1990), Fama and Bliss (1987), Froot (1989), Campbell and Shiller (1991), and Campbell (1995).

[5]The effect of convexity bias is fairly easy to take into account. For more details about the effect of convexity, see Cox, Ingersoll, and Ross (1981), Ho (1990), and Gilles (1994).

French (1989))–take this to indicate the existence of time-varying risk premium. Therefore, we should ask if there are models which are capable of generating similar risk premiums to the ones observed in the real time series.

This question is broaded in Backus, Gregory, and Zin (1989). Using a complete markets model, tehy conclude that the model can account for neither the sign nor the magnitude of average risk premiums in forward prices and holding-period returns. Similar puzzles have been obtained for equity premiums by Mehra and Prescott (1985) and for holding-period yields by Grossman, Melino, and Shiller (1987).

A recent study by Heaton and Lucas (1992) may provide an answer to these puzzles. They use a three-period incomplete markets model with trading costs to address the same question. Their answer is that "uninsurable income shocks may help explain one of the more persistent term structure puzzles" but "the question remains whether the prediction of a relatively large forward premium will obtain in a long horizon model."[6]

[6]See also Shenn and Starr (1994) for evidence about importance of trading costs.

Chapter 4

Conclusions

We have described and compared two different methods to describe the term structure of interest rates. For our purposes an extended Nelson-Siegel approach seems to be preferred methodology. If, however, a spline methodology is used one must be careful with parametrization strategy.

We analyzed the distribution of the pricing errors of a simple model and found that they do not follow normal distribution. Some explanation of non-normality was explained by the economic and legislative environment as well as from the behaviour of the market participants.

After deciding on the Nelson-Siegel estimation methodology is suitable as a base of further analysis, we turned our focus on the interpretation of the term structure. A couple of simple models of the predicted evolution of the rates were introduced. After that we went through some of the components that affect and distort the basic models. As a conclusion we found that a more comprehensive model that allows time-varying risk premia and captures all market pecularities is needed to analyze the behaviour of forward rates.

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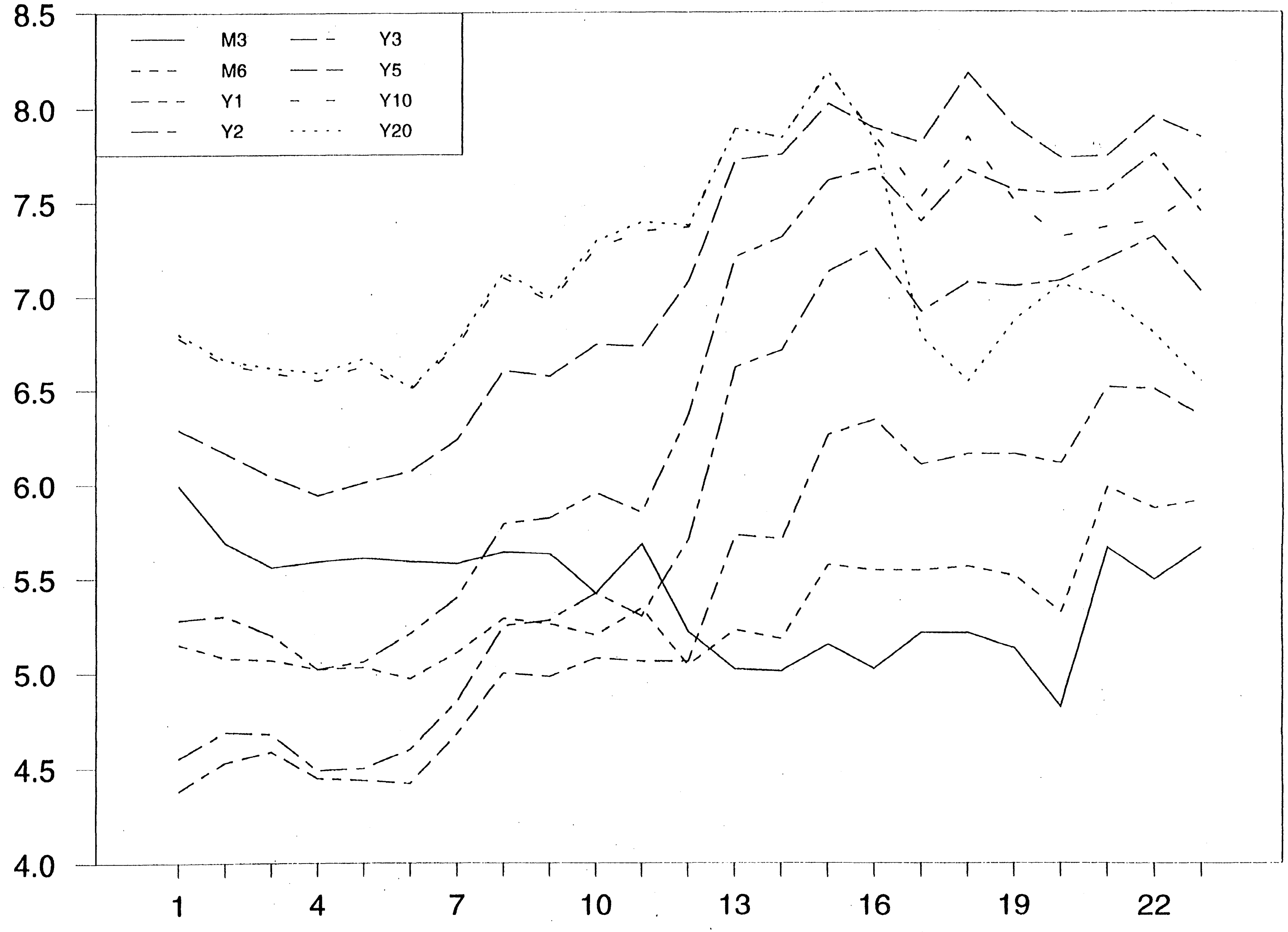
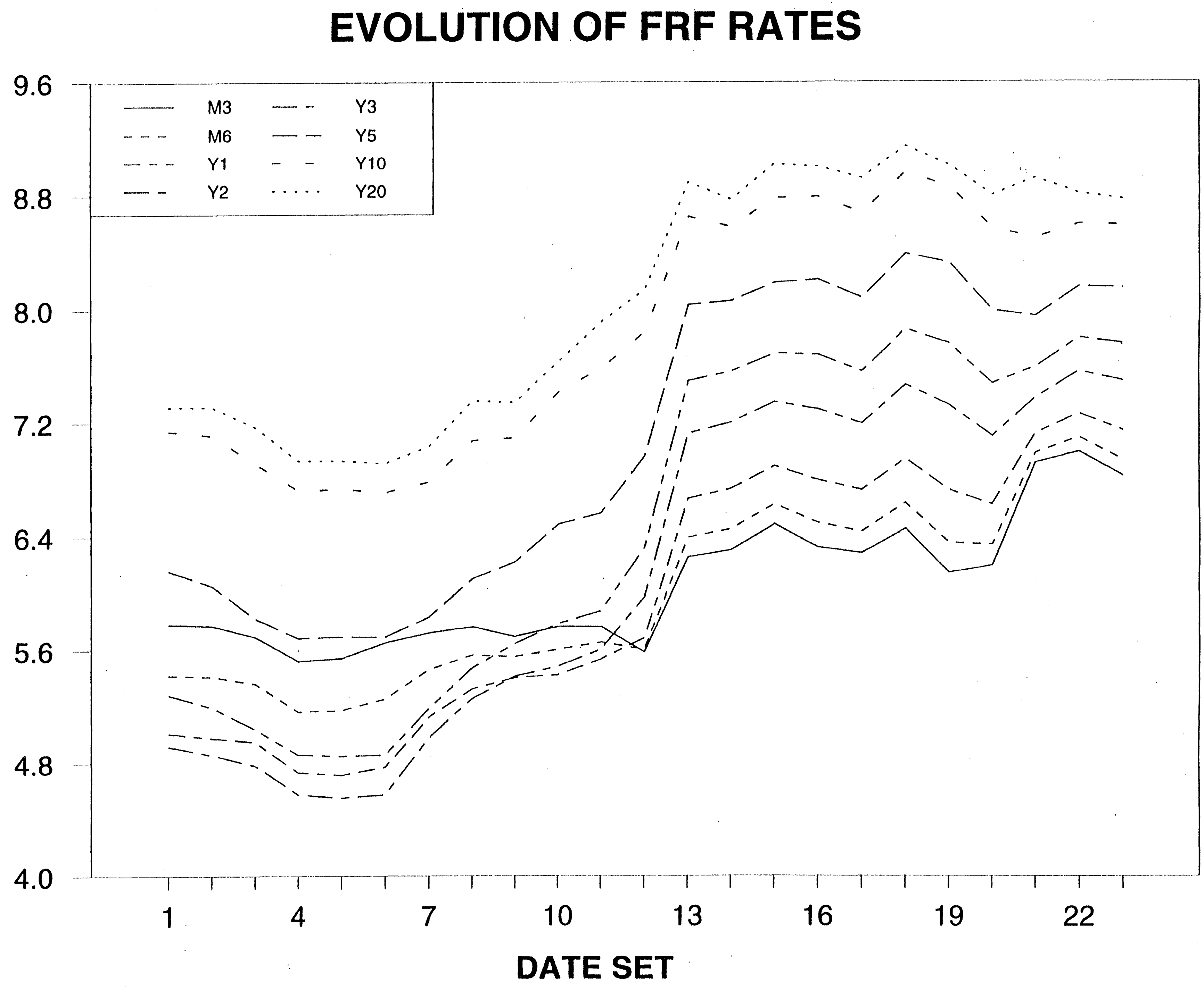
Figure A.1: An example of relative value report

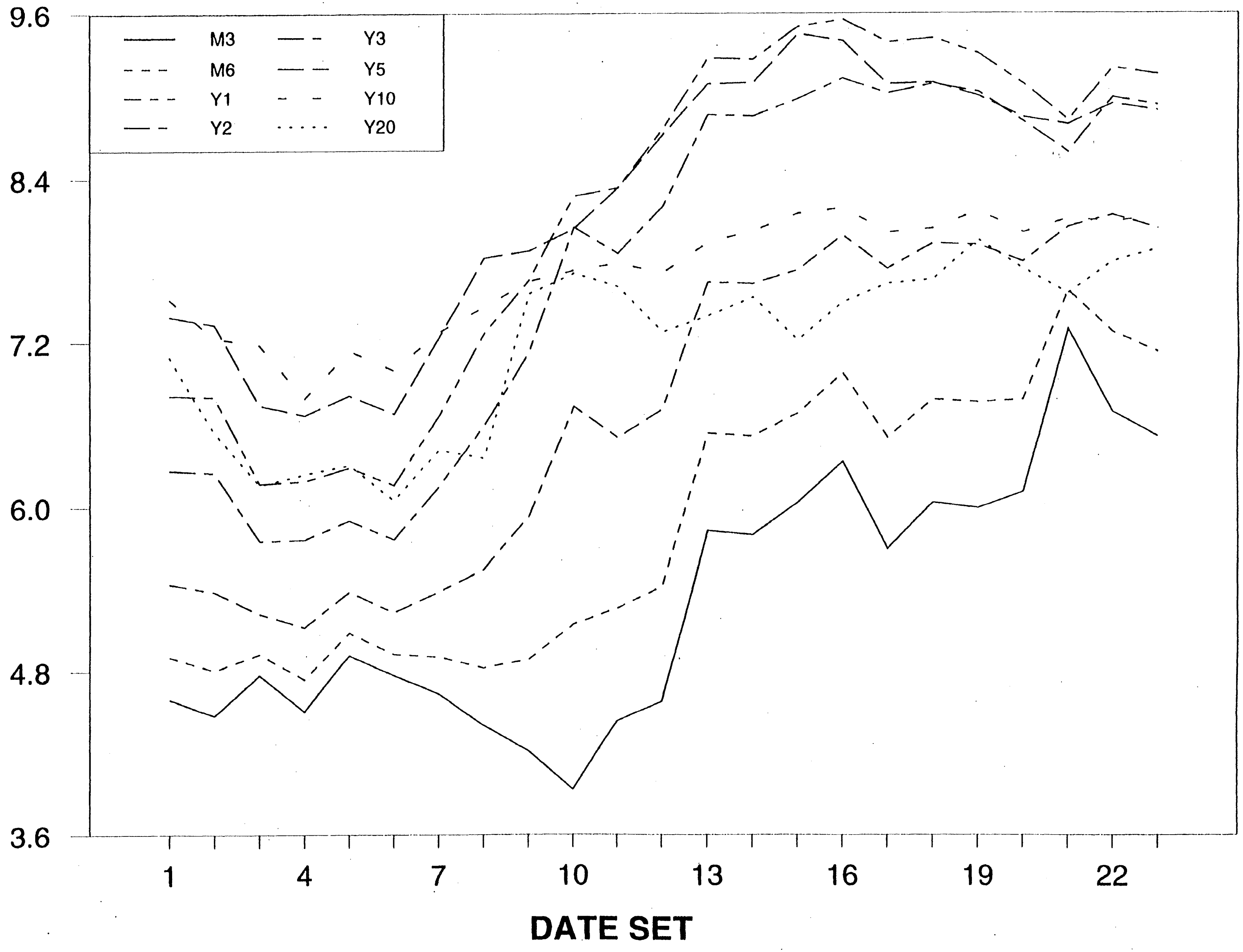
Ticker	Maturity	Cp	Bid	Ask	MD	YTM	Model bps	Avg bps	Std bps	Min bps	Max bps	Z-value
T_4.2500_10/94	31.10.1994	4.25	100.59	100.63	0.88	3.58	5.5	5.4	3.4	1.4	14.6	0.04
T_11.625_11/94	15.11.1994	11.6	107.44	107.47	0.91	3.59	6.2	9.7	7.2	3.6	30.8	-0.49
S_0_11/15/94	15.11.1994	0	96.648	96.73	0.93	3.62	2.9	2.3	2.7	0.0	7.9	0.25
T_6_11/15/94	15.11.1994	6	102.19	102.22	0.92	3.64	2.1	3.4	4.0	0.0	14.8	-0.32
T_8.2500_11/94	15.11.1994	8.25	104.27	104.3	0.91	3.64	1.5	3.8	5.0	-0.8	19.2	-0.46
T_10.125_11/94	15.11.1994	10.1	106	106.06	0.91	3.64	1.2	3.8	5.4	0.0	19.8	-0.48
T_4.6250_11/94	30.11.1994	4.63	100.95	100.97	0.96	3.64	4.2	-7.6	52.7	-225.3	9.4	0.22
T_4.6250_12/94	31.12.1994	4.63	100.98	101.02	1.02	3.68	7.2	7.3	2.1	4.8	12.2	-0.04
T_7.6250_12/94	31.12.1994	7.63	104.11	104.14	1.01	3.69	5.9	8.3	3.1	3.8	14.9	-0.78
T_8.6250_01/95	15.1.1995	8.63	105.27	105.3	1.04	3.76	0.2	2.2	2.6	0.0	8.7	-0.79
T_4.2500_01/95	31.1.1995	4.25	100.56	100.59	1.11	3.75	3.5	3.0	1.7	0.0	6.9	0.29
T_3_02/15/95	15.2.1995	3	100.3	102.3	1.17	2.75	105.6	115.8	3.8	108.0	122.3	-2.69
T_7.8750_02/00	15.2.1995	7.88	104.41	104.53	1.12	4.07	-16.7	-20.5	12.7	-40.5	-1.1	0.29
T_5.5000_02/95	15.2.1995	5.5	102	102.03	1.14	3.78	2.7	1.0	1.5	-0.3	5.0	1.13
T_11.250_02/95	15.2.1995	11.3	108.69	108.72	1.11	3.77	2.2	2.5	3.0	-0.4	9.9	-0.11
T_7.7500_02/95	15.2.1995	7.75	104.61	104.64	1.13	3.78	1.8	1.7	2.0	-0.3	6.1	0.07
S_0_02/15/95	15.2.1995	0	95.585	95.686	1.18	3.8	0.6	0.0	0.0	0.0	0.0	55.10
T_10.500_02/95	15.2.1995	10.5	107.78	107.84	1.11	3.8	0.0	1.6	2.7	0.0	8.8	-0.60
T_3.8750_02/95	28.2.1995	3.88	100.08	100.09	1.19	3.81	0.7	0.5	1.1	-0.9	3.6	0.14
T_3.8750_03/95	31.3.1995	3.88	100.02	100.03	1.27	3.86	0.2	0.6	1.0	-0.3	3.7	-0.40
T_8.3750_04/95	15.4.1995	8.38	105.88	105.91	1.28	3.91	0.0	0.4	1.1	-1.3	3.8	-0.33
T_3.8750_04/95	30.4.1995	3.88	99.953	99.969	1.35	3.91	0.0	-0.4	0.7	-1.9	0.5	0.59
T_12.625_05/95	15.5.1995	12.6	112.23	112.3	1.34	3.87	3.6	1.9	3.5	-2.1	10.7	0.47
T_5.8750_05/95	15.5.1995	5.88	102.72	102.73	1.38	3.93	-0.3	-1.0	1.5	-5.4	0.3	0.43
T_11.250_05/95	15.5.1995	11.3	110.25	110.31	1.35	3.91	0.0	0.9	2.6	-3.1	6.9	-0.35
T_10.375_05/95	15.5.1995	10.4	109.02	109.08	1.36	3.92	0.0	0.4	2.1	-1.8	6.7	-0.21
T_8.5000_05/95	15.5.1995	8.5	106.36	106.42	1.37	3.94	0.0	-0.3	1.7	-4.4	3.2	0.16
S_0_05/15/95	15.5.1995	0	94.519	94.64	1.42	3.93	0.0	0.0	0.0	0.0	0.0	0.00
T_4.1250_05/95	31.5.1995	4.13	100.25	100.27	1.43	3.95	0.0	-8.2	31.5	-138.0	0.0	0.26
T_4.1250_06/95	30.6.1995	4.13	100.2	100.23	1.49	3.99	0.0	-0.3	0.5	-1.6	0.0	0.65
T_8.8750_07/95	15.7.1995	8.88	107.45	107.52	1.47	4.06	-1.9	0.6	1.7	-0.7	6.6	-1.47
T_4.2500_07/95	31.7.1995	4.25	100.31	100.36	1.57	4.05	-1.8	-1.5	0.6	-2.6	0.0	-0.56
T_4.6250_08/95	15.8.1995	4.63	100.91	100.97	1.6	4.07	0.0	-0.7	1.0	-3.3	0.0	0.67
T_10.500_08/95	15.8.1995	10.5	110.52	110.58	1.54	4.03	0.0	0.9	2.2	0.0	8.4	-0.43
T_8.5000_08/95	15.8.1995	8.5	107.23	107.3	1.56	4.05	0.0	0.3	1.2	-0.9	4.7	-0.27
S_0_08/15/95	15.8.1995	0	93.422	93.563	1.67	4.04	0.0	0.0	0.0	0.0	0.1	-0.31
T_3.8750_08/95	31.8.1995	3.88	99.672	99.703	1.65	4.07	0.0	-0.1	0.6	-1.6	1.9	0.12
T_3.8750_09/95	30.9.1995	3.88	99.594	99.625	1.73	4.11	1.0	-0.2	0.9	-2.6	1.9	1.40
T_8.6250_10/95	15.10.1995	8.63	107.98	108.05	1.72	4.13	0.0	0.8	1.6	0.0	5.5	-0.49
T_3.8750_10/95	31.10.1995	3.88	99.484	99.516	1.82	4.16	-0.6	-1.2	1.0	-3.5	0.0	0.61
S_0_11/15/95	15.11.1995	0	92.337	92.496	1.91	4.13	2.8	0.7	1.0	0.0	2.2	2.19
T_9.5000_11/95	15.11.1995	9.5	109.95	110.02	1.79	4.13	0.8	1.3	1.9	0.0	6.4	-0.27
T_8.5000_11/95	15.11.1995	8.5	108.05	108.11	1.8	4.16	0.0	0.5	1.2	-0.1	4.6	-0.39
T_11.500_11/95	15.11.1995	11.5	113.64	113.7	1.77	4.14	0.0	0.7	1.9	-1.5	6.3	-0.37
T_5.1250_11/95	15.11.1995	5.13	101.78	101.84	1.84	4.16	0.0	0.3	0.8	0.0	3.4	-0.33
T_4.2500_11/95	30.11.1995	4.25	100.11	100.14	1.89	4.19	-0.3	0.0	0.0	0.0	0.0	-25.40
T_9.2500_01/96	15.1.1996	9.25	110.03	110.09	1.88	4.24	0.0	-0.6	0.6	-2.0	0.0	1.04
T_7.5000_01/96	31.1.1996	7.5	106.61	106.67	1.95	4.26	0.0	-0.1	0.6	-1.3	2.0	0.11

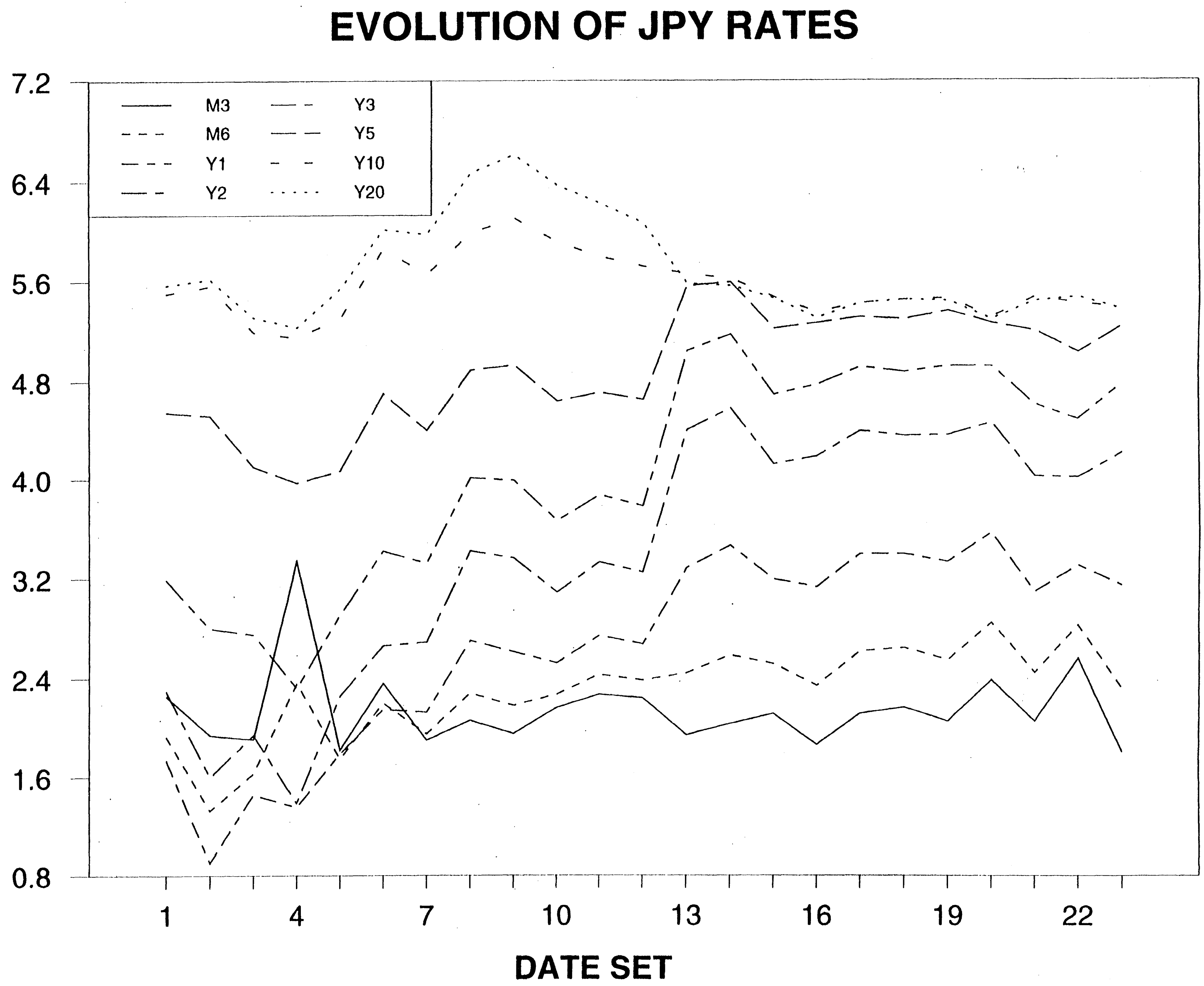
EVOLUTION OF DEM RATES



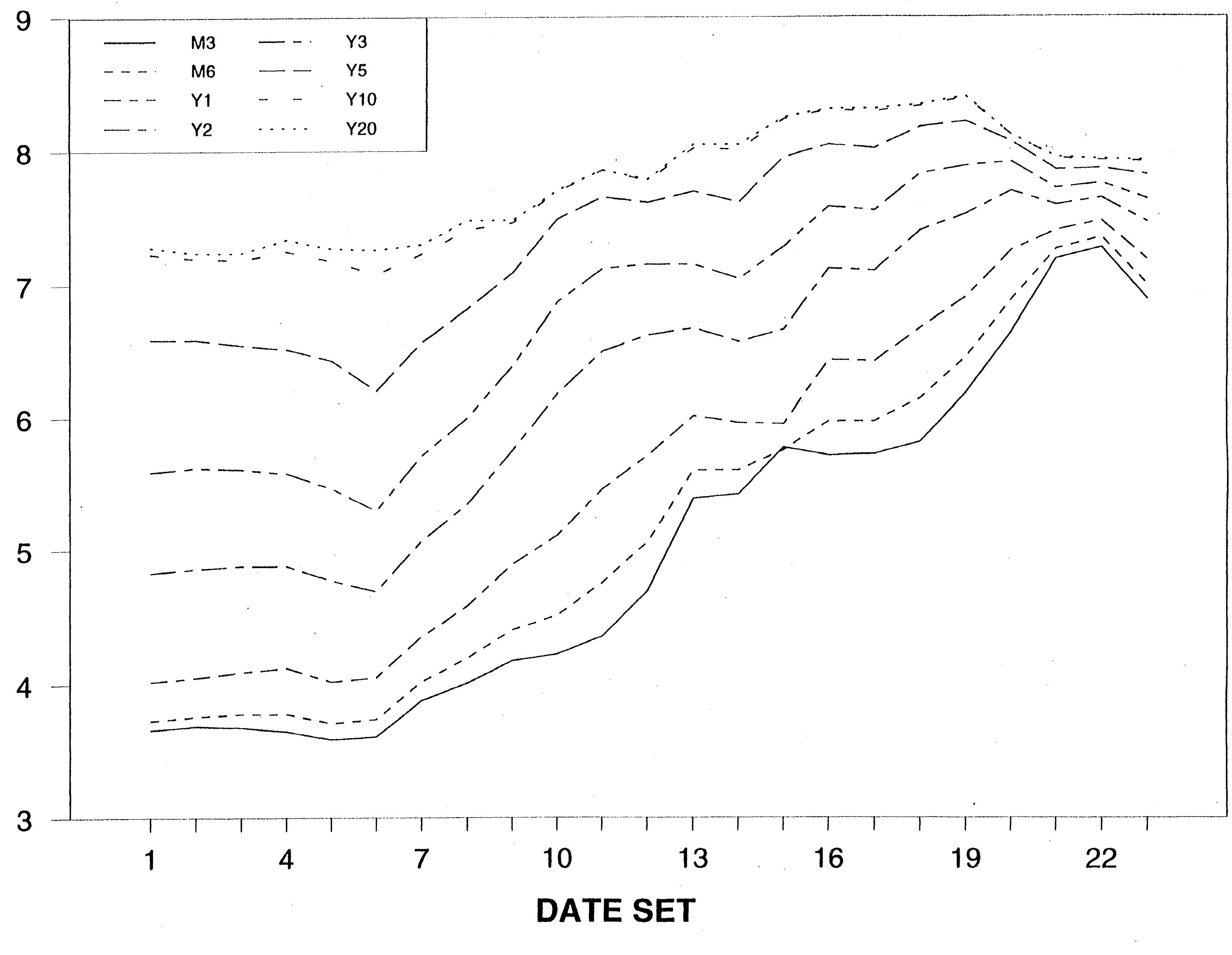

EVOLUTION OF FRF RATES
9.6
8.8
8.0
7.2
6.4
5.6
4.8
4.0
M3
M6
Y1
Y2
Y3
Y5
Y10
Y20
1
4
7
10
13
16
19
22
DATE SET

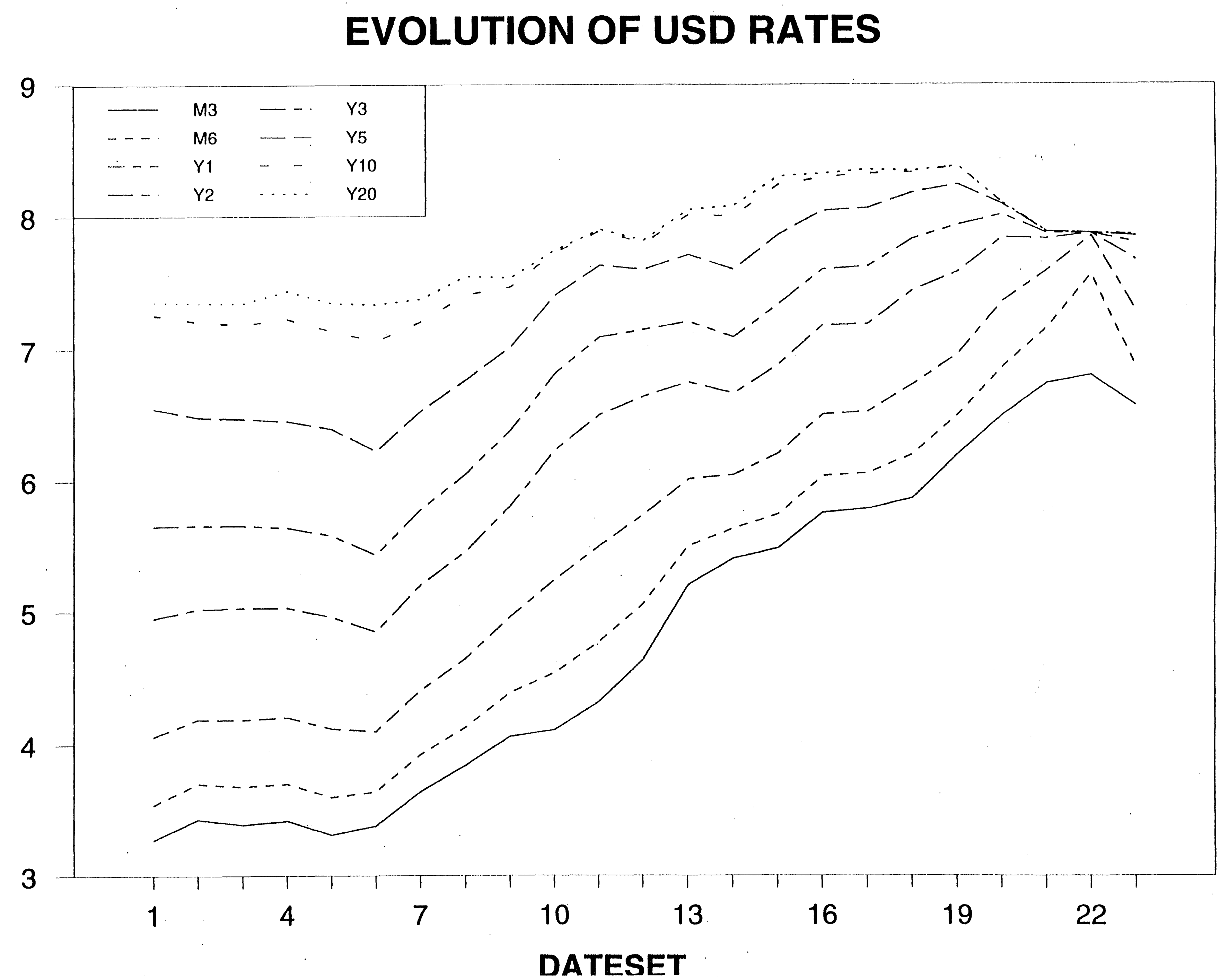
EVOLUTION OF GBP RATES




EVOLUTION OF JPY RATES
M3
M6
Y1
Y2
Y3
Y5
Y10
Y20
7.2
6.4
5.6
4.8
4.0
3.2
2.4
1.6
0.8
1
4
7
10
13
16
19
22
DATE SET

EVOLUTION OF STR RATES




EVOLUTION OF USD RATES
M3
M6
Y1
Y2
Y3
Y5
Y10
Y20
9
8
7
6
5
4
3
1
4
7
10
13
16
19
22
DATESET

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